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Robust Stabilization of Nonlinear Systems: The LMI Approach

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This paper presents a new approach to robust quadratic stabilization of nonlinear systems within the framework of Linear Matrix Inequalities (LMI). The systems are composed of a linear constant part perturbed by an additive nonlinearity which depends discontinuously on both time and state. The only information about the nonlinearity is that it satisfies a quadratic constraint. Our major objective is to show how linear constant feedback laws can be formulated to stabilize this type of systems and, at the same time, maximize the bounds on the nonlinearity which the system can tolerate without going unstable.

We shall broaden the new setting to include design of decentralized control laws for robust stabilization of interconnected systems. Again, the LMI methods will be used to maximize the class of uncertain interconnections which leave the overall system connectively stable. It is useful to learn that the proposed LMI formulation "recognizes" the matching conditions by returning a feedback gain matrix for any prescribed bound on the interconnection terms. More importantly, the new formulation provides a suitable setting for robust stabilization of nonlinear systems where the nonlinear perturbations satisfy the generalized matching conditions.

Keywords: Robust stabilization; LMI; Generalized matching conditions; Interconnected systems; Decentralized control

1. INTRODUCTION

The subject of this paper is the robust quadratic stability and feedback stabilization of a class of linear constant systems under additive

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perturbations which are nonlinear and discontinuous functions in time and state of the system. The perturbations are uncertain and all we know about them is that they are contained within quadratic bounds. Conceptually, our mathematical models have the structure of Lur'e-Postnikov systems, and it is natural that we attempt to solve the new stability problems in the framework of absolute stability (Popov [1], Yakubovich [2]). A crucial difference, however, is in that we do not impose any structure on the way the nonlinear functions depend on the state of the system. This distinction prevents us from using the powerful methods of absolute stability in solving robust stability problems in the new setting.

Another important distinction of our results within the vast literature on robust control (*e.g.*, see the surveys by Šiljak [3], Leitmann [4], Kokotović and Arcak [5], as well as the book by Battacharya *et al.* [6]), is that we want to determine a linear control which stabilizes the system and, at the same time, maximizes the class of uncertain perturbations which can be tolerated by the stabilized (closed-loop) system. We do not assume that the linear part is stable, nor that the perturbations satisfy the matching conditions. The main objective of this work is to show how a solution to such a complex control problem can be obtained by using the full extent of the S-procedure [7] and applying the efficient tools of Linear Matrix Inequalities (LMI) (Boyd *et al.* [8]). The LMI methods are flexible in allowing inclusion of a wide variety of additional design requirements, such as the size and structure of the gain matrices, degree of exponential stability, and time delays, to mention a few. In the subsequent paper [9], we shall extend the present framework to include robust stability and stabilization of discrete-time systems (see also the paper by Oliveira *et al.* [10]).

Of our special interest in this paper is to exploit the extraordinary ability of the LMI approach to accommodate the decentralized information structure constraints imposed on the gain matrices (Geromel and Bernussou [11], Geromel *et al.* [12], Ikeda *et al.* [13], Cao *et al.* [14], Geromel *et al.* [15]). In control applications of large scale systems, the controllers are allowed to use only the locally available state variables [16]. In interconnected systems this translates into a restriction that the control law for each subsystem contains only the state of that subsystem. By recognizing the well known fact (*e.g.* [17]) that interconnections can be considered as perturbations of subsystem dynamics, we shall compute the robust decentralized control laws using our new LMI problem formulation. The resulting closed-loop interconnected systems will be connectively stable with the maximal bounds on the interconnection terms which can change discontinuously as functions of time and state of the system.

2. ROBUST STABILITY

Let us consider a nonlinear system described by the differential equation

$$\dot{x} = Ax + h(t, x) \tag{2.1}$$

where $x \in \mathbf{R}^n$ is the state of the system, A is an $n \times n$ constant matrix, and $h: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$ is the nonlinearity. In this section, we assume that the matrix A is Hurwitz stable, that is, it has all eigenvalues with negative real parts. We shall remove this assumption in the later sections, when we use linear state feedback to stabilize an otherwise unstable matrix A.

With regard to the nonlinearity $h(t, x)$ we assume that it is a piecewise-continuous function in both arguments t and x (Filippov [18]). Notice that piecewise-continuity of $h(\cdot,\cdot)$ implies the same property of the right hand side $Ax+h(t,x)$ of Eq. (2.1), and their domains of continuity coincide.

The crucial assumption about nonlinear function $h(t, x)$ is that it is uncertain and all we know is that, in the domains of continuity, it satisfies the quadratic inequality

$$h^T(t, x)h(t, x) \leq \alpha^2 x^T H^T H x \tag{2.2}$$

where $\alpha > 0$ is the bounding parameter and H is a constant $\ell \times n$ matrix. We immediately note that for any given H, inequality (2.2) defines a class of piecewise-continuous functions

$$\mathbf{H}_\alpha = \{h: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n | h^T h \leq \alpha^2 x^T H^T H x \text{ in the domains of continuity}\}. \tag{2.3}$$

The class $\mathbf{H}_\alpha$ is comprised of functions that satisfy $h(t,0)=0$ in their domains of continuity, and $x=0$ is an equilibrium of system (2.1). The main objective of this paper is to establish stability of $x=0$, and we introduce the following:

DEFINITION (2.4) System (2.1) is robustly stable with degree α if the equilibrium $x=0$ is globally asymptotically stable for all $h(t,x)\in\mathbf{H}_\alpha$.

Definition (2.4) is similar to the standard definition of absolute stability (*e.g.* [19]). A significant difference, however, is that we do not assume here any structure of dependence of $h(t,x)$ on the state x; the function $h(t,x)$ depends on the state rather than the output of the system.

To establish robust stability in the sense of Definition (2.4), we use a quadratic Liapunov function

$$V(x) = x^T P x, \tag{2.5}$$

where P is a symmetric positive definite matrix ($P>0$). We assume that $V(x)$ is a $\mathbf{C}^1$ function satisfying for all $x\in\mathbf{R}^n$

$$\phi_1(\|x\|) \le V(x) \le \phi_2(\|x\|), \tag{2.6}$$

where $\phi_1,\phi_2\in K_\infty$ are Hahn's functions (*e.g.*, Khalil [19]). Using the well-known results of Filippov [18], we can establish stability of Definition (2.4) by negative definiteness of the derivative

$$\dot V(x)_{(2.1)} = (\mathrm{grad}_x V)\{Ax + h(t,x)\} \le -\phi_3(\|x\|) \tag{2.7}$$

in the domains of continuity of $h(\cdot,\cdot)$. Thus, from this point on we shall consider all equations and inequalities involving $h(\cdot,\cdot)$ only in the domains of continuity without any further mentioning of this fact.

Now, we want to show how the LMI approach (Boyd *et al.* [8]) is ideally suited to analyze the introduced notion of robust stability. We start with the fact that constraint (2.2) is equivalent to the quadratic inequality

$$\begin{bmatrix} x \\ h \end{bmatrix}^T \begin{bmatrix} -\alpha^2 H^T H & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} \le 0. \tag{2.8}$$

To show stability we compute

$$\dot{V}(x)_{(2.1)} = x^T(A^TP + PA)x + h^TPx + x^TPh \tag{2.9}$$

and require

$$P > 0, \quad x^T(A^TP + PA)x + h^TPx + x^TPh < 0, \tag{2.10}$$

or

$$P > 0, \quad \begin{bmatrix} x \\ h \end{bmatrix}^T \begin{bmatrix} A^TP + PA & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ h \end{bmatrix} < 0. \tag{2.11}$$

By using the S-procedure (Yakubovich [7]) we conclude that, when (2.8) is satisfied, (2.11) is equivalent to the existence of P and a number $\tau > 0$ such that

$$\begin{gathered} P > 0 \\ \begin{bmatrix} A^TP + PA + \tau\alpha^2 H^TH & P \\ P & -\tau I \end{bmatrix} < 0 \end{gathered} \tag{2.12}$$

which is further equivalent to the existence of a matrix Y so that

$$\begin{gathered} Y > 0 \\ \begin{bmatrix} AY + YA^T + \alpha^2 YH^THY & I \\ I & -I \end{bmatrix} < 0, \end{gathered} \tag{2.13}$$

where $Y = \tau P^{-1}$.

Relying on the Schur complement formula, (2.13) can be rewritten as

$$\begin{gathered} Y > 0 \\ \begin{bmatrix} AY + YA^T & I & YH^T \\ I & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} < 0, \end{gathered} \tag{2.14}$$

where $\gamma = 1/\alpha^2$.

Our interest is in establishing robust stability for as large a class $\mathbf{H}_\alpha$ as possible. We assume that the matrix H is selected and maximize parameter α by solving the following LMI problem in Y and γ

$$\begin{aligned} &\text{minimize} && \gamma \\ &\text{subject to} && Y > 0 \\ & && \begin{bmatrix} AY + YA^T & I & YH^T \\ I & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} < 0. \end{aligned} \tag{2.15}$$

We arrive at

THEOREM (2.16) *System* (2.1) *is robustly stable with degree α if problem* (2.15) *is feasible.*

To illustrate the benefit of the LMI formulation (2.15) let us consider the following simple example [16]:

Example (2.17) A system is given as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \mu(t)x, \tag{2.18}$$

where $\mu(t)$ is an uncertain time-varying parameter. To estimate the largest bound on the perturbation function,

$$|\mu(t)| \leq \alpha \tag{2.19}$$

which can be tolerated by stability of the nominal linear part in (2.18), the following problem was formulated in (Šiljak [17]):

$$\begin{aligned} &\max_{Q} \left\{ \frac{\lambda_m(Q)}{\lambda_M(P)} \right\} \\ &\text{subject to} \quad A^T P + PA = -Q. \end{aligned} \tag{2.20}$$

The maximization problem was solved in (Patel and Toda [20]), where it was shown that $\bar{Q} = I$ is the optimal choice.

In the case of system (2.18), we solve the Liapunov matrix equation in (2.20) for $\bar{Q} = I$ and obtain

$$\bar{P} = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}, \quad \bar{Q} = I \tag{2.21}$$

resulting in

$$\bar{\alpha} = \frac{1}{2}\frac{\lambda_m(\bar{Q})}{\lambda_M(\bar{P})} = 0.3810. \tag{2.22}$$

On the other hand, solving the LMI problem (2.15) for $H=I$ by the LMI Toolbox (Gahinet *et al.* [21]), we get the bound

$$\hat{\alpha} = 0.5397 \tag{2.23}$$

with matrices

$$\hat{P} = \begin{bmatrix} 0.7774 & 0.2604 \\ 0.2604 & 0.1868 \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} 1.0428 & 0.3783 \\ 0.3783 & 0.5993 \end{bmatrix}, \tag{2.24}$$

which is more than 40% larger than the bound $\bar{\alpha} = 0.3810$ obtained for $\bar{Q} = I$.

If we apply the transformation (Šiljak [16])

$$x = T\tilde{x}, \quad T = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \tag{2.25}$$

to improve the bound α, we obtain

$$\tilde{P} = I, \quad \tilde{Q} = 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{2.26}$$

producing the bound

$$\tilde{\alpha} = \frac{1}{2}\frac{\lambda_m(\tilde{Q})}{\lambda_M(\tilde{P})} = 1 \tag{2.27}$$

which is almost twice the bound $\hat{\alpha} = 0.5397$ obtained in the original space.

When the LMI solution is computed in the transformed space, it reproduces the maximal bound $\tilde{\alpha} = 1$ confirming at the same time the fact that the LMI formulation (2.15) of robust stability margin is "coordinate dependent".

3. ROBUST STABILIZATION

When the linear part of the system (2.1) is not stable we can introduce feedback to stabilize the overall system and, at the same time, maximize its tolerance to uncertain nonlinear perturbations. The system with control input is described by equation

$$\dot{x} = Ax + Bu + h(t, x) \tag{3.1}$$

where B is $n \times m$ constant matrix, $u: \mathbf{R}^n \to \mathbf{R}^m$ is the linear feedback control law

$$u(x) = Kx, \tag{3.2}$$

and K is an $m \times n$ constant gain matrix. We assume that the pair (A, B) is stabilizable.

When we apply the feedback (3.2) to the open-loop system (3.1), we obtain the closed-loop system

$$\dot{x} = \hat{A}x + h(t, x) \tag{3.3}$$

where

$$\hat{A} = A + BK \tag{3.4}$$

is the closed-loop system matrix.

Definition (3.5) System (3.1) is robustly stabilized by the control law (3.2) if the closed-loop system (3.3) is robustly stable with degree α.

Using the quadratic function $V(x)$ and computing the derivative $\dot{V}(x)_{(3.3)}$, we can imitate the process that led to problem (2.15) and formulate the following problem

$$\begin{aligned} &\text{minimize} \quad \gamma \\ &\text{subject to} \quad Y > 0 \\ &\begin{bmatrix} AY + YA^T + BKY + YK^TB^T & I & YH^T \\ I & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} < 0 \end{aligned} \tag{3.6}$$

which is not an LMI in Y and K, but can be made so by introducing the change of variable (Bernussou *et al.* [22])

$$KY = L, \tag{3.7}$$

or

$$K = LY^{-1}. \tag{3.8}$$

The transformed problem is now of LMI variety,

$$\begin{aligned}
&\text{minimize} \quad \gamma \\
&\text{subject to} \quad Y > 0 \\
&\qquad \begin{bmatrix} AY + YA^T + BL + L^T B^T & I & YH^T \\ I & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} < 0.
\end{aligned} \tag{3.9}$$

We have the following:

THEOREM (3.10) *System* (3.1) *is robustly stabilized by control law* (3.2) *if problem* (3.9) *is feasible.*

Example (3.11) Let us consider the same system of Example (2.17), but add control to get

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \mu(t)x. \tag{3.12}$$

We choose $u = u(x)$ as the feedback control (3.2) with the gain matrix $K = [k_{11}, k_{12}]$. Since the open-loop matrix is stable, the objective is to apply feedback in order to increase the uncertainty bound α. Solving the problem (3.9) with bounding matrix $H = I$ we get $\alpha = 0.9998$ which is an improvement over the open-loop bound $\hat{\alpha} = 0.5397$ obtained in Example (2.17), but not an improvement we expect to have when we use feedback. Furthermore, the gain matrix is exceedingly large

$$K = 10^8[-8.9716, -0.0006]. \tag{3.13}$$

The bound does not change when we solve the problem (3.9) in the transformed space as in Example (2.17). The outcome remains the same even if we start with the unstable open-loop matrix A in

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \mu(t)x \tag{3.14}$$

and use feedback $u(x) = Kx$ to stabilize A and, at the same time, maximize the uncertainty bound α. We obtain the same α and the gain matrix

$$K = 10^6[-1.2860, -0.0185] \tag{3.15}$$

which has again a large size as K in (3.13).

When we change matrix B in (3.12) to get the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \mu(t)x \tag{3.16}$$

we increase the bound to $\alpha = 3.1623$ with the gain matrix

$$K = 10^3[-6.996, 3.5013] \tag{3.17}$$

having a reduced size.

The situation changes drastically when we choose $B = I$. Solving problem (3.9) for the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u + \mu(t)x \tag{3.18}$$

we obtain γ close to zero implying an arbitrarily large α. The gain matrix

$$K = 10^9 \begin{bmatrix} -5.1195 & 0.001 \\ 0.0001 & -5.1189 \end{bmatrix} \tag{3.19}$$

has elements even larger than K in (3.13). System (3.18) satisfies the matching conditions and, as expected, we get arbitrarily small γ with very large size of K. That the matching conditions always lead to this LMI result needs a proof which we provide later on in this section. Of our immediate concern is the size of the feedback gain matrix.

To make the outcomes of problem (3.9) practical, we must limit the size of the gain matrix K while, at the same time, guaranteeing a prescribed uncertainty bound $\bar{\alpha}$. We can restrict the size of K by constraining L and Y^{-1} (Chen *et al.* [23]). We set

$$L^T L < \kappa_L I, \quad \kappa_L > 0 \tag{3.20}$$

which is equivalent to the LMI

$$\begin{bmatrix} -\kappa_L I & L^T \\ L & -I \end{bmatrix} < 0. \tag{3.21}$$

Similarly, we assume

$$Y^{-1} < \kappa_Y I, \quad \kappa_Y > 0 \tag{3.22}$$

which can be represented as an LMI,

$$\begin{bmatrix} \kappa_Y I & I \\ I & Y \end{bmatrix} > 0. \tag{3.23}$$

From constraints (3.20) and (3.22), we get the desired bound

$$K^T K = Y^{-1} L^T L Y^{-1} < \kappa_L Y^{-1} Y^{-1} < \kappa_L \kappa_Y^2 I, \tag{3.24}$$

which has the LMI representation (3.21) and (3.23).

In order to guarantee a desired value $\bar{\alpha}$, we recall that $\gamma = 1/\alpha^2$, and require that $\gamma - 1/\bar{\alpha}^2 < 0$.

With these modifications the optimization problem (3.9) becomes

$$\begin{aligned} \text{minimize} \quad & \gamma + \kappa_L + \kappa_Y \\ \text{subject to} \quad & Y > 0 \\ & \begin{bmatrix} AY + YA^T + BL + L^T B^T & B & YH^T \\ B^T & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} < 0 \\ & \gamma - \frac{1}{\bar{\alpha}^2} < 0 \\ & \begin{bmatrix} -\kappa_L I & L^T \\ L & -I \end{bmatrix} < 0 \\ & \begin{bmatrix} Y & I \\ I & \kappa_Y I \end{bmatrix} > 0. \end{aligned} \tag{3.25}$$

THEOREM (3.26) *System* (3.1) *is robustly stabilizable with prescribed degree* $\bar{\alpha}$ *by control law* (3.2) *if problem* (3.25) *is feasible.*

Example (3.27) To illustrate the effect of the added features to problem (3.9), we solve the new optimization problem for the system (3.18) of Example (3.11), with the same bounding matrix $H = I$, but include the fixed bound

$$\bar{\alpha} = 2. \tag{3.28}$$

We obtain $\alpha = 2.004$ with the gain matrix

$$K = \begin{bmatrix} -1.8759 & -0.1206 \\ -0.0974 & -0.5553 \end{bmatrix} \tag{3.29}$$

having a much smaller size than that of gain matrices in Example (3.11).

Raising the requirement on guaranteed α to

$$\bar{\alpha} = 10, \tag{3.30}$$

produces the stabilizing gain matrix

$$K = \begin{bmatrix} -10.8005 & 0.1949 \\ 0.2881 & -9.0440 \end{bmatrix} \tag{3.31}$$

which has (expectedly) a larger size than the previous solution (3.29) obtained for smaller $\bar{\alpha}$. In this way, the optimization problem (3.25) offers a useful trade-off between the guaranteed robustness bound $\bar{\alpha}$ and the size of the stabilizing gain matrix K.

3.1. The Matching Conditions

The fact that solution of problem (3.9) for system (3.3), where no constraints were placed on the gain matrix K, produced arbitrarily large uncertainty bound α has been attributed to the *matching conditions* being satisfied by the matrix B and the perturbation $\mu(t)x$. To show that this connection is true in general let us consider the system (3.1) in the form

$$\dot{x} = Ax + Bu + Bg(t, x), \tag{3.32}$$

where we replaced $h(t, x)$ by $Bg(t, x)$, with $g(t, x)$ satisfying the same general conditions as $h(t, x)$. Since the control input u and the perturbation function $g(t, x)$ enter the system through the same input matrix B, the system satisfies the matching conditions (Leitmann [24]). It is well-known that when the matching condition is present, there always exists a stabilizing feedback control $u(x)$ regardless of the size of the perturbation. What we want to show is that our LMI formulation of robust stabilization "recognizes" the matching conditions, and that a stabilizing gain matrix can always be computed by the LMI method.

The quadratic constraints are imposed on the perturbation function $g(t, x)$,

$$g^T(t, x)g(t, x) \leq \alpha^2 x^T H^T H x \tag{3.33}$$

where we assume that the constant matrix H has full rank, that is, $H^T H$ is positive definite.

As for the linear part of the system we require that the pair (A, B) be stabilizable and the pair (A, H) is detectable.

Remark (3.34) We note that we assumed (without loss of generality) that both the input u and perturbation g enter the system through a single matrix B. The matching conditions hold even if u and g enter *via* two different matrices B and $\tilde{B}$, provided $\operatorname{Im} B \supset \operatorname{Im} \tilde{B}$. Then, there is a constant matrix C of appropriate dimension such that $\tilde{B} = BC$ and, when C is absorbed by g, we are back to (3.32). A similar argument holds in case of the decentralized matching conditions considered in Section 5, Remark (5.26).

We prove the following:

THEOREM (3.35) *System* (3.32) *is robustly stabilizable with arbitrarily large degree* α *by the control law* (3.2).

Proof For the new description (3.32) of the open-loop system (3.1), the LMI problem (3.9) becomes

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & Y > 0 \\ & \begin{bmatrix} AY + YA^T + BL + L^T B^T & B & YH^T \\ B^T & -I & 0 \\ HY & 0 & -\gamma I \end{bmatrix} < 0 \end{array} \tag{3.36}$$

where again $L = KY$ and $\gamma = 1/\alpha^2$.

First, we note that inequality constraints in (3.36) are equivalent to

$$\begin{array}{l} \gamma > 0 \\ Y > 0 \\ AY + YA^T + BL + L^T B^T + \dfrac{1}{\gamma} YH^T HY + BB^T < 0. \end{array} \tag{3.37}$$

To prove feasibility of (3.37) let us consider the Riccati equation

$$A^T X + XA - XBB^T X + \frac{1}{\tilde{\gamma}} H^T H = 0, \tag{3.38}$$

where $0 < \tilde{\gamma} < \gamma$. Since the pair (A, H) is detectable, there exists a unique positive definite solution X to Eq. (3.38) such that $(A - BB^T X)$ is a stable matrix (*e.g.*, Zhou *et al.* [25]). Now, premultiplying and postmultiplying (3.38) by X^{-1} we obtain

$$X^{-1} A^T + AX^{-1} - BB^T + \frac{1}{\tilde{\gamma}} X^{-1} H^T H X^{-1} = 0 \tag{3.39}$$

or

$$AX^{-1} + X^{-1} A^T + B(-B)^T + (-B)^T B + \frac{1}{\tilde{\gamma}} X^{-1} H^T H X^{-1} + BB^T = 0. \tag{3.40}$$

Since $\gamma > \tilde{\gamma}$, where $\tilde{\gamma}$, can be arbitrarily small number, and $H^T H$ is positive definite, it follows from (3.40) that

$$AX^{-1} + X^{-1} A^T + B(-B)^T + (-B)^T B + \frac{1}{\gamma} X^{-1} H^T H X^{-1} + BB^T < 0. \tag{3.41}$$

By comparing (3.41) with (3.37) and choosing

$$L = -B^T, \quad Y = X^{-1} \tag{3.42}$$

where X is the positive definite solution of (3.38), we get a solution to the problem (3.37) for any $\gamma > 0$. Therefore, the solution of the optimization problem (3.36) is

$$\gamma = \inf_{\tilde{\gamma} > 0} \tilde{\gamma} = 0. \tag{3.43}$$

Q.E.D.

As we pointed out in Example (3.11), this solution may be impractical requiring high feedback gains. Therefore, problem (3.36) should be appended by restrictions on both γ and K, as in problem (3.25).

4. DECENTRALIZED CONTROL

Our crucial assumption in this section is the presence of *decentralized information structure constraints* on feedback control [16]. This simply means that not all state variables are available for control at every point of the system. The linear feedback control law for the system

$$\dot{x} = Ax + Bu + h(t,x) \tag{4.1}$$

has the familiar block diagonal form

$$u(x) = K_D x, \tag{4.2}$$

where $m \times n$ feedback gain matrix

$$K_D = \{K_1, K_2, \ldots, K_N\} \tag{4.3}$$

has diagonal blocks K_i's of dimensions $m_i \times n_i$. The size of the blocks are predetermined by the control information structure, and $\sum_{i=1}^{N} m_i = m$, $\sum_{i=1}^{N} n_i = n$.

We assume again that the nonlinear function $h(t,x)$ satisfies the quadratic constraint (2.2). As for the linear part of the system the stabilizability property of the pair (A, B) is not appropriate because of the information structure constraints. It is replaced by the requirement that the closed-loop matrix $A + BK_D$ has no unstable decentralized fixed modes [16]. This means that there are no unstable eigenvalues of $A + BK_D$ which are invariant to any and all changes of elements of the gain matrix K_D. This is a difficult condition to test and it is important to note that for LMI approach to work we do not need to test the condition. Within the limits of our LMI formulation of the decentralized stabilization problem, the feasibility solution is answered by solving the related LMI problem.

By applying the decentralized control law (4.2) to the open-loop system (4.1) we get the closed-loop system

$$\dot{x} = \hat{A}x + h(t,x) \tag{4.4}$$

where

$$\hat{A} = A + BK_D \tag{4.5}$$

is the closed-loop matrix.

We use again the change of variables (Bernussou *et al.* [22])

$$K_D Y_D = L_D, \tag{4.6}$$

and express K_D as

$$K_D = L_D Y_D^{-1}. \tag{4.7}$$

By using the Schur complement formula (Boyd *et al.* [8]), we reformulate problem (3.9) as

$$\begin{aligned}
&\text{minimize} \quad \gamma \\
&\text{subject to} \quad Y_D > 0 \\
&\begin{bmatrix} AY_D + Y_D A^T + BL_D + L_D^T B^T & I & Y_D H^T \\ I & -I & 0 \\ HY_D & 0 & -\gamma I \end{bmatrix} < 0.
\end{aligned} \tag{4.8}$$

THEOREM (4.9) *System* (4.1) *is robustly stabilized with degree* α *by decentralized control law* (4.2) *if problem* (4.8) *is feasible.*

Example (4.10) Let us consider the system of Example (3.11) with a different control matrix B,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} u + \mu(t)x. \tag{4.11}$$

Since the system satisfies the matching condition the solution of problem (3.9) with $H = I$ yields a large uncertainty bound $\alpha = 1.4826 \times 10^5$ and the gain matrix

$$K = 10^9 \begin{bmatrix} 4.6785 & -2.7920 \\ -3.8646 & 0.6845 \end{bmatrix}. \tag{4.12}$$

When we impose the diagonal structure on the feedback gain matrix, $K_D = \text{diag}\{k_1, k_2\}$ and solve problem (4.8), we get a drastically reduced uncertainty bound $\alpha = 1.8877$, with a smaller size of gain matrix

$$K_D = \begin{bmatrix} -524.0131 & 0 \\ 0 & 0.0349 \end{bmatrix}. \tag{4.13}$$

The reason for reduction in α is that, in order for the matching conditions to produce arbitrarily large α with high gains, diagonal solutions L_D and Y_D of problem (4.8) need to exist for the entire range of variation of α. In any given situation, this flexibility of the formulation (4.8) may not be available. A stabilizing K_D may exist but the corresponding L_D and Y_D, as solutions of (4.8), may not.

Recalling inequality (3.41) and relations (3.42), we conclude that the matching conditions would work if the control matrix $B = B_D$ is compatible with K_D and the system matrices A and B allow for the existence of the corresponding diagonal matrices L_D and Y_D. This takes place if we change the system (4.11) back to (3.18), that is,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u + \mu(t)x. \tag{4.14}$$

Solving again problem (4.8) with $H = I$, we get $\alpha = 1.7001 \times 10^5$ and

$$\begin{aligned} L_D &= 10^8 \operatorname{diag}\{-5.5266,\ -5.5234\} \\ Y_D &= \operatorname{diag}\{0.1817,\ 0.1802\} \end{aligned} \tag{4.15}$$

resulting in an impractical gain matrix

$$K_D = 10^9 \operatorname{diag}\{-3.0414,\ -3.0655\}. \tag{4.16}$$

Decentralized feedback gains may be unacceptably high even in the absence of the matching conditions. For this reason we need to append problem (4.8) with constraints

$$L_D^T L_D < \kappa_L I, \quad Y_D^{-1} < \kappa_Y I \tag{4.17}$$

where κ_L and κ_Y are positive numbers. By imitating problem (3.25) in the present context, we get the expanded version of problem (4.8) as

$$\begin{aligned} &\text{minimize} \quad \gamma \\ &\text{subject to} \quad Y_D > 0 \\ &\qquad \begin{bmatrix} AY_D + Y_D A^T + BL_D + L_D^T B^T & I & Y_D H^T \\ I & -I & 0 \\ HY_D & 0 & -\gamma I \end{bmatrix} < 0 \\ &\qquad \gamma - \frac{1}{\bar{\alpha}^2} < 0 \\ &\qquad \begin{bmatrix} -\kappa_L I & L_D^T \\ L_D & -I \end{bmatrix} < 0 \\ &\qquad \begin{bmatrix} Y_D & I \\ I & \kappa_Y I \end{bmatrix} > 0. \end{aligned} \tag{4.18}$$

Example (4.19) We want to show now that even if we have a block-diagonal B_D we may not be able to compute K_D because the matrix pair (A, B_D) does not permit diagonal matrices L_D and Y_D as a part of the solution to problems (4.8) and (4.18). To illustrate this fact let us consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u + h(t,x). \tag{4.20}$$

Since the pair (A, B) is stabilizable, we can solve problem (3.25) with $H = I$ and without restriction on γ to get $\alpha = 0.5818$ and full gain matrix

$$K = \begin{bmatrix} 0.4218 & -1.9333 & -1.2357 & -0.1966 \\ -1.2357 & -0.1966 & 0.4218 & -1.9333 \end{bmatrix}. \tag{4.21}$$

When we attempt to solve decentralized problem (4.8) or (4.18) with a block-diagonal matrix

$$K_D = \operatorname{diag}\{K_1,\ K_2\} \tag{4.22}$$

having diagonal blocks

$$K_1 = [k_{11}, k_{12}]^T, \quad K_2 = [k_{21}, k_{22}]^T \tag{4.23}$$

the problems are found to be infeasible. To explain this result let us recall that the system (4.20) was considered in (Sezer and Huseyin [26]) as a composition of two interconnected subsystems

$$\begin{aligned} \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 \\ \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2 \end{aligned} \tag{4.24}$$

when the perturbation term $h(t,x)$ is set to zero, that is, $h(t,x) \equiv 0$. For the decentralized control laws

$$u_1 = K_1 x_1, \quad u_2 = K_2 x_2 \tag{4.25}$$

to stabilize system (4.24), it is necessary that

$$k_{12} k_{22} < 0, \tag{4.26}$$

which implies that at least one of the decoupled closed-loop subsystems

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ k_{11} & k_{12} \end{bmatrix} x_1, \quad \dot{x}_2 = \begin{bmatrix} 0 & 1 \\ k_{21} & k_{22} \end{bmatrix} x_2 \tag{4.27}$$

must be unstable. As we recall from [16] this fact is not conducive to existence of block-diagonal solutions to Liapunov matrix equations. Yet, if K_D is to robustly stabilize system (4.20) it must be stabilizing for system (4.24) which is the system (4.20) when $h(t,x) \equiv 0$.

Let us note that system (4.20) was carefully chosen not to satisfy the *generalized matching conditions* (Ikeda and Šiljak [27], Šiljak [16]). One way to make problem (4.8) feasible is to change the interconnection terms in (4.21) from

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x_2, \quad \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x_1 \tag{4.28}$$

to

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x_2, \quad \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} x_1, \tag{4.29}$$

so that interconnections satisfy the generalized matching conditions. Solving problem (4.18) with $H = I$ and without restrictions on γ, we get $\alpha = 0.2157$ having the gain matrix

$$K_D = \begin{bmatrix} -26.3549 & -15.1577 & 0 & 0 \\ 0 & 0 & -13.9516 & -19.3660 \end{bmatrix}. \tag{4.30}$$

Obviously, with this K_D both docoupled closed-loop subsystems (4.27) are stable. The matrices L_D and Y_D are obtained as

$$\begin{aligned} L_D &= \begin{bmatrix} -2.9233 & -5.2388 & 0 & 0 \\ 0 & 0 & 0.7605 & -5.9507 \end{bmatrix} \\ Y_D &= \begin{bmatrix} 2.1485 & -3.5427 & 0 & 0 \\ -3.5427 & 6.5054 & 0 & 0 \\ 0 & 0 & 1.2455 & -0.9366 \\ 0 & 0 & -0.9366 & 0.9820 \end{bmatrix} \end{aligned} \tag{4.31}$$

resulting in $K_D = L_D Y_D^{-1}$ of (4.30).

Another way to make problem (4.8) feasible is to reduce the size of interconnections (4.28), so that the interconnected subsystems (4.24) become *weakly coupled* [16]. When we replace interconnections (4.28) by

$$\begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} x_2, \quad \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} x_1, \tag{4.32}$$

and solve problem (4.18) with $H = I$ and without the restrictions on γ, we get $\alpha = 0.3347$ and the gain matrix

$$K_D = \begin{bmatrix} -2.8793 & -2.8614 & 0 & 0 \\ 0 & 0 & -2.8793 & -2.8614 \end{bmatrix}. \tag{4.33}$$

With this K_D both subsystems (4.27) are stable. Problem (4.18) has block-diagonal solutions for L_D and Y_D as

$$\begin{aligned} L_D &= \begin{bmatrix} -0.4410 & -1.7232 & 0 & 0 \\ 0 & 0 & -0.4410 & -1.7232 \end{bmatrix} \\ Y_D &= \begin{bmatrix} 1.8694 & -1.7541 & 0 & 0 \\ -1.7541 & 2.3673 & 0 & 0 \\ 0 & 0 & 1.8964 & -1.7541 \\ 0 & 0 & -1.7541 & 2.3673 \end{bmatrix} \end{aligned} \tag{4.34}$$

producing $K_D = L_D Y_D^{-1}$ in (4.33).

5. INTERCONNECTED SYSTEMS

When a dynamic system is modeled from the outset of stability analysis as an interconnection of a number of subsystems, the problem of structural perturbations arises in a natural way. We want the interconnected system to remain stable despite subsystems being disconnected and again connected during operation; we want the system to be *connectively stable* (Šiljak [16, 17], Lakshmikantham *et al.* [28], Stipanović and Šiljak [29]). In this section, we consider a collection of linear subsystems with nonlinear interconnections and compute linear feedback which connectively stabilizes the overall system. The model is justified by the fact that in most practical situations reliability of subsystems is much higher than that of the interconnections [16].

Let us consider the interconnected system

$$\dot{x}_i = A_i x_i + B_i u_i + h_i(t, x), \quad i \in \mathbf{N} \tag{5.1}$$

which is composed of N linear time-invariant subsystems

$$\dot{x}_i = A_i x_i + B_i u_i, \quad i \in \mathbf{N} \tag{5.2}$$

where $x_i \in \mathbf{R}^{n_i}$ are the states, $u_i \in \mathbf{R}^{m_i}$ are the inputs, $h_i : \mathbf{R}^{n+1} \to \mathbf{R}^{n_i}$ are the interconnections, and $\mathbf{N} = \{1, 2, \ldots, N\}$.

For the linear part of the system we require that all pairs (A_i, B_i) be stabilizable. Besides general assumptions concerning the nonlinear interconnection functions stated in Section 1, we require that they all satisfy the quadratic constraints

$$h_i^T(t, x) h_i(t, x) \leq \alpha_i^2 x^T H_i^T H_i x \tag{5.3}$$

where $\alpha_i > 0$ are interconnection parameters and H_i are fixed matrices. The constraints can be interpreted as

$$\|h_i(t, x)\| \leq \alpha_i \|H_i x\|. \tag{5.4}$$

where $\|\cdot\|$ is the Euclidean norm. If we define the constant matrix H_i as a block matrix

$$H_i = [H_{i1}, H_{i2}, \ldots, H_{iN}] \tag{5.5}$$

with the blocks H_{ij} compatible with the subsystems state vectors x_i, we can rewrite (5.4) as

$$\|h_i(t, x)\| \leq \alpha_i \left\| \sum_{j=1}^{N} H_{ij} x_j \right\| \leq \alpha_i \sum_{j=1}^{N} \|H_{ij}\| \|x_j\| \tag{5.6}$$

and arrive at the inequality

$$\|h_i(t, x)\| \leq \alpha_i \sum_{j=1}^{N} \xi_{ij} \|x_j\| \tag{5.7}$$

which is the standard interconnection constraint with $\xi_{ij} = \|H_{ij}\|$ (*e.g.*, Šiljak [16]).

The overall interconnected system can be rewritten in a *compact form*

$$\dot{x} = A_D x + B_D u + h(t, x), \tag{5.8}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the input, and $A_D = \text{diag}\{A_1, A_2, \dots, A_N\}$ and $B_D = \text{diag}\{B_1, B_2, \dots, B_N\}$ are constant matrices of appropriate dimensions. We assume that the subsystems are *disjoint*, that is, $x = (x_1^T, x_2^T, \dots, x_N^T)^T$ and $u = (u_1^T, u_2^T, \dots, u_N^T)^T$. In the compact notation (5.8), the interconnection function $h: \mathbf{R}^{n+1} \to \mathbf{R}^n$, $h = (h_1^T, h_2^T, \dots, h_N^T)^T$, is constrained as

$$h^T(t, x)h(t, x) \leq x^T \left(\sum_{i=1}^{N} \alpha_i^2 H_i^T H_i \right) x. \tag{5.9}$$

Our crucial assumption in this section is that the feedback control law has to obey the *decentralized information structure constraint*, that is, each subsystem is controlled using only its locally available state. The requirement implies that the i-th subsystem is controlled by the local control law

$$u_i(x_i) = K_i x_i, \quad i \in \mathbf{N} \tag{5.10}$$

which we require to be linear and time-invariant, that is, K_i is an $m_i \times n_i$ constant matrix. The control law for the overall system (5.8), which is a collection of the individual local laws, has the familiar block-diagonal form

$$u(x) = K_D x, \tag{5.11}$$

where $K_D = \text{diag}\{K_1, K_2, \dots, K_N\}$ is a constant $m \times n$ matrix with diagonal blocks compatible with those of A_D and B_D.

To compute the gain matrix K_D, so that the closed-loop system

$$\dot{x} = (A_D + B_D K_D)x + h(t, x) \tag{5.12}$$

is robustly asymptotically stable in the large under the constraint (5.9) on the interconnection function $h(t, x)$, we use again the change of variables (Bernussou *et al.* [22])

$$K_D Y_D = L_D \tag{5.13}$$

and express K_D as

$$K_D = L_D Y_D^{-1}. \tag{5.14}$$

Then, by applying repeatedly the Schur complement formula we can reformulate problem (3.9) in the decentralized context as

$$\begin{aligned}
&\text{minimize } \sum_{i=1}^{N} \gamma_i \\
&\text{subject to } Y_D > 0 \\
&\begin{bmatrix}
A_D Y_D + Y_D A_D^T + B_D L_D + L_D^T B_D^T & I & Y_D H_1^T & \cdots & Y_D H_N^T \\
I & -I & 0 & \cdots & 0 \\
H_1 Y_D & 0 & -\gamma_1 I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
H_N Y_D & 0 & 0 & \cdots & -\gamma_N I
\end{bmatrix} < 0
\end{aligned} \tag{5.15}$$

where $\gamma_i = 1/\alpha_i^2$.

At this point we are ready to provide a generalization of Theorem (3.10) for the interconnected systems in the following form:

THEOREM (5.16) *Interconnected system* (5.8) *is robustly stabilized with degree vector* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)^T$ *by control law* (5.11), *if problem* (5.15) *is feasible.*

Before we use problem (5.15) to compute the local feedback matrices K_i for each subsystem, we should note that the formulation of the constraints (5.3) on the interconnection terms $h_i(t, x)$ includes the case when any and/or all $h_i(t, x) \equiv 0$ guaranteeing that the overall closed-loop system (5.12) is *connectively stable* (Šiljak [16]). Let us also note that in solving problem (5.15) we can encounter the same problem of existence of matrices L_D and Y_D in pretty much the same way as in solving problems (4.8) and (4.18), as illustrated in Examples (4.10) and (4.19).

5.1. The Decentralized Matching Conditions

We want to show that if the matching conditions are satisfied by each subsystem individually, then the system can be stabilized by linear decentralized feedback with unlimited bounds on the interconnection functions. This means that, in this case, we can guarantee existence of L_D and Y_D when solving problem (5.15).

An interconnected system satisfying decentralized matching conditions can be described by the equations

$$\dot{x}_i = A_i x_i + B_i u_i + B_i g_i(t,x), \quad i \in \mathbf{N} \tag{5.17}$$

where the uncertain functions $g_i\colon \mathbf{R}^{n+1} \to \mathbf{R}^{m_i}$ are the interconnections between the subsystems (5.2). We again assume that the pairs (A_i, B_i) are stabilizable and the interconnections $g_i(t,x)$ satisfy the usual quadratic constraints

$$g_i^T(t,x) g_i(t,x) \le \alpha_i^2 x^T G_i^T G_i x. \tag{5.18}$$

In the compact form, Eq. (5.17) are

$$\dot{x} = A_D x + B_D u + B_D g(t,x), \tag{5.19}$$

which is the same as (5.8) except for fact that the interconnection term $h(t,x)$ is replaced by $B_D g(t,x)$. The subsystems are interconnected through the control matrix B_D and, therefore, satisfy the matching conditions. The quadratic constraints (5.18) are collectively formulated as a single inequality

$$g^T(t,x) g(t,x) \le x^T \left(\sum_{i=1}^{N} \alpha_i^2 G_i^T G_i \right) x. \tag{5.20}$$

where $g = (g_1^T, g_2^T, \ldots, g_N^T)^T$.

Notice that the inequality constraints in (5.15) are equivalent to

$$\begin{aligned}
&\gamma_i > 0, \quad \text{for all } i \in \mathbf{N} \\
&Y_D > 0 \\
&A_D Y_D + Y_D A_D^T + B_D L_D + L_D^T B_D^T + Y_D \left\{ \sum_{i=1}^{N} \gamma_i^{-1} G_i^T G_i \right\} Y_D + B_D^T B_D < 0.
\end{aligned} \tag{5.21}$$

Now, we will prove that the set of inequalities (5.21) is feasible for all $\gamma_i > 0$, $i \in \mathbf{N}$, when all the pairs (A_i, B_i), $i \in \mathbf{N}$, are stabilizable and for the appropriate choice of the matrix $G = [G_1^T, G_2^T, \ldots, G_N^T]^T$. Let us choose G_i's to be block diagonal such that G has a full column rank. Block diagonal structure is required to secure existence of the decentralized feedback control.

First notice that if the pairs (A_i, B_i), $i \in \mathbf{N}$, are stabilizable then the pair (A_D, B_D) is stabilizable. Also, if G has a full column rank then for any positive γ_i's, the matrix

$$\tilde{G} = \left[\frac{1}{\sqrt{\gamma_1}} G_1^T, \frac{1}{\sqrt{\gamma_2}} G_2^T, \ldots, \frac{1}{\sqrt{\gamma_N}} G_N^T\right]^T \tag{5.22}$$

has full column rank as well. Thus, in the last inequality in (5.21) we observe that the following is satisfied:

$$\sum_{i=1}^{N} \gamma_i^{-1} G_i^T G_i = \overline{G}^T \overline{G} > 0 \tag{5.23}$$

where $\overline{G}^T \overline{G}$ has a block diagonal structure. From positive definiteness of $\sum_{i=1}^{N} \gamma_i^{-1} G_i^T G_i$ it follows that $\overline{G}$ can always be chosen as a block diagonal matrix $\overline{G}_D$ having a full column rank. This further implies that the pair $(A_D, \overline{G}_D)$ is detectable for any positive γ_i's. Thus, the left-hand side of the third inequality in (5.21) has a block diagonal structure, that is, (5.21) may be decoupled into N inequalities. Using the similar arguments to those used in the nondecentralized case of Section 3, when the matching conditions were satisfied, we conclude that the solution to optimization problem (5.15) is given by

$$\inf \sum_{i=1}^{N} \gamma_i = \inf_{\substack{\gamma_i > 0 \\ i \in \mathbf{N}}} \sum_{i=1}^{N} \gamma_i = 0. \tag{5.24}$$

From this result we see that the decentralized matching conditions are "recognized" by the LMI problem (5.15). Therefore, in solving (5.15), when this kind of matching is present, we always have a stabilizing decentralized control law but should expect high feedback gains. To limit the gains and, at the same time, guarantee a certain degree of robustness at each subsystem, we propose the following decentralized version of problem (3.25):

$$\begin{aligned}
&\text{minimize} \quad \sum_{i=1}^{N} \gamma_i + \sum_{i=1}^{N} \kappa_{Yi} + \sum_{i=1}^{N} \kappa_{Li} \\
&\text{subject to} \quad Y_D > 0 \\
&\begin{bmatrix} A_D Y_D + Y_D A_D^T + B_D L_D + L_D^T B_D^T & B_D & Y_D G_1^T & \cdots & Y_D G_N^T \\ B_D^T & -I & 0 & \cdots & 0 \\ G_1 Y_D & 0 & -\gamma_1 I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_N Y_D & 0 & 0 & \cdots & -\gamma_N I \end{bmatrix} < 0 \\
&\gamma_i - \frac{1}{\bar{\alpha}_i^2} < 0, \quad i \in \mathbf{N} \\
&\begin{bmatrix} -\kappa_{Li} I & L_i^T \\ L_i & -I \end{bmatrix} < 0, \; i \in \mathbf{N} \\
&\begin{bmatrix} Y & I \\ I & \kappa_{Yi} I \end{bmatrix} > 0, \quad i \in \mathbf{N}.
\end{aligned} \tag{5.25}$$

Remark (5.26) By recalling Remark (3.34), we can assume that our interconnected system is given by

$$\dot{x}_i = A_i x_i + B_i u_i + \tilde{B}_i g_i(t, x), \quad i \in \mathbf{N}. \tag{5.27}$$

Using constraints (5.18), we can reformulate the optimization problem (5.25) to obtain

$$\begin{aligned}
&\text{minimize} \quad \sum_{i=1}^{N} \gamma_i + \sum_{i=1}^{N} \kappa_{Yi} + \sum_{i=1}^{N} \kappa_{Li} \\
&\text{subject to} \quad Y_D > 0 \\
&\begin{bmatrix} A_D Y_D + Y_D A_D^T + B_D L_D + L_D^T B_D^T & \tilde{B}_D & Y_D G_1^T & \cdots & Y_D G_N^T \\ \tilde{B}_D^T & -I & 0 & \cdots & 0 \\ G_1 Y_D & 0 & -\gamma_1 I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_N Y_D & 0 & 0 & \cdots & -\gamma_N I \end{bmatrix} < 0 \\
&\gamma_i - \frac{1}{\bar{\alpha}_i^2} < 0, \quad i \in \mathbf{N}
\end{aligned}$$

$$\begin{bmatrix} -\kappa_{Li} I & L_i^T \\ L_i & -I \end{bmatrix} < 0, \quad i \in \mathbf{N}$$
$$\begin{bmatrix} Y & I \\ I & \kappa_{Yi} I \end{bmatrix} > 0, \quad i \in \mathbf{N} \tag{5.28}$$

where $\tilde{B}_D = \operatorname{diag}\{\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_N\}$. Again, we cannot guarantee the solution of (5.28) unless $\operatorname{Im} \tilde{B}_D \subset \operatorname{Im} B_D$.

Example (5.29) To illustrate the application of problem (5.28) we consider the system composed of two coupled inverted penduli, which was discussed in [16]. The system is interesting in the LMI environment because we can compute decentralized feedback gains to make the system connectively stable. The penduli are coupled by a sliding spring, the position of which is uncertain. In the present control design we allow the sliding of the spring to be a discontinuous function of both time and state of the system.

Let us describe the motion of two penduli (Fig. 1) as two interconnected subsystems (for details, see [16]),

$$\begin{aligned} \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 + e \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x_1 + e \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_2 \\ \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2 + e \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_1 + e \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x_2 \end{aligned} \tag{5.30}$$

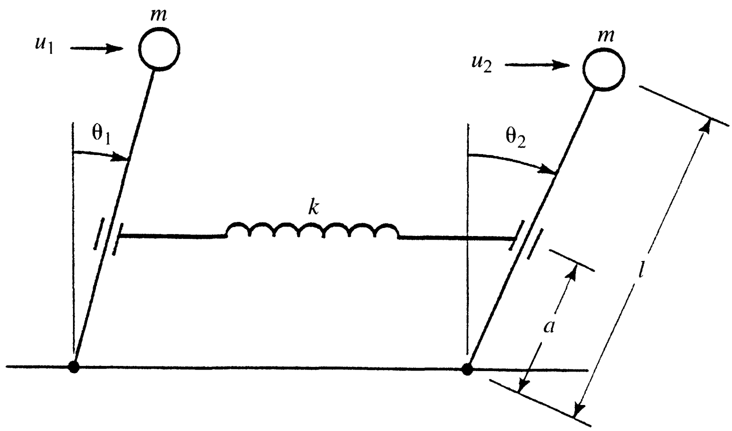

FIGURE 1 Inverted penduli.

where $g/\ell = 1$, $1/m\ell^2 = 1$, and $\bar{a}^2 k/m\ell^2 = 1$. We assume that the spring can slide up and down the rods of the penduli in sudden (discontinuous) jumps of unpredictable size and direction between the support and a height $\bar{a}$. This means that $a(t,x)$ is assumed as a piecewise continuous function in both time and state, such that $0 \le a(t,x) \le \bar{a}$, To study connective stability of the penduli under structural perturbations caused by the jumps of the coupled spring, a normalized interconnection parameter $e\colon \mathbf{R}^5 \to [0,1]$ is defined as $e(t,x) = a(t,x)/\bar{a}$. We want to compute the linear decentralized control laws

$$u_1(x_1) = K_1 x_1, \quad u_2(x_2) = K_2 x_2 \tag{5.31}$$

to robustly stabilize the system (5.30) for all values $e(t,x) \in [0,1]$ and, at the same time maximize the uncertainty bounds α_1 and α_2 on the interconnections

$$\begin{aligned} h_1(t,x) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} e(t,x)x \\ h_2(t,x) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} e(t,x)x. \end{aligned} \tag{5.32}$$

By choosing $G_1 = G_2 = I$ and setting the minimal limits on α_1 and α_2 as

$$\bar{\alpha}_1 = \bar{\alpha}_2 = 1 \tag{5.33}$$

we can solve problem (5.28) to get the decentralized gains

$$K_1 = [-3.0080 \;\; -3.0032], \quad K_2 = [-3.0080 \;\; -3.0032]. \tag{5.34}$$

The resulting quadratic Liapunov function

$$V(x) = x^T P_D x, \tag{5.35}$$

which establishes connective stability of the interconnected penduli (5.30), has the block-diagonal form

$$P_D = \begin{bmatrix} 3.0820 & 0.9408 & 0 & 0 \\ 0.9408 & 2.0565 & 0 & 0 \\ 0 & 0 & 3.0820 & 0.9408 \\ 0 & 0 & 0.9408 & 2.0565 \end{bmatrix} \tag{5.36}$$

as expected.

5.2. The Generalized Matching Conditions

A generalization of the standard matching conditions introduced in (Šiljak and Vukčević [30]) was subsequently broadened by many people (Ikeda and Šiljak [27], Sezer and Šiljak [31], Ikeda *et al.* [32], Shi and Gao [33], Yang and Zhang [34], Zhang *et al.* [35]; see also the book [16] and references therein). In this section, we consider the example from (Ikeda and Šiljak [27]), which satisfies the generalized matching conditions, and demonstrate the fact that the LMI does not "recognize" the conditions, as it was the case with the standard matching conditions. We cannot drive the robustness bound to infinity by raising up the feedback gains. This means that, if we set the minimum uncertainty bound, the constrained problem (5.28) may not be feasible.

Example (5.37) Let us consider the interconnected system

$$\begin{aligned}
\dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 + e_{13}(t,x) \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x_3 \\
\dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2 + e_{23}(t,x) \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} x_3 \\
\dot{x}_3 &= \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_3 + e_{31}(t,x) \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} x_1 + e_{32}(t,x) \begin{bmatrix} 1 & 5 \\ 4 & 6 \end{bmatrix} x_2.
\end{aligned} \tag{5.38}$$

As shown in (Ikeda and Šiljak [27]), this system is decentrally stabilizable for any numbers e_{ij} replacing functions $e_{ij}(t,x)$. Now, let us select matrices G_i as

$$\begin{aligned}
G_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 3 & 4 \end{bmatrix} \\
G_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix} \\
G_3 &= \begin{bmatrix} 3 & 0 & 1 & 5 & 0 & 0 \\ 2 & 1 & 4 & 6 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{5.39}$$

In (5.38) we consider the interconnection parameters e_{ij} as uncertain piecewise-continuous functions $e_{ij}\colon \mathbf{R}^7 \rightarrow [0,1]$ setting our sight on connective stability of system (5.38). When we choose the minimal bounds $\bar{\alpha}_i = 1$ on the uncertain interconnection functions $e_{ij}(t, x)$, problem (5.25) is infeasible. It is infeasible even if the functions $e_{ij}(t, x)$ are replaced by numbers e_{ij}. The system (5.38) is quadratically stabilizable by the decentralized feedback laws

$$u_i(x_i) = K_i x_i, \quad i = 1, 2, 3 \tag{5.40}$$

as shown in [27], yet, we cannot compute K_i's by solving problem (5.28). Let us note at this point that problem formulation (5.28) is more suitable for this example than (5.15). We again attribute the failure to the fact that quadratic formulation of the problem does not recognize the structure that satisfies the generalized matching conditions because of the restrictive formulation $K_D = L_D Y_D^{-1}$ of K_D.

Removing the constraints on bounds α_i and solving again problem (5.28), we obtain the decentralized control laws (5.40) with gain matrices appearing as diagonal blocks in the overall gain matrix $K_D = \mathrm{diag}\{K_1, K_2, K_3\}$ having the numerical representation

$$K_D = \begin{bmatrix} -5.88 & -6.66 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.67 & -1.48 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6.67 & -8.82 \end{bmatrix}. \tag{5.41}$$

The overall Liapunov matrix P_D obtained as

$$P_D = \begin{bmatrix} 3.47 & 2.62 & 0 & 0 & 0 & 0 \\ 2.62 & 2.90 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.87 & 0.83 & 0 & 0 \\ 0 & 0 & 0.83 & 1.80 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.77 & 2.84 \\ 0 & 0 & 0 & 0 & 2.84 & 3.97 \end{bmatrix} \tag{5.42}$$

guarantees robust stability of system (5.38) for the following bounds on the interconnection functions

$$\begin{aligned} |e_{13}(t, x)| &\leq 0.6481, \quad |e_{23}(t, x)| \leq 0.4007, \\ |e_{31}(t, x)| &\leq 0.2192, \quad |e_{32}(t, x)| \leq 0.2192. \end{aligned} \tag{5.43}$$

As expected, the individual decoupled closed-loop subsystems, which are obtained from (5.38) by using K_D of (5.41) and setting $e_{ij}(t, x) \equiv 0$, are all stable for any and all combinations of the subscripts $i, j = 1, 2, 3$.

6. CONCLUDING REMARKS

A class of nonlinear systems has been considered which are composed of a linear (possibly unstable) constant part and an uncertain additive nonlinearity which is a discontinuous function of time and state of the system. We have shown how the LMI methods can be used to stabilize the system and, at the same time, maximize its robustness to nonlinear perturbations. An attractive feature of the proposed framework is its flexibility in accommodating various design constraints involving matching and generalized matching conditions, size of the gain matrices, and the bounds on the nonlinear terms. Most importantly, the framework is suitable for designing of feedback control laws for robust stabilization of interconnected systems with decentralized information structure constraints resulting in connectively stable systems.

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