Mathematical Problems in Engineering, 2002, Vol. 8(2), pp. 161–168 Taylor & Francis Taylor & Francis Group

Numerical Method of Identification of an Unknown Source Term in a Heat Equation

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(Received 5 January 2000)

A numerical procedure for an inverse problem of identification of an unknown source in a heat equation is presented. Approach of proposed method is to approximate unknown function by polygons linear pieces which are determined consecutively from the solution of minimization problem based on the overspecified data. Numerical examples are presented.

Key words: Parabolic equation; Inverse problem; Unknown source; Finite difference method

Classification: 35K15, 35Q99, 39A10

1 INTRODUCTION

In this paper we solve the problem of structural identification of an unknown source term in a heat equation subject to the specification of the solution at the boundary. This problem is described by the following inverse problem:

Find $u = u(x, t)$ and $F = F(u)$ which satisfy

$$u_t(x, t) = u_{xx}(x, t) + F(u(x, t)), \quad (x, t) \in Q_T = (0, 1) \times (0, T), \tag{1}$$

$$u(x, 0) = 0, \quad x \in (0, 1), \tag{2}$$

$$u_x(0, t) = g(t), \quad t \in (0, T), \tag{3}$$

$$u_x(1, t) = 0, \quad t \in (0, T), \tag{4}$$

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ISSN 1024-123X print; ISSN 1563-5147 online

subject to the overspecification

$$u(0, t) = f(t), \quad t \in (0, T) \tag{5}$$

where $f(t)$ and $g(t)$ are known functions.

In the context of heat conduction and diffusion when u represents temperature and concentration, the unknown function $F(u)$ is interpreted as a heat and material source, respectively, while in a chemical or biochemical application F may be interpreted as a reaction term. Although the results in this paper apply to each of these interpretations, the unknown function $F(u)$ will be referred to here as a source term.

The problem above in such formulation have been studied by authors [1] recently.

One approach to solve this problem referred to in the literature as the method of output least squares is to assume that the unknown function is a specific functional form depending on some parameters and then seek to determine optimal parameter values so as to minimize an error functional based on the overspecified data. However, this approach has some drawbacks. For example, it is usually not evident that the solution to the optimization problem solves the original inverse problem and the error functional may be based on data which do not uniquely determine the unknown function. Another methods to solve this problem are residual update methods such as Newton, Homotopy and FPP (Fixed Point Projection) methods [4]. The main difficulty with these methods is the form of the nonlinearity. The approach of method presented in this paper is not of these types.

The strategy used here is to approximate unknown function by polygons linear pieces which are determined consecutively from the solution of minimization problem associated with finding minimum of function based on the additional condition. This paper is organized as follows. In section 2 we give the formulation of the direct and inverse problems and the properties of solution of the direct problem. In section 3 we describe a numerical procedure for the solution of the formulated inverse problem. Numerical examples are presented in section 4.

2 DIRECT AND INVERSE PROBLEMS

Let F is continuous and piecewise differentiable on R and $g \in C(0, \infty)$ with $g(0) = 0$ functions. Then initial boundary value problem (1)–(4) has a unique classical solution in Q'_T for sufficiently small T' (local existence of a solution) [3]. It is known that if the local solution is known to satisfy an a priori estimate then the local solution may be extended to a global solution. In particular, if it is known a priori that the solution of (1)–(4) satisfies

$$|u(x, t)| \leq C_1 \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{and} \quad 0 \leq t \leq T \tag{6}$$

for some$T > 0$, then it can be concluded that $T' = T$. One refers to (1)–(4) as a direct problem.

Let $u = u(x, t; F, g)$ denote the solution of (1)–(4) for boundary input $g = g(t)$ and source term $F = F(u)$. Assume that this solution is known to satisfy (6) for a fixed $T > 0$ so that $u(x, t)$ is then a solution in Q_T. Then $u(x, t)$ will be said to be a solution of the direct problem. The function $f(t) = u(0, t; F, g)$ will be viewed as an output corresponding to the input $g(t)$ in the presence of the source term F. In [1] the following properties were deduced for the direct problem solution:

Suppose that $g(t) < 0$ for $0 < t < T$. Then

a) for each $t, 0 < t < T$, $f(t) = u(0,t) > u(x,t) > u(1,t)$ for $0 < x < 1$. If, in addition, $F(0) \geq 0$ with $F(u) \geq 0$ for $0 \leq u \leq U_1$ for some $U_1 > 0$, then exists $T_1 > 0$ such that $f'(t) > 0$ for $0 < t < T_1$ and
b) $f(t) = u(0,t) > u(x,t) > u(1,t) \geq 0$ for $0 < x < 1, 0 < t < T_1$.

We shall use these properties to construct numerical procedure for the solution of considered problem.

Now the inverse problem can be defined as follows: suppose $g \in C[0,T]$, $g(0) = 0, g(t) < 0$ and $f \in C'[0,T]$, $f(0) = 0, f'(t) > 0$ for $0 < t < T$. Then the problem of determining $F(u)$ on an interval $[0, f(T)]$ from the data $f(t)$ and $g(t)$ whose values are known on the interval $[0, T]$ will be said the inverse problem. The uniqueness of formulated and similar inverse problem has been established in [1,5].

3 PROCEDURE OF NUMERICAL SOLUTION

Let $\tau = \Delta t > 0$ and $h = \Delta x > 0$ be step length on time and space coordinate, $\{0 = t_0 < t_1 < \cdots < t_M = T\}$ and $\{0 = x_0 < x_1 < \cdots < x_N = 1\}$ denote partitions of the $[0, T]$ and $[0, 1]$ respectively. It follows from the properties of direct problem solution that $\{u_0 \leq u_1 \leq \cdots \leq u_M\}$, where $u_i = f(t_i)$, defines a corresponding partition of the interval $[u_0, f(T)]$, where $u_0 = \inf_{(x,t)\in Q_T} u(x,t)$ and $f(T) = \sup_{(x,t)\in Q_T} u(x,t)$. We replace the region Q_T by a set of grid points (x_i, t_n) denoted by (i, n).

We may write the problem (1)–(5) at the grid points (i, n) as

$$\frac{\partial u_{i,n}}{\partial t} = \frac{\partial^2 u_{i,n}}{\partial x^2} + F(u_{i,n}), \quad 1 \leq i \leq N-1, \; 1 \leq n \leq M, \tag{7}$$

$$u_{i,0} = 0, \quad 0 \leq i \leq N \tag{8}$$

$$\frac{\partial u_{0,n}}{\partial x} = g(t_n), \quad 1 \leq n \leq M \tag{9}$$

$$\frac{\partial u_{N,n}}{\partial x} = 0, \quad 1 \leq n \leq M, \tag{10}$$

and

$$u_{0,n} = f(t_n), \quad 1 \leq n \leq M \tag{11}$$

The implicit finite difference approximation of this system may be written in the form

$$\frac{U_{i,n}-U_{i,n-1}}{\tau}=\frac{U_{i+1,n}-2U_{i,n}+U_{i-1,n}}{h^2}+F(U_{i,n}),$$
$$1\le i\le N-1,\quad 1\le n\le M, \tag{12}$$

$$U_{i,0}=u_0,\quad 0\le i\le N, \tag{13}$$

$$\frac{U_{1,n}-U_{0,n}}{h}=g(t_n),\quad 1\le n\le M, \tag{14}$$

$$\frac{U_{N,n}-U_{N-1,n}}{h}=0,\quad 1\le n\le M, \tag{15}$$

$$U_{N,n}=f(t_n),\quad 1\le n\le M, \tag{16}$$

where $U_{i,n}$ is the approximate value of $u_{i,n}$. The difference scheme (12)–(16) has a second order approximation on x on the interior nodal points and first order approximation in t.

Define $P_n(u)$ a polygonal (*i.e.*, continuous and piecewise linear) approximation of $F(u)$ $[u_0, u_n]$ as follows. For each $m = 1, 2, \ldots, n$, let $P_n(u)$ given for $u_{m-1} < u \le u_m$ by

$$P_n(u)=F_{m-1}\frac{u_m-u}{u_m-u_{m-1}}+F_m\frac{u-u_{m-1}}{u_m-u_{m-1}} \tag{17}$$

where $F_m = F(u_m)$. $P_0(u) = F_0 = const.$ Assume that $F(u(x, 0)) = F_0$ is known. This value can be found from the solution of (7)–(11) by using finite difference approximation (12)–(16) with $n = 1$ if we assume that $F(u)$ is constant on the initial time segment $(0, t_1)$, *i.e.*, $F(u) = F_0 = const.$ on $[u_0, u_1]$. We shall determine graph of unknown function $F(u)$ on the consecutive segments $[u_1, u_2), [u_2, u_3), \ldots, [u_{M-1}, u_M]$ in the following manner.

First it will be shown how $F_2 = F(u_2)$ is determined and then the procedure will be generalized to the case of subsequent l's.

Let $\beta \le F'(u) \le \gamma$ for any u and let $u(x, t; \alpha)$, where $\alpha \in (\tan^{-1}\beta,\ \tan^{-1}\gamma)$, denote the solution of (1)–(4) for

$$F(u)=\begin{cases} F(u;\alpha)=F_1+tg\alpha(u-u_1), & \text{if } u\ge u_1,\\ F_0, & \text{if } \in [u_0,u_1), \end{cases} \tag{18}$$

on the time segment $[0, t_2]$. Let $\alpha_2 = \arg\min_\alpha I(\alpha)$, where $I(\alpha) = |u(0, t_2; \alpha) - f(t_2)|$. Then we can define $F_2 = F_1 + tg\alpha_2(u_2 - u_1)$ and on $[u_1, u_2]$ unknown $F(u)$ will be approximated by $P_2(u)$.

To find F_l one the first l constants $F_0, F_1, \ldots, F_{l-1}$ are known, let $u(x, t; \alpha)$ denote the solution of (1)–(4) for

$$F(u)=\begin{cases} F(u;\alpha)=F_{l-1}+tg\alpha(u-u_{l-1}), & \text{if } u\ge u_{l-1},\\ P_{l-1}(u), & \text{if } u\in [u_0,u_{l-1}), \end{cases} \tag{19}$$

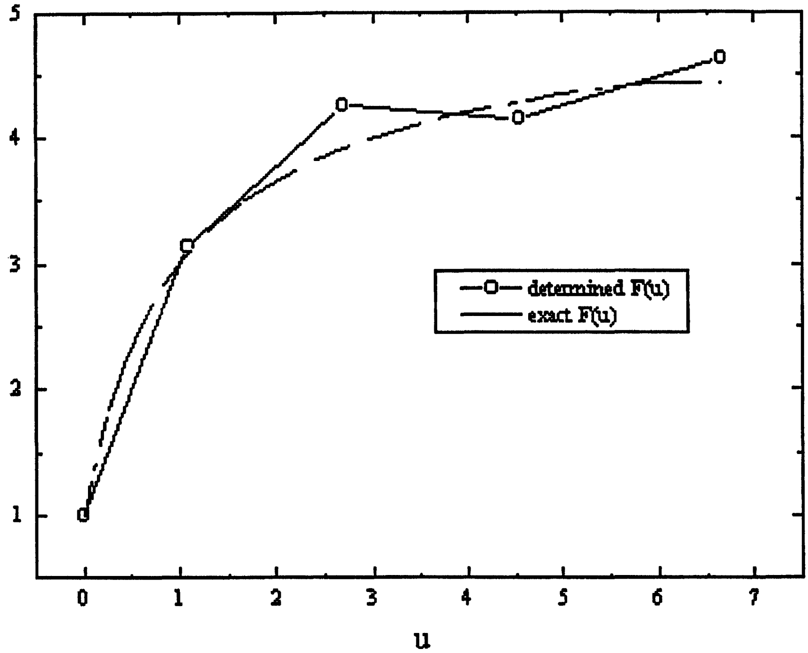

FIGURE 1

on the interval $[0, t_l]$, and $\alpha_l = \arg\min_\alpha I(\alpha)$, where

$$I(\alpha) = |u(0, t_l; \alpha) - f(t_l)| \tag{20}$$

Then $F_l = F_{l-1} + tg\alpha_1(u_l - u_{l-1})$ and $P_l(u)$ approximate the unknown $F(u)$ on $[u_0, u_l)$.

Executing this procedure for $l = 1$ to $l = M$ generates a polygonal approximation $P_M(u)$ for $F(u)$ on the partition $[u_0, u_1, \ldots, u_M]$.

Numerical experiments to test the effectiveness of this algorithm which will be discussed in the following section.

4 NUMERICAL EXAMPLES

In this section we report some results of our numerical calculations using the numerical algorithm proposed in the previous section.

The data function $g(t)$, the source $F(u)$, u_0 and T were given by

$$g(t) = -3t, \quad F(u) = 5 - 4(u + 1), \quad T = 1, \quad u_0 = 0.$$

By solving the direct problem with these data by using implicit finite-difference approximation (12)–(16), the solution values of $f(t)$ were recorded. Then the inverse problem was solved with this overspecification to determine the unknown source $F(u)$. For minimization of (20) here the method of golden section search is used. Results of determination of $F(u)$ by the presented numerical procedure, are illustrated in Figures 1–3, corresponds to results with grids $N \times M = 50 \times 5, 50 \times 10, 50 \times 30$, respectively, where the symbols correspond to approximate results and the ones without symbols correspond to exact $F(u)$. It is seen that

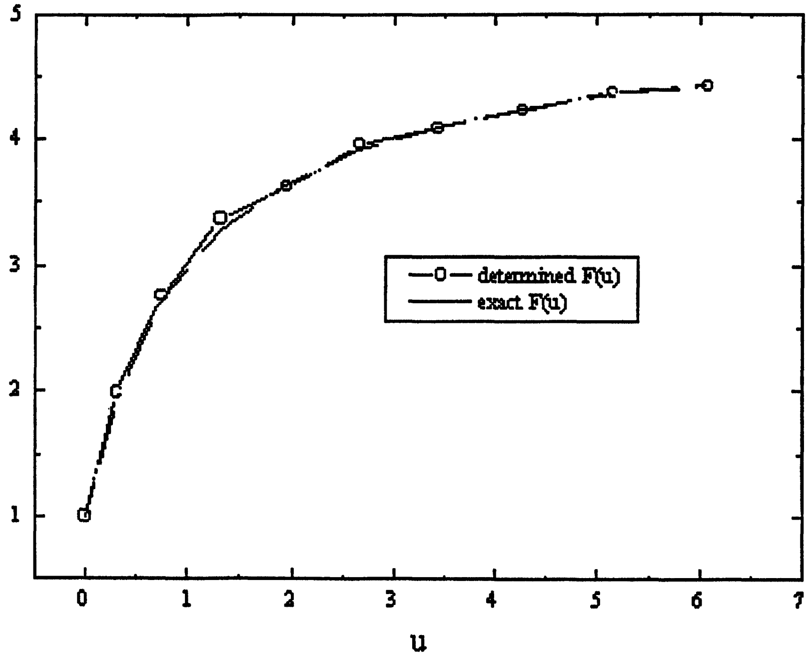

FIGURE 2

approximation of $F(u)$ is improved by increasing the number of nodes and that for sufficiently large number of nodes the agreement between numerical and exact solution becomes uniformly good.

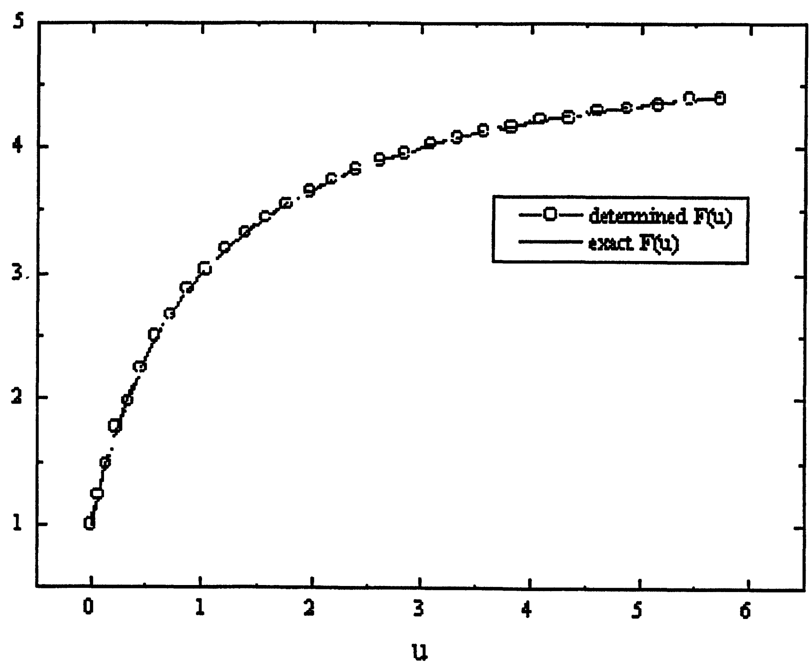

FIGURE 3

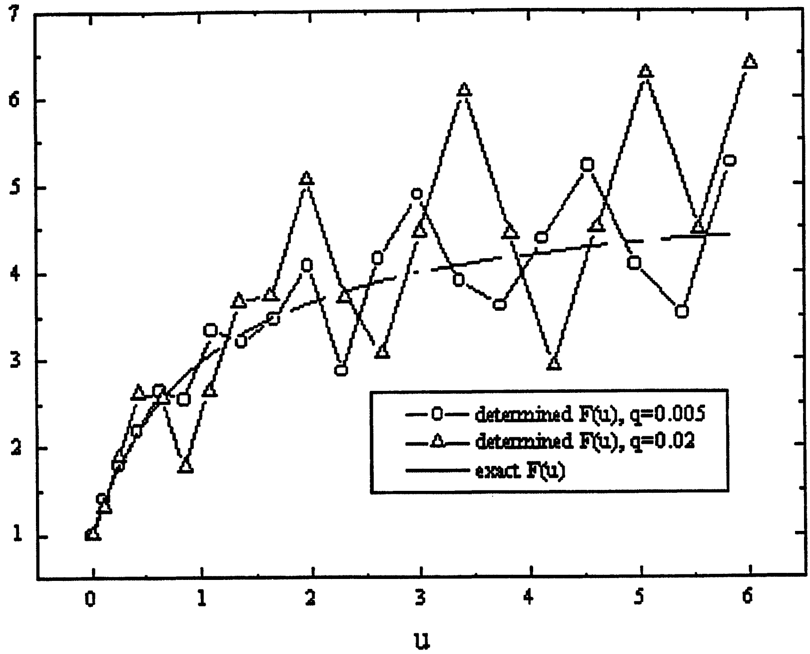

FIGURE 4

In the next example input data were used from the previous example. Artificial errors were introduced into the overspecified data by defining function

$$f^*(t_n) = f(t_n)(1 + d(t_n, q)), \quad n = 1, 2, \ldots, M.$$

Here $d(t_n, q)$ represents the level of relative error in the corresponding piece of data. Calculation results with grid $N \times M = 50 \times 20$ are presented in Figure 4 according to cases with $d(t, 0.02)$, and $d(t, 0.04)$. As seen from the figure that in this case results are worsening and there are approximations in some integral norm. It is also seen that $P_M(u)$ appears to be a relative good approximation for small values of artificial errors.

5 CONCLUSION AND FUTURE DIRECTIONS

Considered problem in this paper has such properties that presented method is readily available. Numerical experiments show effectiveness of the presented method in determining close estimates of unknown source term in a heat equation. In considered problem additional condition is given on boundary, but the proposed procedure may be used for the solution of problems where additional conditions are given on the interior points. The presented method can be applied for the solution of class of inverse problem associated with parabolic equation

$$u_t = F(x, t, u, u_{xx}, p(u))$$

with appropriate initial, boundary and additional conditions, where $u(x, t)$ and $p(u)$ are to be determined. This method can be extended also for the solution of multidimensional and multiple coefficient inverse problems.

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