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Adaptive Stabilization of Continuous-time Systems Guaranteeing the Controllability of a Modified Estimation Model

M. DE LA SEN

Instituto de Investigación y Desarrollo de Procesos IIDP, Facultad de Ciencias. Universidad del Pais Vasco, Leioa (Bizkaia) – Aptdo. 644 de Bilbao. Spain

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This paper presents an indirect adaptive control scheme for continuous-time systems. The estimated plant model is controllable while the estimation model is free from singularities. Such singularities are avoided through a modification of the estimated plant parameter vector so that its associated Sylvester matrix is guaranteed to be nonsingular. This property is achieved by ensuring that the absolute value of its determinant does not lie below a prescribed positive threshold. A switching rule is used in the estimates modification algorithm to ensure the controllability of the modified estimated model while avoiding possible chattering. For that purpose, the switching rule takes values at two possible distinct prefixed thresholds. In the event when the Sylvester determinant takes the current value of the switching function then that one switches to the alternative threshold. The convergence of both the unmodified and modified estimates to finite limits guarantees that switching ends in finite time. Thus, the solution to the controlled plant exist so that all the signals within the loop are well-posed.

Keywords: Adaptive control; Controllability; Estimation scheme; Estimation modification scheme; Stability; Sylvester matrix

AMS Classification Numbers: 93C40, 93D21, 93D10, 93C10, 93E24

INTRODUCTION

The adaptive stabilization and control of linear continuous and discrete systems has been successfully developed in the two last decades [1–4]. Usually, the plant is assumed to be stable inverse and its relative degree and its high-frequency gain sign are assumed to be known together with an absolute upper-bound for that gain in the discrete case. The assumption on the knowledge of the order can be relaxed by assuming a nominal known order and considering the exceeding modes as unmodeled dynamics [11, 12, 15]. The assumption on the knowledge of the high frequency gain has been removed in [4] and [13]. The controllability of the estimated plant model has been successfully guaranteed in the discrete case and, more recently, in the continuous one [7 – 12]. The problem is solved by using either excitation of the plant signals or a modification of the least-squares estimation by exploiting the properties of the standard least-squares covariance matrix [8 – 12, 14]. In [8 – 13], such a controllability of the estimated plant model is guaranteed by using either excitation of the plant signals or estimates modification. The estimates modification has been addressed either by incorporating hysteresis switching functions in the estimation schemes or guaranteeing the determinant of the Sylvester's matrix to be nonzero while exploiting the properties of the covariance matrix [8 – 12]. This paper presents an adaptive stabilization algorithm of pole-placement type for continuous-time systems. Generally speaking, strategies of modification of the estimation are important since they allow the maintenance the controllability of the estimated model so that singularities in the control law are avoided since the diophantine equation associated with the controller synthesis is solvable for all time and at the limit (see [9 – 12, 15] and references therein). The modification mechanism which has been used in those references is related to the use of a hysteresis switching function that guarantees that the Sylvester matrix of the modified estimated model is nonsingular and the modified estimates converge to finite limits as the unmodified ones do. *The adaptive scheme proposed in this paper uses a parameter modification mechanism while guaranteeing that the absolute value of the determinant of the Sylvester matrix associated with the parameter estimates is bounded from below by a positive threshold.* The boundedness and convergence of all the unmodified and modified estimates and controller parameters are guaranteed. The plant input and output are bounded and converge to zero in the ideal perfectly modelled case. The second section is devoted to the synthesis of the adaptive stabilizer in the perfectly modelled case for unknown continuous-time plants. The estimation scheme, used prior to the modification procedure, is of least-squares type. The third section presents the convergence and stability properties of the proposed scheme. A numerical example is then given and, finally, conclusions end the paper. The proofs of the results are developed in Appendix A.

ADAPTIVE STABILIZER FOR A CONTINUOUS-TIME PLANT

In the sequel, the time-argument is suppressed unless confusion can arise and the constant parameters are denoted by a superscript '*'. Consider the following continuous-time controllable system

$$A^*(D)y(t) = B^*(D)u(t);\ D^i y(0) = y_0^{(i)} \quad (i = 0, 1, \ldots, n-1) \tag{1}$$

where $D^i \equiv (d^i/dt^i)(i = 0, 1, \ldots, n-1)$ is the ith time-derivative operator, $A^*(D) = D^n + \sum_{i=1}^{n} a_i^* D^{n-i}$ and $B^*(D) = \sum_{i=0}^{m} b_i^* D^{m-i}$ with $n \geq m$. Since (1) is controllable then its associated Sylvester resultant matrix

$$S(\theta_0^*) = \begin{array}{c}\begin{bmatrix} 1 & 0 & \cdots & 0 & b_0^* & \cdots & 0 \\ a_1^* & 1 & & \vdots & b_1^* & & 0 \\ & & & & \vdots & & \\ \vdots & a_1^* & & 0 & b_m^* & & \vdots \\ a_n^* & \vdots & \ddots & 1 & & \ddots & b_0^* \\ 0 & a_n^* & & a_1^* & & & b_1^* \\ \vdots & 0 & \ddots & \vdots & 0 & \ddots & \vdots \\ & & & & \vdots & & \\ 0 & \cdots & \cdots & a_n^* & 0 & \cdots & b_m^* \end{bmatrix} \\ \qquad\quad \underbrace{\qquad}_{m} \qquad\qquad \underbrace{\qquad}_{n} \end{array}$$

is nonsingular. Define the subsequent filtered signals:

$$E^*(D)u_f = u;\ E^*(D)y_f = y;\ E^*(D) = D^n + \sum_{i=1}^{n-1} e_i^* D^{n-i} \tag{2}$$

with $E^*(D)$ being a strictly Hurwitz polynomial. The filtered control law for a known plant (1) is generated as

$$S^*(D)u_f = -R^*(D)y_f \tag{3}$$

where $S^*(D) = D^n + \sum_{i=1}^{n} s_i^* D^{n-i}$, $R^*(D) = D^n + \sum_{i=0}^{m-1} r_i^* D^{m-i-1}$ satisfy the diophantine equation:

$$A^*(D)S^*(D) + B^*(D)R^*(D) = C^*(D)$$

where

- $C^*(D) = D^n + \sum_{i=1}^{n^*-1} c_i^* D^{n^*-i}$ of prefixed degree fulfilling the constraint $n^* \leq n + \deg(S^*(D)) \leq 2n$ is a strictly Hurwitz polynomial (*i.e.*, with roots in Re $D < 0$) which defines the closed-loop dynamics being suitable for appropriate system's performance; and
- $S^*(D)$ and $R^*(D)$ are polynomials being the unique solution to the above diophantine equations since $A^*(D)$ and $B^*(D)$ are coprime because of the controllability of (1) and they satisfy the constraints $\deg(S^*(D)) \leq \deg(E^*(D)) \leq n$ and $\deg(R^*(D)) < \deg(A^*(D))$. [In particular, if $E^*(D)$ satisfies $\deg(E^*(D)) \leq n-1$ then its exceeding coefficients in (2) are zeroed]. In the following, the time argument t is omitted in time-dependent signals and time-dependent parameters for the shake of notation abbreviation unless it is necessary for a precise meaning in some expressions which require to emphasize time dependence of some terms against possible constant terms.

Equation (3) is equivalent to its unfiltered version:

$$u = (E^*(D) - S^*(D))u_f - R^*(D)y_f \tag{4}$$

The control objective in the adaptive case for unknown plant is to update the controller parameter s_i and r_j $(i = 1, 2, \ldots, n; j = 0, 1, \ldots, m)$ in an adaptive way so that the plant (1), subject to the control law (4) when replacing the parameters by their estimates, is asymptotically stable in the large in the absence of disturbances. Simple direct calculus with (1)–(2) yields for filtered signals the following reformulated version of the plant (1):

$$D^n y_f = \theta^{*T}\varphi \Leftrightarrow A^*(D)y_f = A^*(D)u_f + (\varepsilon_0^{*T} i_\varphi) \tag{5}$$

with

$$\begin{aligned}\theta^* &= [\theta_0^{*T} \vdots \varepsilon_0^{*T}]^T \\ &= [\theta_1^*, \theta_2^*, \ldots, \theta_{n+m+1}^* \vdots \theta_{n+m+2}^*, \\ &\quad \theta_{n+m+3}^*, \ldots, \theta_{2n+m+1}^*]^T \\ &= [b_0^*, b_1^*, \ldots, b_m^*, a_1^*, a_2^*, \ldots, a_n^* \vdots \\ &\quad \varepsilon_{01}^*, \varepsilon_{02}^*, \ldots, \varepsilon_{0n}^*]^T \end{aligned} \tag{6.a}$$

$$\begin{aligned}\varphi(t) &= [\varphi_0^T(t) \vdots i_\varphi^T(t)]^T \\ &= [D^m u_f, D^{m-1} u_f, \ldots, u_f, -D^{n-1} y_f, \\ &\quad -D^{n-2} y_f, \ldots, -y_f \vdots i_1, i_2, \ldots, i_n]^T. \end{aligned} \tag{6.b}$$

where $\varepsilon_0^{*T} i_\varphi(t)$ is an exponentially decaying scalar signal that depends on initial conditions associated with the various filters $1/E^*(D)$, and parametrized as the (nonnecessarily known) ε_0^*-vector, used to get filtered signals (2) used for the process description. Each component $i_j(t)$ of the vector signal $i_\varphi(t)$ is known and it has the form $t^\ell e^{\lambda_k^* t}$ for $\ell = 0, 1, \ldots, m_k - 1$ with m_k being the multiplicity of the root λ_k^* of $E^*(D)$. There are m_k terms $i_{(.)}(t)$ of such a form for each λ_k^*. The parameter vector θ^* is estimated by using a standard least-squares algorithm of covariance matrix $P(t)$ and estimated vector $\theta(t) = (\theta_0^T(t), \varepsilon_0^T(t))^T$ with $\varepsilon_0(t)$ being the estimated function of the initial conditions ε_0^*. The incorporation of the estimation of the parameter vector ε_0^* to the estimation scheme arises from the fact that (1) and its filtered version (5) are not equivalent if the signal in parenthesis associated with initial conditions in the state-space realization of (2) is omitted when the initial conditions of (2) are nonzero. If such a signal is not included in (5) then the closed-loop system may also be stabilized at the expense of a worse transient performance.

The estimation algorithm below consists of a least-squares type estimation procedure together with a rule to modify the estimates to ensure the controllability of the modified estimates model:

Parameter Estimation

Introduce the adaptation algorithm as follows:

$$e = D^n y_f - \theta^T \varphi \text{ (prediction error)} \tag{7.a}$$

$$\dot{\theta} = P\varphi e \tag{7.b}$$

$$\dot{P} = -P\varphi\varphi^T P; \;\; P(0) = P^T(0) > 0. \tag{7.c}$$

The basic modification of the estimated plant model is performed when necessary to maintain the controllability of the estimated model in the sense that $|\mathrm{Det}(S(\bar{\theta}_0))| \geq \rho > 0$ even if $|\mathrm{Det}(S(\theta_0))| < \rho$ for some positive real constant ρ while the Sylvester resultant matrices of the estimates and modified estimates have the same structures as $S(\theta_0^*)$ and their values are obtained by replacing θ_0^* with estimates $\theta_0(t)$ and modified estimates $\bar{\theta}_0(t)$, respectively.

Features in the Modification Philosophy

The estimates modification philosophy is simply in summary as follows for each $t \geq 0$:

(a) If $|\mathrm{Det}(S(\theta_0))| \geq \rho$ then the estimated model remains unmodified since it is controllable, *i.e.*, $\theta_0(t) = \bar{\theta}_0(t) \Rightarrow \theta(t) = \bar{\theta}(t)$ and then the control signal is generated based on this estimated vector since the diophantine equation associated with the controller synthesis is solvable with solution uniqueness.

(b) If $|\mathrm{Det}(S(\theta_0))| < \rho$ then the estimated model is modified from $\theta_0(t)$ to $\bar{\theta}_0(t)$ since it is either uncontrollable or close to loose controllability, in order to achieve $|\mathrm{Det}(S(\bar{\theta}_0))| \geq \rho$, implying the replacement $\Rightarrow \theta(t) \to \bar{\theta}(t)$. As a result, the diophantine equation is singularity-free and, then, uniquely solvable after replacing the estimates by the modified estimates. Then, the plant control signal is generated based on the modified vector of estimates.

(c) In order to avoid a possible chattering behavior (implying infinitely many switches) in view of the above features (a)–(b) caused when $|\mathrm{Det}(S(\theta_0(t)))| \to \rho$, from the left and the right, over time intervals of nonzero measure in the above situation, two distinct thresholds ρ and ρ' are used so that when the determinant is fixed to any of those values the switching function $h(t)$ defined in the subsequent subsection below changes of value until another switch, if any, occurs. Since the absolute Sylvester determinant is a continuous function of the parameter estimates, it may be ensured that it cannot be close to the other threshold value during some nonzero finite interval $(t,\ t')$.

(d) Since the determinant of any square matrix is an analytic function of the entries of such a matrix, the modified Sylvester determinant may be calculated as a Taylor series expansion around the unmodified one as $\mathrm{Det}(S(\bar{\theta}_0)) = \mathrm{Det}(S(\theta_0)) + \Delta(\theta_0, \bar{\theta}_0)$. The known formula from Linear Algebra $(d/d\theta_{0i})(\mathrm{Det}(S(\theta_0)))|_{\theta_0=q_0} = \mathrm{Trace}(S_{\theta_{0i}}(\theta_0) \cdot \tilde{S}(\theta_0))$ $\tilde{S}(\theta_0)$ denoting matrix of cofactors [19], with subscripts denoting partial derivatives with respect to the components of the parameter vector may be used to calculate $\Delta(\theta_0, \bar{\theta}_0)$ from which higher-order derivatives with respect to the various parameter vector components can be directly obtained as well.

(e) If the estimated model is close to loose controllability then the modified estimation is performed by a linear rule of the type $\bar{\theta} = \theta + \bar{\delta} = (\theta_0 + \bar{\delta}_0 \vdots \varepsilon_0^T)$ with $\bar{\theta}_0 = \theta_0 + \alpha(\sigma_1, \ldots, \sigma_{n+m+1})^T$ with σ_i taking values in the set $\{-1, 0, 1)$, the estimates of the initial conditions remaining unmodified since they do not affect to the solvability of the diophantine equation. The α-function is calculated so that $|\mathrm{Det}(S(\bar{\theta}_0)) - \mathrm{Det}(S(\theta_0))| = |\Delta(\theta_0, \bar{\theta}_0)|$ is maximized for each t such that $h(t)$ switches for all the set of possible values of σ_i in $\{-1, 0, 1\}$ ($i = 1, 2, \ldots, n+m+1$). That means that each estimate is modified by $\pm\alpha$ or it remains unmodified. The strategy allows the modified determinant to take values far from those implying uncontrollability of the estimation model. The details of addressing the above ideas (a) to (e) are detailed in the subsequent subsection.

The modification scheme to calculate $\bar{\theta}$ from θ is implemented according to the following scheme:

Modification Procedure of the Estimation

The plant parameter estimates through the algorithm (7) are modified as follows. First, define the switching rule through the functions δ_α and

h as follows:

$$\delta_\alpha(t)=\begin{cases}\dfrac{3h(t)-\mathrm{Det}(S(\theta_0(t)))}{\bar{C}} \\ \quad =\dfrac{3h(t)-|\mathrm{Det}(S(\theta_0))|\mathrm{Sign}(\bar{C})\mathrm{Sign}(\mathrm{Det}(S(\theta_0)))}{|\bar{C}|} & \text{if } |\mathrm{Det}S(\theta_0)|<h(t) \\ 0 \ \text{ if } |\mathrm{Det}S(\theta_0)|\geq h(t) \end{cases} \tag{8.a}$$

where $h(0^-)=\rho$; and

$$h(t^+)=\begin{cases} h(t^-) & \text{if } |\mathrm{Det}\,S(\theta_0(t^-))|\neq\rho \text{ and } |\mathrm{Det}\,S(\theta_0(t^-))|\neq\rho' \\ \rho & \text{if } |\mathrm{Det}\,S(\theta_0(t^-))|=\rho' \\ \rho & \text{if } |\mathrm{Det}\,S(\theta_0(t^-))|=\rho \end{cases} \tag{8.b}$$

for some small positive real constants $\rho'\geq 2\rho$, so that $h(t)\geq\rho>0$ for all $t\geq 0$ even at discontinuity points, where $h(t^+)\neq h(t^-)$, with ρ fulfilling $\rho<(|\sigma|/6(n+m))$ in (8.b) and $-|\sigma|$ being the stability abscissa of the polynomial $C^*(D)$ (which defines the control objective), with

$$\bar{C}:=C(\bar{\sigma}_1,\bar{\sigma}_2,\ldots,\bar{\sigma}_{n+m+1}) \tag{8.c}$$

$$C(\sigma_1,\ldots,\sigma_{n+m+1})=\sum_{k=1}^{n+m}\sum_{i_1,i_2,\ldots,i_k=1}^{n+m+1}\frac{1}{k!}$$
$$\mathrm{Trace}(S_{\theta_{i_1}}(\theta_0)\tilde{S}_{\theta_{i_1}\ldots\ldots\theta_{i_k}}(\theta_0))\prod_{j=i_1}^{i_k}[\sigma_j];\qquad \sigma_i\in\{0,-1,1\} \tag{8.d}$$

$$(\bar{\sigma}_1,\bar{\sigma}_2,\ldots,\bar{\sigma}_{n+m+1}):=\Big\{\mathrm{Arg}(\sigma_1,\sigma_2,\ldots,\sigma_{n+m+1}):|C(\sigma_1,\sigma_2,\ldots,\sigma_{n+m+1})|$$
$$=\underset{\sigma_i\in\{0,-1,1\}}{\mathrm{Max}}|C(\sigma_1,\ldots,\sigma_{n+m+1})|\Big\}. \tag{8.e}$$

with $\tilde{S}(\theta_0)$ being the matrix of cofactors of $S(\theta_0)$ whose first and higher-order partial derivatives are denoted below by subscripts with respect to the respective arguments. The first-order derivatives with respect to the vector of parameter estimates are sparse matrices defined by:

$$S_{a_i}(\theta_0)=\frac{dS}{da_i}|_{\theta_0}=\begin{array}{l}\left[\begin{array}{c}0_{i\times(n+m)}\\ \cdots\cdots\cdots\cdots\\ I_m\ \ 0_{m\times n}\\ \cdots\cdots\cdots\cdots\\ 0_{(n-i)\times(n+m)}\end{array}\right]\end{array}\begin{array}{l}\\ \\ \leftarrow(i+1)\text{-th row}(i=1,\ldots,n)\\ \\ \\ \end{array}$$

$$S_{b_j}(\theta_0)=\frac{dS}{db_j}|_{\theta_0}=\begin{array}{l}\left[\begin{array}{c}0_{j\times(n+m)}\\ \cdots\cdots\cdots\cdots\\ 0_{n\times m}\ \ I_n\\ \cdots\cdots\cdots\cdots\\ 0_{(m-j)\times(n+m)}\end{array}\right]\end{array}\begin{array}{l}\\ \\ \leftarrow(j+1)\text{-th row}(j=0,1,\ldots,m)\\ \\ \\ \end{array}\tag{8.f}$$

where the subscripts in the above zeros denote their orders as block matrices and I_m is the m-identity matrix. The modified estimates are generated as follows:

$$\bar{\theta}=\theta+\bar{\delta}=(\bar{\theta}_0^T,\varepsilon_0^T)^T=[\bar{b}_0,\bar{b}_1,\ldots,\bar{b}_m,\bar{a}_1,\bar{a}_2,\ldots,\bar{a}_n\vdots\varepsilon_0^T]^T \tag{9.a}$$

$$\bar{\delta}=[\delta b_0,\delta b_1,\ldots,\delta b_m,\delta a_1,\delta a_2,\ldots,\delta a_n,0,\ldots,0]^T=[\bar{\delta}_0^T,0^T]^T \tag{9.b}$$

and the $n+m+1$ first components of the estimated vector $\theta(t)$ are modified according to the rule

$$\begin{aligned}\bar{a}_i&=a_i+\delta a_i=a_i+\alpha\bar{\sigma}_i;\\ \bar{b}_j&=b_j+\delta b_j=b_j+\alpha\bar{\sigma}_{n+1+j};\ i=1,2,\ldots,n;\ j=0,1,\ldots,m\end{aligned}\tag{9.c}$$

$$\alpha=\begin{cases}\delta_\alpha & \text{if } \delta_\alpha\geq 1\\ (\delta_\alpha)^{(1/n+m)} & \text{if } \delta_\alpha<1\end{cases}\tag{9.d}$$

by using (8) in order to obtain the modified estimated vector $\bar{\theta}(t)$ in (9.a).

Note that $\bar{\sigma}=(\bar{\sigma}_1,\bar{\sigma}_2,\ldots,\bar{\sigma}_{n+m+1})^T$ is a non necessarily unique vector, whose components take values in the set $\{0,-1,1\}$ which maximizes the function $|C(\sigma_1,\sigma_2,\ldots,\sigma_{n+m+1})|$ for the of constraints

$\sigma_i \in \{0, -1, 1\}$ for $i = 1, 2, \ldots, n+m+1$. The idea behind the above modification method (8)–(9) is the following. Two different thresholds ρ and $\rho' \geq 2\rho$ are used to modify the parameter components in (8.a). The use of two distinct thresholds is only made for purposes of avoiding chattering by involving the mechanism of switching between them each time a discontinuity in the modification occurs. These thresholds are sufficiently small compared to the stability abscissa of the objective polynomial $C^*(D)$ in order to guarantee the closed-loop stability (see Assumption 1 in the second section). Each absolute value of a parameter estimate is either modified with an absolute amount $\alpha(t)$ or it becomes unmodified (see (9.c)–(9.d)). The maximum value of the modified Sylvester determinant depends on the threshold sizes (see (8.a)–(8.b) and (9.d)). The mechanism which ensures that the absolute value of the modified Sylvester determinant exceeds the size of the minimum threshold ρ is the manipulation of its Taylor expansion around its unmodified value. This is performed by checking the maximum allowable absolute increasing the determinant through the modification process by varying each estimate in $\pm\alpha$ or leaving it unmodified.

In particular, assume that each ith parameter component of θ_0 is modified by an additive increment $\alpha\,\sigma_i$ so that the modification scheme is $\bar{\theta}_0 = \theta_0 + \alpha(\sigma_1, \ldots, \sigma_{n+m+1})^T$ with each $\sigma_{(\cdot)} \in \{1, 0, -1\}$. A well-known equation from Linear Algebra is $(d/d\theta_{0i})(\mathrm{Det}(S(\theta_0)))\big|_{\theta_0 = q_0} = \mathrm{Trace}(S_{\theta_{0i}}(\theta_0) \cdot \tilde{S}(\theta_0))$, [19], from which higher-order derivatives with respect to the various parameter vector components can be directly obtained as well. Thus, a series Taylor expansion of the analytic multivariable function of the modified estimates $\mathrm{Det}(S(\bar{\theta}_{01}, \ldots, \bar{\theta}_{0,n+m+1}))$ around $\mathrm{Det}(S(\theta_{01}, \ldots, \theta_{0,n+m+1}))$ (later denoted as $\mathrm{Det}(S(\theta_0))$ for notation simplicity purposes) is stated. Such an expansion is given by the identity $\mathrm{Det}(S(\bar{\theta}_0)) = \mathrm{Det}(S(\theta_0)) + \Delta(\theta_0, \bar{\theta}_0)$ with $|\Delta(\theta_0, \bar{\theta}_0)| \geq |\delta_\alpha \bar{C}|$, with the value $\bar{C}$ being calculated from (8.a)–(8.e) since each parameter estimate has a variation $\pm\alpha$ (*i.e.*, $\sigma_i = \pm 1$) or zero (*i.e.*, $\sigma_i = 0$).

Note from (8.b) that $h(\cdot)$ is a piecewise constant function which only takes values at ρ and ρ' and changes of value only when the absolute Sylvester determinant reaches the current threshold ρ or ρ' to which the function was previously set at the preceding switching time. Such switches have as objective of avoiding chattering so that the existence of solution is ensured for all time. Chattering could potentially arise if the Sylvester determinant would converge to a constant function h while, at the same time, its time-derivative converges to zero with changing sign. This phenomenon is avoided in this approach by using the switching rule (8.a) by taking advantage of the fact that the unmodified and modified parameter estimates converge asymptotically to finite limits. Thus, if the Sylvester determinant converges to ρ (or, respectively, ρ') it cannot converge to ρ' (or, respectively, ρ) since it remains in a certain small neighborhood centered at ρ (or, respectively ρ') after a large but finite time. The avoidance of chattering guarantees directly the existence of solution. These features will be proved in the following section.

It is proved in Appendix A, as an intermediate step in the proof of the subsequent controllability result, that $\bar{C} \neq 0$ for all time because not all the derivatives in (8.f) with respect to the estimates evaluated at the parameter vector estimated from the algorithm (7) *are zero*. This feature makes possible that the Sylvester determinant of the modified estimates can always be modified with respect to its value prior to modification. It becomes obvious from the above modification philosophy that $|\bar{C}|$ can be replaced with any value of $|C|$ which be bounded from below by a positive constant.

The use of switching functions for the estimation modification procedure was used in [9–12] to ensure the controllability of the modified estimated plant model. The mechanism used in those papers to prove the boundedness and convergence properties of the modified estimates was to take advantage of the properties of the least-squares estimation. The procedure used to guarantee the above properties in the modification algorithm of this paper is based on the properties of the Sylvester matrices. It also involves the use of a rule with switching functions to avoid possible chattering (see (8.a)–(8.b) and (9.d)). The $h(\cdot)$-function takes only two possible prescribed small threshold values so as to avoid chattering caused by a possible convergence of the unmodified Sylvester determinant to a discontinuity point. The switching mechanism operates as follows. In the eventual case when the Sylvester determinant of the unmodified estimates takes any of both prefixed thresholds the switching function takes the alternative threshold for the Sylvester determinant and a switching takes place. Since both the unmodified and modified estimates converge to finite limits, what is proved in the next section, it is impossible the simultaneous convergence of the absolute value of the Sylvester determinant to both thresholds. The estimates convergence ensures, in addition, that switching ends in finite time. In summary, it can be said that, in a similar way as the modification schemes proposed in [9–12], the modified estimates obtained by the procedure proposed in this paper have the same convergence properties as the unmodified ones due to two facts, namely:

(a) Switches in (8), if any, end in finite time as a direct consequence of the convergence properties of the unmodified estimates since the modifications are obtained by incremental values which are calculated with the unmodified estimates through (8)–(9). This property is proved in the subsequent section.
(b) The parameter estimates variations converge by construction as the unmodified estimates converge.

In the proposed estimation-modification scheme, the covariance matrix is not used to calculate the modifications, as it was in [9–12], but it still plays a crucial role in guaranteeing the convergence of the unmodified estimates which is a key point to ensure that of the modified ones. Note also that the key point in the approaches used in [9–12] was to exploiting the properties of the least-squares estimation while the key point in the current approach is the use of the properties of the Sylvester matrix to maintain it nonsingular after estimates modification what ensures that the modified estimate model remains controllable for all time and at the limit. In [9] and [11, 12], the use of hysteresis switching functions was proposed to guarantee the controllability of the modified estimated plant model without chattering of the parametrical estimation (*i.e.*, the abrupt switching between two values of the estimated parameter vector). In [9] and [11], the estimation modification method based on the use of a hysteresis switching function is applied to a class of hybrid systems which consist of coupled continuous and digital substates. In [10], a direct manipulation of the Sylvester determinant was proposed to guaranteeing for such a determinant to be sufficiently far away from zero. Only the case of first-order plants was considered with the eventual modification of only one estimated parameter to guarantee the estimation model controllability.

In this paper, a nth-order plant has been considered in contrast with [10] and all its estimated parameters can potentially be modified in order to guarantee the controllability of the estimated plant model. Furthermore, the parameter modification technique applies to all the parameters looking for the modified Sylvester determinant being less close to zero according to prescribed thresholds of all possible obtainable modified Sylvester's determinants. The analysis technique used is based on the analiticity of the unmodified Sylvester determinant with respect to its parametrization. Also, the signs of the modification amounts may be positive or negative for each individual parameter estimate in order to look for the highest degree of controllability among all possible modified estimated plant models. Furthermore, two distinct thresholds ρ and ρ' are used in (8.a) in order to avoid possible chattering so that if the absolute determinant is converging to any of these thresholds, the test threshold is changed *via* the switching function $h(\cdot)$.

Stabilizing Adaptive Control Law

Introducing (9.a) into (7.a), we obtain

$$\begin{aligned} D^n y_f &= e + \theta^T \varphi = e + (\bar{\theta}^T - \bar{\delta}^T)\varphi \\ &= e + A(D,t)y_f + B(D,t)u_f + \varepsilon_0^T(t) i_\varphi(t) \end{aligned} \tag{10}$$

with $A(D,t)$ and $B(D,t)$ being time-varying polynomials associated with the estimates, obtained from (7), which define the estimated model of the plant prior to eventual modification, and whose adjustable parameters are the components of the unmodified estimated vector θ. *The filtered and unfiltered control inputs are generated from the adaptive version of* (3)–(4),

$$S(D,t)u_f(t) = -R(D,t)y_f(t) \tag{11}$$

$$u(t) = (E^*(D) - S(D,t))u_f(t) - R(D,t)y_f(t) \tag{12}$$

so that the following closed-loop diophantine equation is satisfied by the controller polynomials $R(D,t)$ and $S(D,t)$ which are calculated from modified parameter estimates:

$$\bar{A}(D,t)S(D,t)+\bar{B}(D,t)R(D,t)=C^*(D) \tag{13}$$

with $\bar{A}(D,t)=A(D,t)+\delta A(D,t)$, $\bar{B}(D,t)=B(D,t)+\delta B(D,t)$, $\delta A(D,t)=\sum_{i=1}^{n}\delta a_i D^{n-i}$ and $\delta B(D,t)=\sum_{i=0}^{m}\delta b_i D^{m-i}$. The solution is unique since the modified plant parameter estimated model is controllable for $t\geq 0$ so that the time-varying polynomials $\bar{A}(D,t)$ and $\bar{B}(D,t)$ are coprime for $t\geq 0$.

CONTROLLABILITY OF THE ESTIMATED MODEL AND STABILITY RESULTS

The following result reflects the feature that the Modification Scheme makes recover the controllability of the Modified Estimated Model, with a prescribed degree, in the case when controllability of the Unmodified Estimated Model is lost or close to be lost.

PROPOSITION 1 *The modified estimation scheme* (8)–(9) *of the plant model estimated from* (7) *fulfils for all time* $|Det(S(\bar{\theta}_0))|\geq\rho>0$ *so that such a model is controllable. Furthermore, there is no chattering caused by switches in the estimates modification rule* (8.*a*)–(8.*b*) *and* (9.*d*) *and then the solution to* (1), *subject to the control law*, (12)–(13) *exists for all time.* ■

The following assumption is explicitly introduced to guarantee the stability of the closed-loop system under modification of the estimates while it is not required for the properties of boundedness and convergence of the estimation algorithm.

ASSUMPTION 1 *The design constant* ρ *in* (8.*a*) *fulfils* $0<\rho<(|\sigma|/6(n+m))$ *in the Estimation Modification Scheme of Section* II *Eqs.* (8)–(9). ■

The constraint $0<\rho<(|\sigma|/6(n+m))$ of Assumption 1 will be used in the stability proof of Theorem 1 below which involves the use of Gronwall's Lemma to an equivalent dynamic system. Such a system describes the combined dynamics of the plant and adaptive controller so that it is a key feature in the proof of closed-loop stability.

The properties of the adaptive scheme are given in the subsequent main result proved in Appendix A.

THEOREM 1 *The adaptive control law* (11)–(13), *under the Estimation Modification Scheme* (7)–(9), *has the following properties when applied to the plant* (1):

(i) θ, $\bar{\theta}$ *and* P *are uniformly bounded and the modified estimated plant model is controllable for all time.*
(ii) e *and* $P\ \varphi$ *are in* L_2.
(iii) θ, P, $Det(S(\theta_0))$, $\bar{\theta}$, $Det(S(\bar{\theta}_0))$, s_i *and* $r_j (i=1,2,\ldots,n; j=0,1,\ldots, m-1)$ *converge asymptotically to finite limits for any bounded initial conditions for the plant and the estimation algorithm.*
(iv) *If the Modification Scheme used satisfies Assumption* 1 *then* $D^i u_f$, $D^i y_f$ $(i=0,1,\ldots,n-1)$ *and* u *and* y *are bounded and converge asymptotically to zero.* ■

Note that $e \in L_2 \cap L_\infty$ from Theorem 1 [(i) and (iv)] so that $e \to 0$ as $t \to \infty$ and $\theta \in L_\infty$ and converges to a finite limit. Also, $\|\dot{\tilde{\theta}}\| \in L_\infty$ from (7.b) since $P \in L_\infty$ and $\varphi \in L_\infty$. These properties guarantee that both θ_0 and $\mathrm{Det}(S(\theta_0))$ are bounded and converge to finite limits so that the modification $\bar{\delta}$ is bounded and converges.

NUMERICAL EXAMPLE

A numerical example is now tested for a nominally unstable and inversely unstable plant (1) parametrized by $A^*(D) = D^4 + 0.75D^3 + 0.5D^2 + 0.25D + 0.25$ and $B^*(D) = 0.75D^3 + (2/3)D^2 + 0.25D + 0.25$ with initial conditions $(-5, -7, 0, 0)^T$ with filter parameter $E^*(D) = (D + 6.93)^2$. A second-order additive unmodeled dynamics is assumed to be present which is given by a second-order differential equation $\ddot{\eta} + 0.12\eta - 7.8 = 7.8u$ under zero initial conditions with η being an additive disturbance signal in the right-hand-side of (1). An absolute overbounding signal $\bar{\eta} = 1.04\left(1 + 10^{-5}\mathrm{Sup}_{0 \le \tau \le t}\left(\|\varphi(\tau)e^{-0.1(t-\tau)}\|\right)\right) \ge |\eta_f|$ is known for all time for the filtered $\eta_f = (1/E^*)\eta$. The Modification Scheme of Section II is used by incorporating a relative normalized adaptation dead-zone so that the parameter estimation (7) is frozen (*i.e.*, $\dot{\theta} = 0$) if $|e| < |\bar{\eta}|$ while (7.b) is replaced with $\dot{\theta} = bP\varphi e$ and

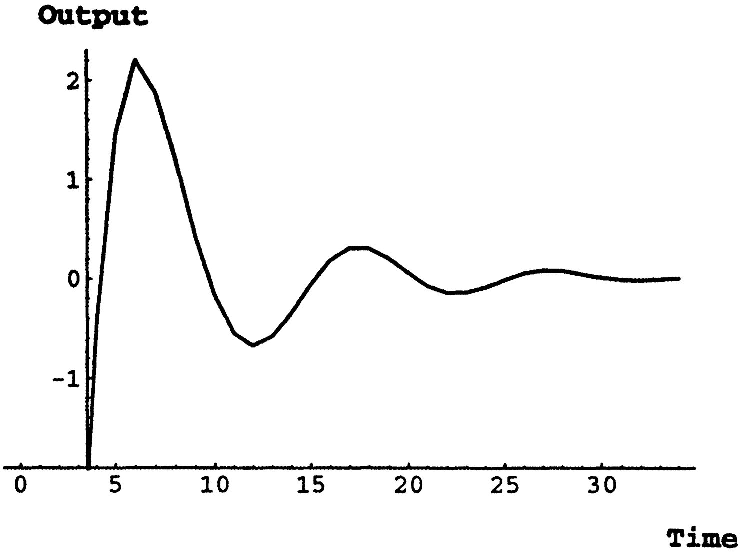

FIGURE 1 Output *versus* time.

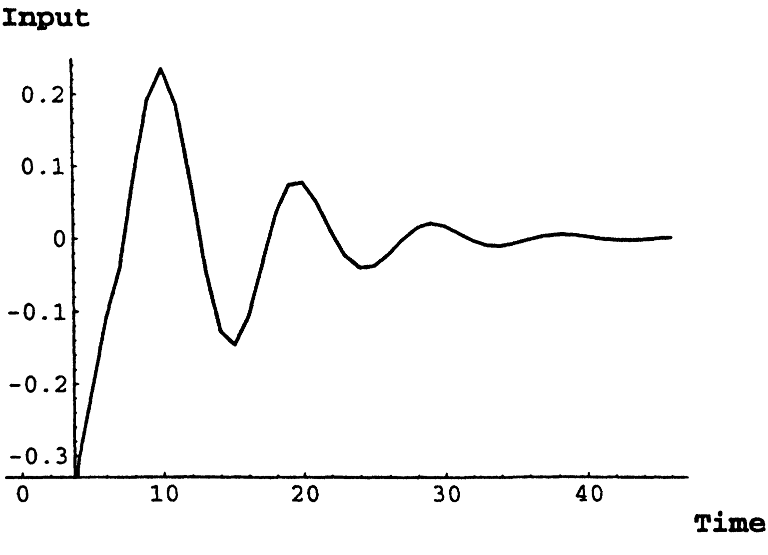

FIGURE 2 Input *versus* time.

$b=(1-|\bar{\eta}/e|)/(1+\varphi^T P\varphi)$, otherwise. This adaptation mechanism ensures the closed-loop stability in the presence of stable unmodeled dynamics with boundedness of the prediction error and regressor and integrability of $b|\eta_f^2-e^2|$ (see, for instance, [12] and [15]). The basic determinant threshold for parameter modification of the estimates is $\rho=0.01$. The adaptive stabilizer satisfies the constraints

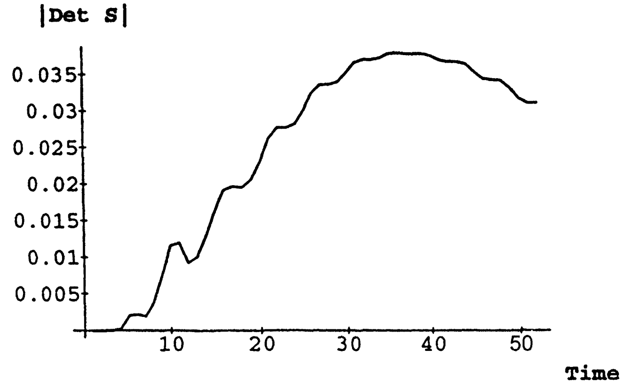

FIGURE 3 Sylvester Determinant of the estimates prior to modification *versus* time.

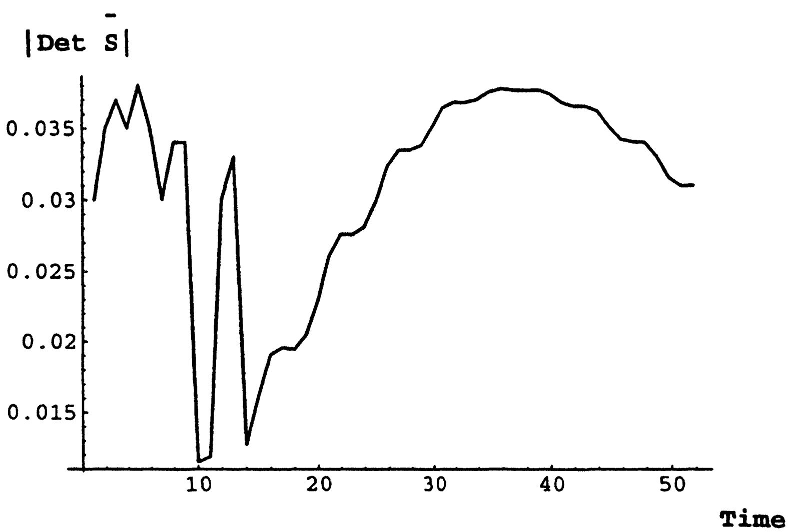

FIGURE 4 Absolute value of the Sylvester determinant of the modified estimates *versus* time.

$\deg(R(D)) = \deg(S(D)) - 1 = 1$. The initialization of the estimation algorithm is $b_0(0) = 1$, $b_1 = -0.008$, $b_2(0) = -0.003$, $a_1(0) = 0.005$, $a_2(0) = -0.005$, $a_3(0) = 0$, $a_4(0) = 0$. The parameter b_3^* is assumed known and deleted from the estimation algorithm. The estimates of the initial conditions of the plant (1) are zero. The covariance matrix is initialized to $P(0) = \text{Diag}\ (10^6)$. The output and input *versus* time are shown on Figures 1–2, respectively. Figures 3 and 4 show $\text{Det}(S(\theta_0))$, whose initial value from initial conditions of the estimate, is -9.85×10^{-9} and $|\text{Det}(S(\bar{\theta}_0))|$, respectively.

CONCLUSIONS

An adaptive stabilizer for a continuous-time plant has been proposed without assuming the inverse stability of the plant, '*a priori*' knowledge on the plant parameters and knowledge of the high-frequency gain sign. The adaptive stabilizer is of pole-placement based type. It consists of a estimation algorithm with covariance matrix adaptation with a subsequent parameter estimation modification of the parameter estimates. A modification scheme has been proposed which ensure the controllability of the modified estimated plant model. The modification mechanism which guarantees such a controllability consists basically of the additive perturbation of nonnecessarily all of the estimated plant parameters according to a test on the Sylvester determinant. In this way, the resulting modified Sylvester matrix becomes nonsingular when the controllability of the unmodified estimation model fails against the determinant test for non singularity. The estimation scheme has suitable stability and convergence properties for both the unmodified and modified estimates. Also, all the relevant closed-loop signals exist since chattering is avoided by using two thresholds in the determinant test as well as switchings from the current determinant threshold to the alternative one when a discontinuity in the test for each current threshold is detected.

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APPENDIX A

Proof of Proposition 1 Firstly, note that the first-order derivatives of the determinant with respect to any parameter estimate are calculated as follows from elementary algebra (see, for instance, [19]):

$$\frac{\partial}{\partial \theta_i} \mathrm{Det}(S(\theta_0)) = \mathrm{Trace}\left(\frac{\partial S(\theta_0)}{\partial \theta_i} \tilde{S}(\theta_0) \right) \tag{A.1}$$

which holds when taking derivatives of determinants with respect to any value of θ_i for $i=1,2,\ldots,n+m+1$. The derivatives are evaluated at θ_0. However, it is clear from (8.e) that $S_{\theta_{i_1},\ldots,\theta_{i_k}} = (\partial^k S(\theta_0))/(\partial\theta_{i_1} \cdots \partial\theta_{i_k}) = 0; k = 2,3,\ldots,n+m+1$ with all the partial derivatives being evaluated at θ_0. Also, since $\tilde{S}(\theta_0)$ is a matrix of cofactors, it contains products of at most $(n+m)$ parameters at each one of its entries. Thus, the matrices of first and higher-order partial derivatives of the matrix of cofactors with respect to the parameter vector components $\tilde{S}_{\theta_{i_1}\ldots\theta_{i_k}}(\theta_0) = 0$ if $k > n+m$ for any integers $i_j \geq 1$ for $j=1,2,\ldots,k$. Now, $\mathrm{Det}(S(\bar{\theta}_0))$ is expanded in Taylor series around $\mathrm{Det}(S(\theta_0))$ by taking successive derivatives with respect to parameter components evaluated at θ_0 by starting with (A.1) while zeroing any derivatives of higher-order than $(n+m)$. One obtains directly from (8.c)–(8.f) and (9.c)–(9.d) that

$$\mathrm{Det}(S(\bar{\theta}_0)) = \mathrm{Det}(S(\theta_0)) + \Delta(\theta_0, \bar{\theta}_0) \tag{A.2a}$$

with

$$\begin{aligned} |\Delta(\theta_0, \bar{\theta}_0)| &= \left| \sum_{k=1}^{n+m} \sum_{i_1,i_2,\ldots,i_k=1}^{n+m+1} \frac{1}{k!} \mathrm{Trace}\ (S_{\theta_{i_1}}(\theta_0) \tilde{S}_{\theta_{i_1}\ldots\theta_{i_k}}(\theta_0)) \prod_{j=i_1}^{i_k} (\bar{\theta}_{0j} - \theta_{0j}) \right| \\ &\geq |C| \operatorname*{Min}_{j=1,2,\ldots,n+m+1} (\alpha^j) = |C| \mathrm{Min}(\alpha, \alpha^{n+m}) = |C\delta_\alpha| \end{aligned} \tag{A.2.b}$$

by using $\prod_{j=i_1}^{i_k} (\bar{\theta}_{0j} - \theta_{0j}) = \prod_{j=i_1}^{i_k} (\sigma_j \alpha)$ for any values $\sigma_{(\cdot)} \in \{0, -1, 1\}$ with $\mathrm{Max}_{\sigma_i \in \{0,-1,1\}} (|\Delta(\theta_0, \bar{\theta}_0)|) \geq |\delta_\alpha \bar{C}|$ since $\alpha^{n+m} = \delta_\alpha < \alpha < 1$ for $\alpha = \delta_\alpha < 1$ and $\alpha \leq \alpha^{n+m} = \delta_\alpha$ for $\delta_\alpha \geq 1$ with $\alpha \geq 1$ from (9.d). Now, it is proved by contradiction that

$$\text{Trace } (S_{\theta_{i_1}}(\theta_0)\tilde{S}_{\theta_{i_1}\ldots\theta_{i_k}}(\theta_0)) = 0$$
$$\text{for all } i_k \in \{1, \ldots, n+m+1\}, \quad k = 1, 2, \ldots, n+m \qquad \text{(A.3)}$$

is impossible since (A.3) depends on the estimates of the plant parameters irrespective of the modification scheme. Assume with no loss in generality, since $\rho' > \rho$, that $h(t) = \rho$ in (8.a). Assume also that $|\text{Det}\, S(\theta_0)| \neq \zeta < \rho$ with $\zeta > 0$. Then, note from the definition of $S(\bar{\theta}_0)$ that $|\text{Det}(S(\bar{\theta}_0))| = \zeta$ with arbitrary nonzero ζ if the subsequent modification rule is used after proceeding with the unmodified least-squares estimation: $\delta a_i = -a_i$, $\delta b_j = -b_j$ and $\delta b_m = \pm\zeta^{(1/n)} - b_m$ for $i = 1, 2, \ldots, n$; $j = 0, 1, \ldots, m$. Assume that (A.3) holds. Since the function $|\text{Det}(S(\theta_0))|$ is analytic in the overall parameter estimates space, all values of $|\text{Det}(S(\bar{\theta}_0))|$ can be obtained from a Taylor series expansion around $|\text{Det}(S(\theta_0))|$, as reflected in (A.2.a)–(A.2.b), with only a finite number of derivatives and higher-order derivatives of $|\text{Det}(S(\theta_0))|$ with respect to the unmodified estimates being structurally nonzero due to the form of the Sylvester determinant. Therefore, if (A.3) holds then $|\text{Det}(S(\bar{\theta}_0))| = |\text{Det}(S(\theta_0))|$ and $\Delta(\theta_0, \bar{\theta}_0) = 0$ in (A.2.a) irrespective of the modification rule used. Thus, one has the impossible relationships $\zeta = |\text{Det}(S(\bar{\theta}_0))| = |\text{Det}(S(\theta_0))| \neq \zeta$ for the maximum variation between both determinants when the unmodified vector of estimates is θ_0. This follows by using a Taylor series expansion in the parameter space of the modified estimates around the estimated ones obtained from (7) according to (A.2). Thus, (A.3) is false, since all the derivatives used in (A.2.b) are not dependent on the estimates modification scheme.

As a result, there is at least one parameter component θ_i of θ_0 for which Trace $(S_{\theta_{i_1}}(\theta_0)\tilde{S}_{\theta_{i_1}\ldots\theta_{i_k}}(\theta_0)) \neq 0$ and then $\bar{C}$ in (8.c)–(8.e) is nonzero. Thus, $\text{Det}(S(\theta_0))$ is not constant for all the values of the components of θ_0 belonging to arbitrary real intervals and it is feasible that a modification $\theta_0 \rightarrow \bar{\theta}_0$ can be potentially carried out to guarantee that $|\text{Det}(S(\bar{\theta}_0))| \geq \rho$ from the analyticity of the function $|\text{Det}(S(\theta_0))|$ in all the space of estimates θ_0. This is now specifically proved.

Note from (8.a)–(8.b) that, if δ_α is continuous at certain time t, then $(2\rho/|\bar{C}|) \leq \delta_\alpha \leq (3\rho/|\bar{C}|)$ and $0 \leq \alpha \leq \text{Max}(|\delta_\alpha|, |\delta_\alpha|^{(1/(n+m))})$ from (9.d) with $\alpha = 0$ if and only if $|\text{Det}(S(\bar{\theta}_0))| \geq \text{Max}(\rho, \rho') \geq \rho$ from (8.b) and (9.d) and no modification is required. If δ_α is discontinuous at t then $|\delta_\alpha(t^+)| \geq (2\rho/|\bar{C}|)$ if $h(t^+)=\rho$ and $h(t^-)=\rho'$ and $|\delta_\alpha(t^+)| \geq (2\rho'/|\bar{C}|)$ if $h(t^+)=\rho'$ and $h(t^-)=\rho$. In any of the above situations $\alpha(t)\neq 0$. The switches in $h(t)$ make this eventual discontinuities to occur only at isolated time instants. Direct calculations with (8.a)–(8.c) and the above considerations yield that if $h(t^+)=\rho$ and $h(t^-)=\rho'=|\text{Det}\, S(\theta_0)|$ then:

$$\begin{aligned}|\text{Det}(S(\bar{\theta}_0))| &\geq \underset{\sigma_i\in\{0,-1,1\}}{\text{Max}}(|\Delta(\theta_0,\bar{\theta}_0)|) - |\text{Det}(S(\theta_0))| \\ &\geq |\delta_\alpha\bar{C}| - |\text{Det}(S(\theta_0))| \\ &\geq 3\rho - \text{Det}(S(\theta_0))\text{Sign}(\bar{C}) - |\text{Det}(S(\theta_0))| \\ &\geq 3h - 2|\text{Det}(S(\theta_0))| \geq \rho > 0 \qquad \text{(A.4)}\end{aligned}$$

for all time since $\text{Max}_{\sigma_i\in\{0,-1,1\}}(|\Delta(\theta_0,\bar{\theta}_0)|) \geq |\delta_\alpha\bar{C}|$ from (A.2.b). Similarly, if $h(t^-)=\rho'$ then (A.4) still holds. Thus, the first part of Proposition 1 has been proved. The absence of chattering and existence of the closed-loop solution follow directly since the eventual switches in (8.a), and then in (8.b) and (9.d), are isolated because of the continuity of the Sylvester determinant function with respect to the unmodified estimates and the fact that the α-function is continuous at $\delta_\alpha=1$ since $\alpha=\delta_\alpha=[(\delta_\alpha)^{(1/(n+m))}]_{\delta_\alpha=1}$. ■

Proof of Theorem 1

(i)–(ii) Note that $\dot{P}^{-1} = -P^{-1}\dot{P}P^{-1} = \varphi\varphi^T$ from (7.c). Define the Lyapunov function candidate $V=\tilde{\theta}^T P^{-1}\tilde{\theta}$ where $\tilde{\theta}=\hat{\theta}-\theta^*$ is the parametrical error before modification of the estimates. Thus, (7.a) can be rewritten as $e=-\tilde{\theta}^T\varphi$ and $\dot{V}=-(\tilde{\theta}^T\varphi)^2=-e^2\leq 0$ after direct calculations with V and (7), [9]. Thus, $e\in L_2$ and $\infty>\tilde{\theta}^T P^{-1}\tilde{\theta}\geq \lambda_{\min}(P^{-1})\tilde{\theta}^T\tilde{\theta}$ with $\lambda_{\min}(P^{-1})$ being the minimum eigenvalue of P^{-1} so that $\tilde{\theta}$ is uniformly bounded since the maximum eigenvalue of P, $\lambda_{\max}(P)$, is upper-bounded by a positive finite constant and then $\lambda_{\min}(P^{-1})=\lambda_{\max}^{-1}(P)>0$ for all $t>0$. Thus, P, θ is uniformly bounded and $||P||$, $||\theta||$ and $||\bar{\delta}||$ are in L_∞ from (9) since $\theta=(\theta_0^T,\varepsilon_0^T)^T$ and θ_0 and $\text{Det}(S(\theta_0))$ are uniformly bounded for all $t\geq 0$. Thus, the modified parameter vector $\bar{\theta}=(\bar{\theta}_0^T,\varepsilon_0^T)^T$ is also uniformly bounded for all $t\geq 0$. The modified estimated plant model is controllable since $\infty>|\text{Det}(S(\bar{\theta}_0))|\geq\rho>0$ from (8)–(9) and the fact that $\bar{\theta}_0$ is uniformly bounded for all $t \geq 0$. On the other hand, $P\varphi \in L_2$ since Trace $(\dot{P}) = -\|P\varphi\|_2^2 \in L_1$ from (7.c) with $\|\cdot\|_2$ denoting the spectral (or Euclidean) vector norm. Thus, propositions (i)–(ii) have been proved.

(iii) It is standard to prove that P and θ converge asymptotically from (7.b) and the fact that $\lim_{t\to\infty}\left(\int_0^t \|\dot{\theta}\|d\tau\right) \leq (1/2)[\lim_{t\to\infty}(\|P\varphi\|^2 d\tau) + \lim_{t\to\infty}\left(\int_0^t e^2 d\tau\right)] < \infty$ since $P\varphi \in L_2$ and $e \in L_2$ what implies $\dot{\theta} \in L_1$ and the θ converges from (ii) (see [18]). Also, θ_0 converges since θ converges and, thus, $\mathrm{Det}(S(\theta_0))$ converge to a finite constant values as time tends to infinity. From the fact that θ_0 converges, the possible switches in (8.a)–(8.b) end in finite time since there exists a large finite time $\bar{t}_0$ such that θ and $\mathrm{Det}(S(\theta_0))$ are close to their limits and the piecewise-constant h-function maintains a constant value (ρ or $\rho' \geq 2\rho$) for all time $t \geq \bar{t}_0$. As a result α, $\sigma_{(\cdot)}$, $\bar{\sigma}_{(\cdot)}$ and $\bar{C}$ converge (see (8.a)–(8.b) and (9.d)). Thus, the modified parameter vector $\bar{\theta}$, and then $\mathrm{Det}(S(\bar{\theta}_0))$, converge asymptotically to finite limits. As a result, each controller parameter, namely, each coefficient of $R(D,t)$ and $S(D,t)$, converges to a finite limit value and (iii) has been proved.

(iv) Note that direct calculation from (12) yields for $m \leq n-1$:

$$D^n y_f = e + (\bar{\theta}^T - \bar{\delta}^T)\varphi = e + \sum_{i=0}^{m} \bar{b}_i D^{m-i} u_f - \sum_{i=1}^{n} \bar{a}_i D^{n-i} y_f - \bar{\delta}_0^T \varphi_0$$

and the substitution $D^n u_f$ made explicit from (13) into (12) yields for $m = n$:

$$\begin{aligned} D^n y_f = e - \bar{b}_0 &\left[\sum_{i=1}^{n} s_i D^{n-i} u_f + \sum_{i=0}^{n-1} r_i D^{n-i-1} y_f\right] \\ &+ \left[\sum_{i=1}^{n} \bar{b}_i D^{n-i} u_f - \sum_{i=0}^{n-1} \bar{a}_{i+1} D^{n-i-1} y_f\right] - \bar{\delta}_0^T \varphi_0 \end{aligned}$$

Thus, the substitution of the above identities together with (13) yield the following extended auxiliary dynamic system which describes the combination of the closed-loop dynamics and control law:

$$\dot{x} = Ax + w \tag{A.5.a}$$

$$\dot{z} = Az + w_1 \tag{A.5.b}$$

with

$$w = [e + \varepsilon_0^T i_\varphi - \bar{\delta}_0^T \varphi_0, 0]^T = \bar{w} + w_1; \qquad \text{(A.6.a)}$$
$$\bar{w} = [-\bar{\delta}_0^T \varphi_0, 0]^T; \ w_1 = [e + \varepsilon_0^T i_\varphi, 0]^T$$

$$A(t) = \begin{bmatrix} & \bar{p}^T & \\ I_{n-1} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 3 & & \\ & \bar{v}^T & \\ 0 & \vdots & I_{n-1} \end{bmatrix}; \ \bar{p} = \begin{cases} \bar{p}^{(1)} & \text{if } m \le n-1 \\ \bar{p}^{(2)} & \text{if } m = n \end{cases} \qquad \text{(A.6.b)}$$

$$\bar{p}^{(1)^T} = [-\bar{a}_1, -\bar{a}_2, \ldots, -\bar{a}_n \vdots 0, \overbrace{\ldots}^{n-m-1}, 0 \vdots \bar{b}_0, \bar{b}_1, \ldots, \bar{b}_m] \qquad \text{(A.6.c)}$$

$$\bar{p}^{(2)^T} = [-(\bar{a}_1 + \bar{b}_0 r_0), -(\bar{a}_2 + \bar{b}_0 r_1), \ldots, -(\bar{a}_n + \bar{b}_0 r_{n-1}) \vdots (\bar{b}_1 - \bar{b}_0 s_1), (\bar{b}_2 - \bar{b}_0 s_2), \ldots, (\bar{b}_n - \bar{b}_0 s_n)] \qquad \text{(A.6.d)}$$

$$\bar{v}^T = [r_0, r_1, \ldots, r_n \vdots s_1, s_2, \ldots, s_n] \qquad \text{(A.6.e)}$$

with $x(0) = z(0) = x_0$, $x = (D^{n-1} y_f, \ldots, Dy_f, y_f, D^{n-1} u_f, \ldots, Du_f, u_f)^T$ and $\varphi_0 = (D^{n-1} y_f, \ldots, Dy_f, y_f, D^n u_f, D^{n-1} u_f, \ldots, Du_f, u_f)^T$. The proof of boundedness and convergence to zero of the input, output, their filtered versions and the time- derivatives of those ones up till $(n-1)$-th order of the closed-loop system is immediate by first proving that (A.6.b) is asymptotically stable in the large. Thus, by vector construction, $|D^n u_f| \le K' \|x\|$ from the controller equation (11) and, then, $\|\varphi_0\| \le \mathrm{Max}(|D^n u_f|, \|x\|) \le K \|x\|$ with $K = 1 + K'$. The eigenvalues of $A(t)$ are less than or equal to $(-\sigma)$ for some real constant $\sigma > 0$ being less than or equal to the minimum absolute value of the roots of the strictly Hurwitz $C^*(D)$-polynomial for all $t \ge 0$ (equality applies when both roots are distinct, [17, 18]). Also, $A(t)$ is uniformly bounded and $\int_t^{t+T_0} \|\dot{A}(\tau)\| d\tau \le \mu T_0 + \mu_0$ for positive constants μ and μ_0, with μ small, all $t \ge 0$ and some finite T_0. This follows directly in the absence of modification on the integration interval since the time-derivative of the estimates and controller parameters are bounded as follows from Theorem 1. Assume that there are $\infty > s_t \geq 0$ modification switches on $[t, t+T_0]$. Their number is finite since the integration interval is finite and $|\mathrm{Det}(S(\theta_0))|$ is a continuous function of time so that existing switches are isolated (*i.e.*, there is no accumulation point of modification switches). Also, their associate discontinuities in $A(t)$ are given by bounded steps, whose norms are upper-bounded by a positive finite constant $\bar{k}$ from Theorem 1 (i), since $\bar{\theta}_0 \in L_\infty$. As a result, the above inequality for the local integral of $\|\dot{A}(\tau)\|$ also holds if there are modification switches on $(t, t+T_0)$ for all $t \geq 0$.

As a result, the common unforced version of both time-varying systems (A.5) is exponentially stable in the large ([12, 15]). Now, direct calculus with the differential systems (A.5.a) and (A.5.b) yields that their solutions are related as follows:

$$x(t) = z(t) + \int_0^t \Psi(t,\tau)\bar{w}(\tau)d\tau \tag{A.7}$$

with $\Psi(t,\tau)$ being the fundamental matrix of the unforced system of both (A.5.a) and (A.5.b), *i.e.*, $x(t) = z(t) = \Psi(t,0)x_0$ for all $t \geq 0$ if $w \equiv w_1 \equiv 0$. Since such a system is exponentially stable in the large, one has for any matrix norm that $\|\Psi(t,\tau)\| \leq K_\Psi e^{-\sigma(t-\tau)}$ for any t and τ fulfilling $t \geq \tau \geq 0$. Since $A(t)$ is exponentially stable and, furthermore, $w_1 \in L_\infty \cap L_2$ from (i)–(ii), $z \in L_\infty \cap L_2$, $\dot{z} \in L_\infty \cap L_2$ and z converges exponentially to zero for any bounded initial condition (see [18]). Thus, one gets directly from the definition of $\bar{w}$ in (A.5.a):

$$\|x(t)\| \leq \|z(t)\| + \int_0^t e^{-\sigma(t-\tau)} K_\Psi \|\bar{\delta}_0\| \|x(\tau)\| d\tau \tag{A.8}$$

Now, $z(t)$ converges to zero exponentially with rate non less than $-|\sigma|$ (*i.e.*, $\|z(t)\| e^{\sigma t} < \infty$ for all $t \geq 0$) since $z \in L_\infty \cap L_2$ and $w_1 \in L_\infty \cap L_2$. This property follows directly, for instance, by applying Bellman-Gronwall's Lemma [17], to the solution of the forced system (A.5.b). Thus, one gets from (A.8) that

$$\begin{aligned} e^{\sigma t}\|x(t)\| &\leq e^{\sigma t} z(t) + \int_0^{t+\tau} K_\Psi \|\bar{\delta}_0\| e^{\sigma\tau} \|x(\tau)\| d\tau \\ &\leq \bar{K} + \int_0^t K_\Psi \|\bar{\delta}_0\| e^{\sigma\tau} \|x(\tau)\| d\tau \end{aligned} \tag{A.9}$$

where $\infty > \bar{K} \geq \mathrm{Sup}_{t \geq 0}(\|z(t)\|e^{\sigma t})$ so that $\|x(t)\| \leq \bar{K}e^{(-|\sigma|+\bar{\delta}_0')t} < \infty$ for all $t \geq 0$ where $\bar{\delta}_0' = K_\Psi \mathrm{Sup}_{0 \leq t < \infty}(\|\bar{\delta}_0(t)\|)$ after applying Bellman-Gronwall's Lemma to (A.9). Thus, $\|x(t)\|$ and $\|\dot{x}(t)\|$ are bounded from (A.8), and the boundedness of both the estimation error and $\bar{\delta}_0'$ and (A.5)–(A.6). Equation (A.9) implies that $x(t) \to 0$ as $t \to \infty$ since $\bar{\delta}_0'$ is bounded from (i).

The proof is completed as follows. From (8.a)–(8.c) and (9), and since the stability property is independent of the used norm, one has for $\|\cdot\|_z$ (*i.e.*, spectral) – matrix norms: $\|\bar{\delta}_0\|_2 \leq |\delta_\alpha \bar{C}|\|\Sigma\|_2 \leq \bar{\delta}_0' \leq 3$ $\mathrm{Max}(\rho, \rho')(n+m) = 6\rho(n+m)$, where $\Sigma = (\Sigma_{ij})$ is the matrix of signs used for modification in (9.c), *i.e.*, Σ_{ij} is one of the elements in the set $\{\bar{\sigma}_i, i = 1, 2, \ldots, n+m+1\}$ with $\bar{\sigma}_i \in \{-1, 1, 0\}(i = 1, 2, \ldots, n+m+1)$, and $S(\bar{\theta}_0) - S(\theta_0) = (\delta_\alpha \bar{C})\Sigma$. As a result, $x \in L_\infty$, $\dot{x} \in L_\infty$ and $x \to 0$ and $\dot{x} \to 0$ as $t \to \infty$. Thus, (iv) follows directly from the calculation of $x : [0, \infty) \to \mathbf{R}^{2n}$ from (A.5.a) for any initial conditions. ■