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On Exponential Stabilizability of Linear Neutral Systems

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In this paper, we deal with linear neutral functional differential systems. Using an extended state space and an extended control operator, we transform the initial neutral system in an infinite dimensional linear system. We give a sufficient condition for admissibility of the control operator B, conditions under which operator B can be acceptable in order to work with controllability and stabilizability. Necessary and sufficient conditions for exact controllability are provided; in terms of a gramian of controllability $N(\mu)$. Assuming admissibility and exact controllability, a feedback control law is defined from the inverse of the operator $N(\mu)$ in order to stabilize exponentially the closed loop system. In this case, the semigroup generated by the closed loop system has an arbitrary decay rate.

Keywords: Linear neutral systems; Boundary control systems; Extended state space; Admissibility; Exact controllability; Exponential stabilizability

AMS Subject Classifications: 93C30, 93C25, 93D15

1. INTRODUCTION

The problem of feedback stabilization of linear infinite dimensional systems is an important domain of investigation since eighties. Several authors consider the case of linear systems with delays. The monograph by Curtain and Zwart [1] contains main references known at that time.

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The case of linear systems of neutral type is of special difficulty because of the possible presence of a chain of an infinite number of unstable modes [9]. This situation implies to use derivative in the feedback control law. In [9] the exact controllabilty is used in order to stabilize the neutral type systems. The main idea is to shift the unstable roots by using state feedback which contain the delayed derivative.

In [2] we used the controllability gramian in order to stabilize linear systems in Hilbert spaces with arbitrary exponential decay rate.

Our purpose is to use the same framework based on semigroup theory for delay systems of neutral type. As exact controllability is possible for this kind of systems, this notion is here described *via* the controllability gramian (in an extended form) and used to define the feedback insuring exponential stabilizability with arbitrary decay rate. This approach make clear the stabilizability of neutral type systems under the condition of exact controllability which was used in [9] as a sufficient condition.

This paper deals with linear neutral functional differential equations that can be written:

$$\begin{cases} (d/dt)\left(x(t)-\sum_{k=1}^{K}A_{-k}x(t-h_k)\right)=\sum_{k=0}^{K}A_k x(t-h_k)+\int_{-h}^{0}E(\theta)x(t+\theta)d\theta+Bu(t) \\ x(t)=\phi(t), \quad t\in[-h,0]. \end{cases} \tag{1}$$

First, let us mention simple notations used later in the paper.

1.1. Notations

$\mathbb{E}^n$ denotes the space of the complex column n-vectors with norm $|x| = (\bar{x}^t x)^{1/2}$ and inner product $\langle x, y\rangle = \bar{x}^t y$ where x^t denotes transpose and $\bar{x}$ denotes complex conjugate. $W_2^{(k)} = W_2^{(k)}([-h,0];\mathbb{E}^n)$ is the Sobolev space of $\mathbb{E}^n$-valued absolutely continuous functions on $[-h,0]$ with square integrable kth derivatives on $[-h,0]$. Norm associated with the Sobolev space $W_2^{(1)}$ is defined by (see [4]):

$$\|\psi\|_{W_2^{(1)}} = \left[|\psi(0)|^2 + \int_{-h}^{0}|\dot{\psi}(\theta)|^2 d\theta\right]^{1/2}.$$

$\mathcal{L}^2_{loc}([0,+\infty);\mathbb{E}^n)$ is the space of $\mathbb{E}^n$-valued functions on $[0,+\infty)$ whose restrictions to finite intervals are square integrable. The space

$M_2([-h,0];\mathbb{E}^n)$ is the product space $\mathbb{E}^n \times \mathcal{L}_2([-h,0];\mathbb{E}^n)$. If Y and Z are normed linear spaces, we define the normed linear space $\mathcal{L}(Y,Z)$ to be the space of bounded linear operators from Y to Z. The class of Lebesgue Z-valued functions with $\int_{\bar{\Omega}} |f(t)|dt < +\infty$ is denoted by $\mathcal{L}_1(\bar{\Omega}, Z)$ where $\bar{\Omega}$ is a closed subset of $\mathbb{R}$.

In the system (1), K stands for a positive integer, h is a fixed finite delay, $0 < h < +\infty, 0 < h_0 < \cdots < h_K = h$. A_k and A_{-k} are real $n \times n$ matrices, B is a real $n \times m$ matrix and E is a real $n \times n$ matrix-valued square integrable function. The initial data ϕ is an element in $W_2^{(1)}$ and the control term u is an element in $\mathcal{L}^2_{loc}([0,+\infty);\mathbb{E}^m)$; in the paper, this space is denoted by Γ.

In the second section, necessary mathematical results are recalled. From a Banach space X, two other Banach spaces X_1 and X_{-1}, complete with respect to adequate norms, are built up. At the end of this section, we recall results needed later for admissibility.

In the third section, we transform the initial neutral system and explain various solutions in order to define an equivalent abstract differential system. For that, we especially used results of Henry [4], O'Connor and Tarn [8] and Weiss [18] and [19].

In the next section, we limit the definition domain of the control operator B in order to obtain an admissible control operator. The definition of a gramian of controllability allows us to give two theorems, one for admissibility and the other for exact controllability. The last section before conclusion is devoted to the stabilization of the neutral system when it is exactly controllable and thus admissible.

2. MATHEMATICAL PRELIMINARIES

In this section, we recall some mathematical results used in the sequel of the paper. The first part is devoted to closedness and extended space. After that, we only recall fundamental results needed for criteria of admissibility.

The first mathematical facts can be find in [16, 18] or [20]. These results are applied to the specifical case used later.

Definition 2.1 [18] Let X be a Banach space and $S(t)$, $t \geq 0$ a strongly continuous semigroup on X with generator A: $D(A) \rightarrow X$.

Let $\beta \in \rho(A)$, the resolvent set of A. We define the space X_1 to be $D(A)$ with the norm

$$\|x\|_1 = \|(\beta I - A)x\|_X$$

and the space X_{-1} to be the completion of X with respect to the norm

$$\|x\|_{-1} = \|(\beta I - A)^{-1}x\|_X.$$

From this definition, we can classify the three spaces X, X_1 and X_{-1}:

$$X_1 \subset X \subset X_{-1}.$$

If X is a reflexive space then X_{-1} can be defined equivalently as the dual of $(X^*)_1$, where $(X^*)_1 = D(A^*)$ with the graph norm. The definition of a closed operator (see [1]) and the following remark about norm $\|\cdot\|_1$ serve to prove the completion of space X_1.

Remark 2.2 [18] It is easy to verify that for different values of β, we get equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_{-1}$. In particular, $\|\cdot\|_1$ is equivalent to the graph norm on $D(A)$, so X_1 is complete.

Dual space of the space X_{-1} is explained especially by Salamon in [16]. The space $X^*{}_{-1}$ is equal to $D(A^*)$ and it becomes an Hilbert space with the graph norm $\|\cdot\|_{A^*}$.

PROPOSITION 2.3 [18] *With the notation of Definition* 2.1, *let* $\mu \in \rho(A)$. *Then the operator*

$$R_\mu = (\mu I - A)^{-1}$$

has a unique continuous extension to an operator in $\mathcal{L}(X_{-1})$, *which we denote by the same symbol.* R_μ *is an isomorphism from* X_{-1} *to* X *and from* X *to* X_1.

If $L \in \mathcal{L}(X)$ *commutes with* A, *i.e., if*

$$LAx = ALx, \quad \forall x \in D(A),$$

then the restriction of L *to* X_1 *belongs to* $\mathcal{L}(X_1)$ *and is the image of* L *via any of the isomorphisms* R_μ. *Further,* L *has a unique continuous extension to an operator in* $\mathcal{L}(X_{-1})$, *which is the image of* L *via any of the isomorphisms* R_μ^{-1}.

Taking $L=S(t)$, $t\geq 0$ in the above proposition, we deduce that $S(t)$, $t\geq 0$ has an extension to a semigroup on X_{-1} whose generator is an extension of A with domain X that is to say $A\in\mathcal{L}(X,X_{-1})$.

In [15], Salamon explains widely existence and relation between spaces X_1 and X defined before. A majority of results about semigroups and infinitesimal generators are also studied.

Results of Weiss stated above are for a Banach space. Similar results can be proved for a Hilbert space (see for example [14] or [16]). In the following, results formulated for Banach spaces are used in the paper for Hilbert spaces with the same associated norms.

The following important theorem states the particular conditions to extract an operator from an integral.

THEOREM 2.4 [1] *Let Z_1 and Z_2 be separable Hilbert spaces, let A be a closed linear operator from $D(A)\subset Z_1$ to Z_2 and let Ω be a closed subset of $\mathbb{R}$. If $f\in\mathcal{L}_1(\Omega;Z_1)$ with $f\in D(A)$ almost everywhere and $Af\in\mathcal{L}_1(\Omega;Z_2)$, then*

$$A\int_E f(t)dt=\int_E Af(t)dt$$

for all measurable subset $E\subset\Omega$.

Proof See Hille and Phillips [6], Theorem 3.7.12 ■

3. DETERMINATION OF THE SOLUTION

The initial system (1) is now transformed in a boundary control system then in a control system with an extended control operator. Successive transformations allow us to define for the neutral functional differential equation (1) a solution, a semigroup, and its infinitesimal generator.

We can rewrite the system (1) in terms of Stieltjes integrals (see [8])

$$\begin{cases} (d/dt)(x(t)-\int_{-h}^{0}d\mu(\theta)x(t+\theta))=\int_{-h}^{0}d\eta(\theta)x(t+\theta)+Bu(t) \\ x(t)=\phi(t), \quad t\in[-h,0] \end{cases} \qquad (2)$$

where μ and η are $n \times n$ matrix functions of bounded variation, continuous from the left, defined by

$$\mu(\theta) = -\sum_{k=1}^{K} A_{-k}\chi_k(\theta), \quad \eta(\theta) = -\sum_{k=0}^{K} A_k\chi_k(\theta) + \int_{-h}^{0} E(\xi)d\xi$$

where χ_k is the characteristic function on the interval $]-\infty, -h_k]$.

This formulation with Stieltjes integrals allows to describe more general systems as those depicted by Eq. (1).

The solution of such systems can be written with the variation of constant formula on the interval $[0, h]$. With this computation, we easily find for all $t \in [0, h]$

$$\begin{aligned} x(t) &= \exp(A_0 t)x_0 \\ &\quad + \int_0^t \exp(A_0(t-\tau))\Bigg[\sum_{k=1}^{K} A_{-k}\dot{\phi}(\tau - h_k) + \sum_{k=1}^{K} A_k\phi(\tau - h_k) \\ &\qquad + \int_{-h}^{0} E(\theta)\phi(\tau+\theta)d\theta + Bu(\tau)\Bigg]d\tau. \end{aligned}$$

The step-by-step method gives the solution for all $t \geq 0$. We refer to [12] for neutral delays systems or [17] for delay differential equation without neutral terms. By the linearity of (1), Henry shows in [4] that there exist bounded linear operators $S(t)$, $t \geq 0$, and $K(t,s)$, $t \geq s \geq 0$, such that

$$x(t) = S(t-s)\phi(s) + K(t,s)u(t), \quad 0 \leq s \leq t \leq h.$$

With $u \equiv 0$ in the last equation, we easily verify that operator $S(t)$, $t \geq 0$ is a strongly continuous semigroup. Its infinitesimal generator A is given by (see [8])

$$D(A) = \left\{\psi \in W_2^{(1)} | \dot{\psi} \in W_2^{(1)}, \dot{\psi}(0) = \int_{-h}^{0} d\mu(\theta)\dot{\psi}(\theta) + \int_{-h}^{0} d\eta(\theta)\psi(\theta)\right\}$$

and

$$A\psi = \dot{\psi}, \quad \forall \psi \in D(A).$$

We can also define the spectrum of A, $\sigma(A)$, which coincides with the roots of the characteristic equation, $\det(\Delta(\lambda))=0$, where the characteristic matrix is defined by

$$\Delta(\lambda) = \lambda\left(I - \int_{-h}^{0} d\mu(\theta)\exp(\lambda\theta)\right) - \int_{-h}^{0} d\eta(\theta)\exp(\lambda\theta). \tag{3}$$

The system (2) cannot be directly transformed in a "classical" distributed system on the space $W_2^{(1)}$ because of the neutral terms. If we use the space $M_2([-h,0];\mathbb{E}^n)$, we can find a transformation from the neutral system to the distributed system with an extended state like in works of Yamamoto and Ueshima (see [21]). Since we chose to stay in the space $W_2^{(1)}$, we transform, as O'Connor and Tarn in [8], system (2) in the following abstract boundary control system:

$$\begin{cases} (d/dt)z(t) = \Lambda z(t), & t > 0, \\ \Xi z(t) = Bu(t), & t \geq 0, \\ z(0) = \phi, & \end{cases} \tag{4}$$

where $\Lambda \in \mathcal{L}(W_2^{(1)}, W_2^{(1)})$ is a closed linear operator defined by

$$\Lambda\psi = \frac{d\psi}{d\theta}, \quad \forall \psi \in D(\Lambda) = W_2^{(2)}$$

$\Xi \in \mathcal{L}(W_2^{(1)}, \mathbb{E}^n)$ is a linear operator expressed by

$$\Xi\psi = \dot{\psi}(0) - \int_{-h}^{0} d\mu(\theta)\dot{\psi}(\theta) - \int_{-h}^{0} d\eta(\theta)\psi(\theta), \quad \forall \psi \in D(\Xi) = W_2^{(2)}.$$

The system (4) allows us to define a strongly continuous semigroup, which is also $S(t)$, $t \geq 0$. In this case, the infinitesimal generator is defined by

$$\Lambda z = Az, \quad \forall z \in D(\Lambda) \cap \ker(\Xi).$$

Now, we explain the relation between systems (2) and (4). Let z be a restriction of x from $W_2^{(1)}$ to $W_2^{(2)}$ such that

$$z(t) = x(t+\theta), \quad \theta \in [-h,0], \quad t \geq 0.$$

From the first equation of (4), we have

$$\frac{dx(t+\theta)}{dt} = \Lambda x(t+\theta) = \frac{dx(t+\theta)}{d\theta}.$$

This equality is trivial for $t \in [0, h]$ but not for all positive t. The general case has been proved by O'Connor and Tarn in [8].

From the second equation of (4), we obtain

$$\Xi x(t+\theta) = Bu(t),$$

which gives

$$\left.\frac{dx(t+\theta)}{d\theta}\right|_{\theta=0} - \int_{-h}^{0} d\mu(\theta)\frac{dx(t+\theta)}{d\theta} - \int_{-h}^{0} d\eta(\theta)x(t+\theta) = Bu(t)$$

$$\left.\frac{dx(t+\theta)}{dt}\right|_{\theta=0} - \int_{-h}^{0} d\mu(\theta)\frac{dx(t+\theta)}{dt} - \int_{-h}^{0} d\eta(\theta)x(t+\theta) = Bu(t)$$

$$\frac{d}{dt}\left(x(t) - \int_{-h}^{0} d\mu(\theta)x(t+\theta)\right) = \int_{-h}^{0} d\eta(\theta)x(t+\theta) + Bu(t).$$

From now, we can easily cross from system (2) to system (4) and conversely.

In order to explain the solution of the system in terms of the semigroup $S(t)$, $t \geq 0$, we make the following definition.

DEFINITION 3.1 [8] The auxiliary boundary operator B_λ associated with system (4) is a bounded linear mapping from $\mathbb{E}^m$ into $W_2^{(1)}$ defined by

$$(B_\lambda u)(\theta) = \exp(\lambda\theta)\Delta(\lambda)^{-1}Bu, \quad \theta \in [-h, 0]$$

where λ is in the resolvent of A, $\rho(A) = \mathbb{C}\backslash\sigma(A)$ and $\Delta(\lambda)$ is the characteristic matrix defined by (3).

The verification of the boundedness of the operator B_λ is trivial. It is also easy to verify that the new operator B_λ lies in the space $W_2^{(2)}$ and that for all $\nu \in \mathbb{E}^m$

$$\Xi(B_\lambda \nu) = B\nu.$$

Assuming that the solution z can be expressed $z(t)=v(t)+B_\lambda u(t)$, $t\geq 0$, with $v\in W_2^{(2)}$, O'Connor and Tarn, in [8], rewrite equations of the abstract boundary control system to prove the following theorem:

THEOREM 3.2 [8] *For all $u\in\mathcal{L}^2_{loc}([0,+\infty);\mathbb{E}^m)$ and $\phi\in W_2^{(1)}$, the state is given by the following variation of constant formula*:

$$x(t)=S(t)\phi+(\lambda I-A)\int_0^t S(t-s)B_\lambda u(s)ds,\quad t\geq 0. \tag{5}$$

We cannot differentiate (5) to obtain an abstract differential equation for the state x since an unbounded operator $(\lambda I-A)$ operates on the integral term. However, we can associate the state x with the system in $W_2^{(1)}$

$$\begin{cases}(d/dt)y(t)=Ay(t)+B_\lambda u(t), & t>0\\ y(0)=(\lambda I-A)^{-1}\phi\\ x(t)=(\lambda I-A)y(t),\end{cases}$$

where the last relation is the output equation. Then, the solution of this new system can be written

$$y(t)=S(t)(\lambda I-A)^{-1}\phi+\int_0^t S(t-s)B_\lambda u(s)ds,\quad t\geq 0$$

As the operator B_λ is compact (see [1]), the operator

$$u\mapsto\int_0^\tau S(\tau-s)B_\lambda u(s)ds$$

is also compact. In this case, we cannot reach the infinite dimensional space $W_2^{(1)}$ through this operator. So, we adopt another way to use Eq. (5).

Using results described in section (2) and taking $X=W_2^{(1)}$, we can easily define spaces X_1 and X_{-1}. In this new space X_{-1}, we can define the system associated with the solution described by (5). We have:

$$\begin{cases}(d/dt)x(t)=Ax(t)+(\lambda I-A)B_\lambda u(t), & t>0\\ x(0)=\phi.\end{cases} \tag{6}$$

As the operator B_λ is bounded in X, the operator $(\lambda I-A)B_\lambda$ is bounded in X_{-1}. With this formulation, the state is in the space

X when computations are in the extended space X_{-1}. The following definition of Weiss allows us to define the system like above in the space X_{-1} although the solution is in the space $W_2^{(1)}$.

Definition 3.3 [18] Let X be a Banach space, let $S(t)$, $t \geq 0$ be a semigroup on X with generator A, and let

$$f \in \mathcal{L}^1_{loc}([0,+\infty), X_{-1}).$$

Then we say that the function

$$x \in \mathcal{L}^1_{loc}([0,+\infty), X)$$

is a strong solution of the differential equation

$$\dot{x}(t) = Ax(t) + f(t)$$

if for any $t \geq 0$

$$x(t) - x(0) = \int_0^t [Ax(s) + f(s)]ds. \tag{7}$$

If additionally, x is continuous in X, *i.e.*, if

$$x \in \mathcal{C}([0,+\infty), X)$$

then we say that x is a continuous state strong solution of the differential equation above.

The assumption (7) is necessary to obtain from system described by (6) the solution (5) of the neutral system (1). This definition of Weiss follows the one of Pazy ([11], page 109) if spaces X_{-1} and X are respectively replaced by X and X_1.

4. ADMISSIBILITY

Now, we must verify in which case the control operator $(\lambda I - A)B_\lambda$ is admissible or not. The problem is: for which operators B_λ admissibility of $(\lambda I - A)B_\lambda$ is verified. The following definition of

Weiss explicits in which cases operator $\tilde{B}$ and semigroup $S(t)$, $t \geq 0$ generate an abstract linear control system.

DEFINITION 4.1 [18] Let U and X be Banach spaces, let $p \in [1, +\infty)$ and let Ω be $\mathcal{L}^p([0, +\infty); U)$. Let S be a strongly continuous semigroup on X and let $\tilde{B} \in \mathcal{L}(U, X_{-1})$. For any $\tau \geq 0$, we define the operator Ψ_τ: $\Omega \to X_{-1}$ by

$$\Psi_\tau u = \int_0^\tau S(\tau - s)\tilde{B}u(s)ds$$

Then we say that $\tilde{B}$ is admissible for the semigroup S if for any $\tau \geq 0$, $\Psi_\tau \in \mathcal{L}(\Omega, X)$.

In Eq. (5), the integral term is as

$$(\lambda I - A) \int_0^\tau S(\tau - s)B_\lambda u(s)ds \tag{8}$$

and not as in the above definition

$$\int_0^\tau S(\tau - s)(\lambda I - A)B_\lambda u(s)ds. \tag{9}$$

Before applying the definition of Weiss, we must show that the two operators (8) and (9) are equivalent. We must show that the operator $(\lambda I - A)$ can be applied inside or outside of the integral and we must show that the range of the operator B_λ is in the definition space of the operator A.

All conditions of Theorem 2.4 are verified: separable Hilbert spaces, closed operator, integrable operators. Therefore, the first part of the problem is solved. As operator B_λ is defined from $\mathbb{E}^m$ into $X = \{D(A), \|\cdot\|_X\}$, semigroup $S(t)$, $t \geq 0$, and isomorphism $R_\mu^{-1} = (\lambda I - A)$ can switch.

Remark 4.2 In the proof of the Theorem 3.2, O'Connor and Tarn cross from one expression to the other, extracting the operator $(\lambda I - A)$ of the integral term.

From now, we have the same condition as in the Definition 4.1. So, we can define a criteria for admissibility of the control operator B_λ of the neutral system.

DEFINITION 4.3 Let Φ_τ be the operator defined from Γ to X_{-1} by

$$\Phi_\tau u = (\lambda I - A)\int_0^\tau S(\tau - s)B_\lambda u(s)ds.$$

Then, we say that the control operator B of the system (1) is admissible if for any $\tau \geq 0$, $\Phi_\tau \in \mathcal{L}(\Gamma, X)$.

Actually, if the operator B is admissible for the semigroup $S(t)$, $t \geq 0$, the operator Φ_τ is bounded in the space X. For more details about admissibility of unbounded control operator, see for example [16, 18] or [19].

5. EXACT CONTROLLABILITY

Now, we will consider that the operator B is admissible, that is to say that condition of Definition 4.3 is verified. In this section, we search conditions for exact controllability of the neutral system (1).

DEFINITION 5.1 [13] A system is said to be exactly controllable if there exists a positive time τ such that for all $x_0, x_1 \in X$ and for some control u, we have:

$$x(\tau) = x(\tau, x_0, u) = x_1.$$

So, from Eq. (5), we can state that the system is exactly controllable if there exist a positive time τ and some control u such that operator Φ_τ is onto that is to say

$$\mathrm{Ran}\{\Phi_\tau\} = X. \tag{10}$$

This last relation is not easy to use and to transform. The following one is better and we prove in the next theorem a necessary and sufficient condition for exact controllability. This theorem, as well as the proof, follows the one given by Curtain and Zwart (see Theorem 4.1.7 in [1]) where authors take a bounded control operator.

THEOREM 5.2 *The state space linear system* $\Sigma(A,(\lambda I-A)B_\lambda)$ *is exactly controllable on* $[0,\tau]$ *if and only if for some* $\gamma>0$ *and for all* $x\in X$:

$$\langle \Phi_\tau\Phi_\tau^* x, x\rangle \geq \gamma\|x\|_X^2.$$

As we saw before, operator Φ_τ is in $\mathcal{L}(\Gamma,X)$. With the boundedness given by admissibility, we are practically in the same situation as a system with a bounded control operator. The most important is not the boundedness of the control operator but that of the control gramian. If the control operator is bounded, it is relatively easy to show the boundedness of the gramian. In our case, admissibility ensures the boundedness of the control gramian. All these remarks allow us to say that our proof will be similar with the one used by Curtain and Zwart in [1].

Let $\Phi_\infty(\mu)$ be the operator defined for all scalar μ and for all $u\in\Gamma$ by

$$\Phi_\infty(\mu)u=(\lambda I-A)\int_0^\infty \exp(-\mu s/2)S(-s)B_\lambda u(s)ds.$$

PROPOSITION 5.3 *If the control operator* B_λ *is admissible, the operator* $\Phi_\infty(\mu)$ *is in* $\mathcal{L}(\Gamma,X)$.

Proof

$$\begin{aligned}\|\Phi_\infty(\mu)u\|_X &= \left\|(\lambda I-A)\int_0^\infty \exp(-\mu s/2)S(-s)B_\lambda u(s)ds\right\|_X\\ &\leq \left\|(\lambda I-A)\int_0^\tau \exp(-\mu s/2)S(-s)B_\lambda u(s)ds\right\|_X\\ &\quad+\left\|(\lambda I-A)\int_\tau^\infty \exp(-\mu s/2)S(-s)B_\lambda u(s)ds\right\|_X\\ &\leq \left\|(\lambda I-A)\int_0^\tau S(-s)B_\lambda u(s)ds\right\|_X\\ &\quad+\left\|(\lambda I-A)\int_\tau^\infty \exp(-\mu s/2)S(-s)B_\lambda u(s)ds\right\|_X.\end{aligned}$$

For all $t\geq 0$, for all $x\in X$, there exist scalars M_ω and ω such that the semigroup $S(t)$ verifies

$$\|S(t)x\|_X\leq M_\omega\exp(\omega t).$$

Using this inequality, we obtain

$$\begin{aligned}\|\Phi_\infty(\mu)u\| &\le M_\omega \exp(-\omega\tau)\left\|(\lambda I - A)\int_0^\tau S(\tau - s)B_\lambda u(s)ds\right\|_X \\ &\quad + \left\|(\lambda I - A)\int_\tau^\infty \exp(-\mu s/2)S(-s)B_\lambda u(s)ds\right\|_X \\ &\le M_\omega \exp(-\omega\tau)\|\Phi_\tau u\|_X \\ &\quad + \left\|(\lambda I - A)\int_\tau^\infty \exp(-\mu s/2)S(-s)B_\lambda u(s)ds\right\|_X \\ &\le M_\omega \exp(-\omega\tau)\|\Phi_\tau u\|_X \\ &\quad + \int_\tau^\infty \|(\lambda I - A)\exp(-\mu s/2)S(-s)B_\lambda u(s)\|_X ds.\end{aligned}$$

For all function f integrable on $[T, +\infty)$,

$$\lim_{T\to+\infty}\int_T^\infty f(s)ds = 0.$$

So, for all constant $c > 0$, there exists a $T_c > 0$ such that

$$\int_{T_c}^\infty f(s)ds \le c.$$

In our case, for all $c > 0$, there exists a $\tau_c > 0$ such that

$$\int_{\tau_c}^\infty \|(\lambda I - A)\exp(-\mu s/2)S(-s)B_\lambda u(s)\|_X ds \le c\|u\|_\Gamma.$$

As the operator B_λ is admissible, there exists a positive scalar δ such that

$$\|\Phi_\tau u\|_X \le \delta\|u\|_\Gamma.$$

Finally, we have

$$\|\Phi_\infty(\mu)u\|_X \le (M_\omega\delta\exp(-\omega\tau_c) + c)\|u\|_\Gamma.$$

■

Let the extended gramian of controllability $N(\mu)$ be defined for all $x \in X$ by

$$N(\mu)x = \Phi_\infty(\mu)\Phi_\infty^*(\mu)x. \tag{11}$$

The next proposition follows from the Theorem 5.2.

PROPOSITION 5.4 *The system* (1) *is exactly controllable if and only if the operator* $N(\mu)$ *is an uniformly positive definite operator, that is*:

$$\langle N(\mu)x, x\rangle \geq \delta\|x\|_X, \quad \forall x \in X, \delta > 0. \tag{12}$$

Proof Let $N\tau(\mu)$ be the operator defined for all x in X and for all positive scalar τ by

$$N_\tau(\mu)x = \Phi_\tau(\mu)\Phi_\tau^*(\mu)x,$$

where

$$\Phi_\tau(\mu)u = (\lambda I - A)\int_0^\tau \exp(-\mu s/2)S(-s)B_\lambda u(s)ds.$$

If the controllability assumption ($\Phi_\tau\Phi_\tau^*$ is positive definite) then the operator $N_\tau(\mu)$ has the same property. The converse can also be checked.

Since

$$\langle N_\tau(\mu)x, x\rangle \leq \langle N(\mu)x, x\rangle,$$

we easily find that the operator $N(\mu)$ is uniformly positive definite if $N_\tau(\mu)$ is.

For the converse, we define the operator $R_\tau(\mu)$ for all u in Γ and for all positive scalar τ by

$$R_\tau(\mu)u = (\lambda I - A)\int_\tau^\infty \exp(-\mu s/2)S(-s)B_\lambda u(s)ds$$

We have

$$\langle N(\mu)x, x\rangle = \langle N_\tau(\mu)x, x\rangle + \langle R_\tau(\mu)R_\tau(\mu)^* x, x\rangle.$$

For the same reason as in the proof of the Proposition 5.3, for all constant c and for all x in X, there exists a $\theta_c > 0$ such that

$$\|R_{\theta_c}^*(\mu)x\|_\Gamma^2 \leq c\|x\|_X^2.$$

As the operator $N(\mu)$ is supposed to be uniformly positive definite, we obtain

$$(\delta - c)\|x\|_X^2 \leq \langle N_{\theta_c}(\mu)x, x\rangle.$$

If we choose $\theta_c > 0$ such that $\delta - c > 0$, we have $N_\tau(\mu)$ uniformly positive definite for all $\tau \geq \theta_c$. ∎

Remark 5.5 If the relation (12) is true then the operator $N(\mu)$ admits a bounded inverse $N(\mu)^{-1}$.

Exact controllability implies complete stabilizability. The converse does not always hold true. In particular conditions, complete stabilizability implies exact controllability. When the system is stabilizable, a feedback can be found in order to put the eigenvalues in the left half plane. For neutral system, another condition of stability is needed: the so called formal stability, defined below for a neutral system like (1) with only one lag ($K = 1$) and without distributed delay:

$$\begin{cases} (d/dt)(x(t) - A_{-1}x(t-h)) = A_0 x(t) + A_1 x(t-h) + Bu(t) \\ x(t) = \phi(t), \quad t \in [-h, 0] \end{cases} \tag{13}$$

Definition 5.6 [5] The system (13) is said formally stable if the operator D, defined by $D\Psi = \Psi(0) - A_{-1}\Psi(-h)$, is stable (in the sense of the formal stability), that is, if there exists $\delta > 0$ such that all solutions of the characteristic equation

$$\det\Delta_0(\lambda) = \det[I - A_{-1}\exp(-\lambda h)] = 0$$

satisfies $\mathrm{Re}(\lambda) \leq -\delta$. In other words, the system is formally stable iff $a_D < 0$, where $a_D = \sup\{Re(z)/det\Delta_0(z) = 0\}$.

The lack of formal stability means that, in order to stabilize the neutral type systems, we need to modify the neutral term. In [8], O'Connor and Tarn use another approach for exact controllability. They give necessary and sufficient algebraic conditions of exact controllability for a neutral system like (13). They obtain the following result.

Proposition 5.7 [8] *System* (13) *is exactly controllable if and only if*

$$\begin{aligned} &Rank[B, A_{-1}B, \ldots, A_{-1}^{n-1}B] = n \\ &Rank[\Delta(\lambda), B] = n, \quad \text{for all } \lambda \in \sigma(A). \end{aligned} \tag{14}$$

The first item is equivalent to (see [9])

$$\mathrm{Rank}[zI - A_{-1}, B] = n, \qquad \text{for all } z.$$

which involve that for all eigenvalues $\lambda_1, \dots, \lambda_n$, we can find an $m \times n$ matrix F_0 such that the $n \times n$ matrix $A_{-1} + BF$ verifies

$$\sigma(A_{-1} + BF) = \{\lambda_1, \dots, \lambda_n\}.$$

The condition (14) proves that we can find a feedback F_0 such that the closed loop system is formally stable by changing the neutral term (see for example [5] or [9]). This means that exact controllability is a sufficiently strong condition to insure stabilizability.

6. STABILIZABILITY

This section deals with exponential stabilization. Exact controllability proves complete stabilizability. So, we search a feedback control law which can stabilize the neutral system (1) *via* the abstract differential system (6).

THEOREM 6.1 *Let the system* (1) *be exactly controllable and let the operator F be defined by*:

$$F = -B_\lambda^*(\lambda I - A)^* N(\mu)^{-1}.$$

Then, the closed-loop system with $u = Fx$ is exponentially stable. Moreover, by means of the choice of μ, the decay rate may be arbitrary:

$$\forall \omega \in \mathbb{R}, \quad \exists \mu, \ \|S_F(t)\| \le M_\omega exp(\omega t). \tag{15}$$

Proof A first step of the proof is to show that the operator F is bounded in appropriate space. Indeed, from a bounded operator, we are sure to be able to generate a semigroup.

We have seen that the operator $(\lambda I - A)B_\lambda$ is bounded in X_{-1}. So, there exists a constant $\tilde{c}$ such that:

$$\|B_\lambda^*(\lambda I - A)^* x\|_\Gamma \le \tilde{c}\|x\|_{X_{-1}^*}.$$

The gramian of controllability $N(\mu)$ is positive definite. In this case, there exists a constant c_1 such that:

$$\|N(\mu)^{-1}x\|_{X_{-1}^*} \le c_1\|x\|_{X_{-1}}.$$

The two above inequalities allow us to conclude about the boundedness of the operator F. In this case, a semigroup $S_F(t)$, $t \geq 0$ with $A+BF$ as infinitesimal generator can be generate in the space X_{-1}.

The relation

$$X \subset X_{-1} = \{D(A), \|\cdot\|_{X_{-1}}\} \tag{16}$$

between spaces X and X_{-1} allows us to say that the semigroup $S_F(t)$, $t \geq 0$ also exists in the space X. From now, we must show that this semigroup is exponentially stable.

The closed loop system with $u = Fx$ can be written:

$$\begin{cases} (d/dt)x(t) = [A - (\lambda I - A)B_\lambda B_\lambda^*(\lambda I - A)^* N(\mu)^{-1}]x(t), & t > 0 \\ x(0) = \phi. \end{cases}$$

For a bounded control operator, Korobov and Sklyar in [7] show that the extended gramian of controllability verifies an Algebraic Riccati Equation. The control operator $(\lambda I - A)B_\lambda$ is not bounded in the space X but is in the space X_{-1}; this equation would be explained in this extended space. After this remark, it is easy to prove that the operator $N(\mu)$ defined by (11) verifies the following Algebraic Riccati Equation

$$AN(\mu) + N(\mu)A^* + \mu N(\mu) = (\lambda I - A)B_\lambda B_\lambda^*(\lambda I - A)^*, \quad \text{in } X_{-1}^*.$$

After a multiplication from the right by the operator $N(\mu)^{-1}$, we obtain:

$$A + N(\mu)A^*N(\mu)^{-1} + \mu I = (\lambda I - A)B_\lambda B_\lambda^*(\lambda I - A)^* N(\mu)^{-1} \quad \text{in } D(A).$$

Then we can obtain

$$\begin{aligned} A - (\lambda I - A)&B_\lambda B_\lambda^*(\lambda I - A)^* N(\mu)^{-1} \\ &= N(\mu)[-A^* - \mu I]N(\mu)^{-1} \quad \text{in } D(A). \end{aligned} \tag{17}$$

From now, we are exactly in the same condition as in [2]. Using the same progression as in the last proof of [2], the semigroup $S_F(t)$, $t \geq 0$ generated by the closed loop system verifies:

$$\|S_F(t)\|_X \leq c_1 c_2 M_\alpha \exp((\alpha - \mu)t)$$

where c_1, c_2, α and M_α respectively verify

$$\|N(\mu)^{-1}\|_X \leq c_1, \quad \|N(\mu)\|_X \leq c_2, \quad \|S(t)\|_X \leq M_\alpha \exp(\alpha t).$$

Choosing μ such that $\omega \leq \alpha - \mu$ and putting $M_\omega = c_1 c_2 M_\alpha$ we get (15), which ends the proof. ■

Remark 6.2 Let us precise that a direct use of Theorem 6.1 is not easy. It is an extension of results by authors in [2] and [3]. Several authors show that for delay system one need use of distributed delays [9, 10] which are not also easy to compute. However, in our case, the fact that the used extended gramian is the solution of an Algebraic Riccati Equation gives the opportunity to make use of numerical methods of approximation of this solution. This problem is under investigation.

7. CONCLUSION

We have derived conditions for admissibility of the control operator B and for exact controllability of the neutral functional differential system. We also found a feedback control law which can stabilize the neutral system in the exponential sense.

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