

# Boundary Value Problems on the Half Line in the Theory of Colloids

RAVI P. AGARWAL<sup>a,\*</sup> and DONAL O'REGAN<sup>b</sup>

<sup>a</sup>Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901-6975, U.S.A.; <sup>b</sup>Department of Mathematics, National University of Ireland, Galway, Ireland

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We present existence results for some boundary value problems defined on infinite intervals. In particular our discussion includes a problem which arises in the theory of colloids.

**Key words:** Boundary value problem, half line, colloids, existence

## 1 INTRODUCTION

In the theory of colloids [4, 7] it is possible to relate particle stability with the charge on the colloidal particle. We model the particle and its attendant electrical double layer using Poisson's equation for a flat plate. If  $\Psi$  is the potential,  $\rho$  the charge density,  $D$  the dielectric constant and  $y$  the displacement, then we have

$$\frac{d^2\Psi}{dy^2} = -\frac{4\pi\rho}{D}.$$

We assume the ions are point charged and their concentrations in the double layer satisfies the Boltzmann distribution

$$c_i = c_i^* \exp\left(\frac{-z_i e \Psi}{\kappa T}\right)$$

where  $c_i$  is the concentration of ions of type  $i$ ,  $c_i^* = \lim_{\Psi \rightarrow 0} c_i$ ,  $\kappa$  the Boltzmann constant,  $T$  the absolute temperature,  $e$  the electrical charge, and  $z$  the valency of the ion. In the neutral case, we have

$$\rho = c_+ z_+ e + c_- z_- e \quad \text{or} \quad \rho = ze(c_+ - c_-)$$

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\* Corresponding author. Tel.: 321-674-8091; Fax: 321-674-7412; E-mail: agarwal@fit.edu

where  $z = z_+ - z_-$ . Then we have using

$$c_+ = c \exp\left(\frac{-ze\Psi}{\kappa T}\right) \quad \text{and} \quad c_- = c \exp\left(\frac{ze\Psi}{\kappa T}\right),$$

that

$$\frac{d^2\Psi}{dy^2} = \frac{8\pi cze}{D} \sinh\left(\frac{ze\Psi}{\kappa T}\right)$$

where the potential initially takes some positive value  $\Psi(0) = \Psi_0$  and tends to zero as the distance from the plate increases *i.e.*  $\Psi(\infty) = 0$ . Using the transformation

$$\phi(y) = \frac{ze\Psi(y)}{\kappa T} \quad \text{and} \quad x = \sqrt{\frac{4\pi cz^2 e^2}{\kappa TD}} y,$$

the problem becomes

$$\begin{cases} \frac{d^2\phi}{dx^2} = 2 \sinh \phi, & 0 < x < \infty \\ \phi(0) = c_1 \\ \lim_{x \rightarrow \infty} \phi(x) = 0, \end{cases} \quad (1.1)$$

where  $c_1 = ze\Psi_0/\kappa T > 0$ . From a physical point of view we wish the solution  $\phi$  in (1.1) to also satisfy  $\lim_{x \rightarrow \infty} \phi'(x) = 0$ .

In this paper using the notion of upper and lower solutions (see [1, 2, 6]) we establish general existence results which guarantee the existence of  $BC[0, \infty)$  solutions to

$$\begin{cases} \frac{1}{p(t)}(p(t)y'(t))' = q(t)f(t, y(t)), & 0 < t < \infty \\ -a_0y(0) + b_0 \lim_{t \rightarrow 0^+} p(t)y'(t) = c_0, & a_0 > 0, \quad b_0 \geq 0 \\ \lim_{t \rightarrow \infty} y(t) = 0; \end{cases} \quad (1.2)$$

here  $BC[0, \infty)$  denotes the space of continuous, bounded functions from  $[0, \infty)$  to  $\mathbf{R}$ . Our theory not only complements some of the known results, *e.g.*, [5, 8], but also automatically produces the existence of a solution to (1.1). To establish these results we recall, for the convenience of the reader, the existence principle [3] we will use in Section 2. Consider the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' = qf(t, y), & 0 < t < \infty \\ -a_0y(0) + b_0 \lim_{t \rightarrow 0^+} p(t)y'(t) = c_0, & a_0 > 0, \quad b_0 \geq 0 \\ y(t) \text{ bounded on } [0, \infty). \end{cases} \quad (1.3)$$

By an upper solution  $\beta$  to (1.3) we mean a function  $\beta \in BC[0, \infty) \cap C^2(0, \infty)$ ,  $p\beta' \in C[0, \infty)$  with

$$\begin{cases} \frac{1}{p}(p\beta')' \leq qf(t, \beta), & 0 < t < \infty \\ -a_0\beta(0) + b_0 \lim_{t \rightarrow 0^+} p(t)\beta'(t) \leq c_0, \\ \beta(t) \text{ bounded on } [0, \infty) \end{cases} \quad (1.4)$$

and by a lower solution  $\alpha$  to (1.3) we mean a function  $\alpha \in BC[0, \infty) \cap C^2(0, \infty)$ ,  $p\alpha' \in C[0, \infty)$  with

$$\begin{cases} \frac{1}{p}(p\alpha')' \geq qf(t, \alpha), & 0 < t < \infty \\ -a_0\alpha(0) + b_0 \lim_{t \rightarrow 0^+} p(t)\alpha'(t) \geq c_0, \\ \alpha(t) \text{ bounded on } [0, \infty). \end{cases} \quad (1.5)$$

**THEOREM 1.1** [3] *Let  $f: [0, \infty) \times \mathbf{R} \rightarrow \mathbf{R}$  be continuous. Suppose the following conditions are satisfied:*

$$q \in C(0, \infty) \text{ with } q > 0 \text{ on } (0, \infty) \quad (1.6)$$

$$p \in C[0, \infty) \cap C^1(0, \infty) \text{ with } p > 0 \text{ on } (0, \infty) \quad (1.7)$$

$$\int_0^\mu \frac{ds}{p(s)} < \infty \text{ and } \int_0^\mu p(s)q(s) ds < \infty \text{ for any } \mu > 0 \quad (1.8)$$

$$\begin{cases} \text{there exists } \alpha, \beta \text{ respectively lower and upper} \\ \text{solutions of (1.3) with } \alpha(t) \leq \beta(t) \text{ for } t \in [0, \infty) \end{cases} \quad (1.9)$$

and

$$\begin{cases} \text{there exists a constant } M > 0 \text{ with } |f(t, u)| \leq M \\ \text{for } t \in [0, \infty) \text{ and } u \in [\alpha(t), \beta(t)]. \end{cases} \quad (1.10)$$

Then (1.3) has a solution  $y \in BC[0, \infty) \cap C^2(0, \infty)$ ,  $py' \in C[0, \infty)$  with  $\alpha(t) \leq y(t) \leq \beta(t)$  for  $t \in [0, \infty)$ . Also there exist constants  $A_0$  and  $A_1$  with  $|p(t)y'(t)| \leq A_0 + A_1 \int_0^t p(s)q(s) ds$  for  $t \in (0, \infty)$ .

## 2 THE BOUNDARY CONDITION AT INFINITY

Motivated by the colloid example [4, 7] we discuss the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' = q(t)f(t, y), & 0 < t < \infty \\ -a_0y(0) + b_0 \lim_{t \rightarrow 0^+} p(t)y'(t) = c_0, & a_0 > 0, \quad b_0 \geq 0, \quad c_0 \leq 0 \\ \lim_{t \rightarrow \infty} y(t) = 0. \end{cases} \quad (2.1)$$

**THEOREM 2.1** *Let  $f: [0, \infty) \times \mathbf{R} \rightarrow \mathbf{R}$  be continuous and suppose the following conditions hold:*

$$q \in C(0, \infty) \text{ with } q > 0 \text{ on } (0, \infty) \quad (2.2)$$

$$p \in C[0, \infty) \cap C^1(0, \infty) \text{ with } p > 0 \text{ on } (0, \infty) \text{ and } \int_0^\infty \frac{ds}{p(s)} = \infty \quad (2.3)$$

$$\int_0^\mu \frac{ds}{p(s)} < \infty \text{ and } \int_0^\mu p(s)q(s) ds < \infty \text{ for any } \mu > 0 \quad (2.4)$$

$$f(t, 0) \leq 0 \text{ for } t \in (0, \infty) \quad (2.5)$$

$$\exists r_0 \geq \frac{-c_0}{a_0} \text{ with } f(t, r_0) \geq 0 \text{ for } t \in (0, \infty) \quad (2.6)$$

$$\exists M > 0 \text{ with } |f(t, u)| \leq M \text{ for } t \in [0, \infty) \text{ and } u \in [0, r_0] \quad (2.7)$$

$$\left\{ \begin{array}{l} \exists \text{ a constant } m > 0 \text{ with } q(t)p^2(t)[f(t, u) - f(t, 0)] \geq m^2u \\ \text{for } t \in (0, \infty) \text{ and } u \in [0, r_0] \end{array} \right. \quad (2.8)$$

$$\int_0^\infty p(x) \exp\left(-m \int_0^x \frac{ds}{p(s)}\right) q(x) |f(x, 0)| dx < \infty \quad (2.9)$$

$$\lim_{t \rightarrow \infty} p^2(t)q(t)f(t, 0) = 0 \quad (2.10)$$

and

$$\left\{ \begin{array}{l} \lim_{t \rightarrow \infty} \left( B_0 \int_\mu^t \frac{1}{p(s)} \int_\mu^s \frac{1}{p(x)} dx ds + C_0 \int_\mu^t \frac{ds}{p(s)} \right) = \infty \\ \text{for any constants } B_0 > 0, C_0 \in \mathbf{R} \text{ and } \mu > 0. \end{array} \right. \quad (2.11)$$

Then (2.1) has a solution  $y \in C[0, \infty) \cap C^2(0, \infty)$  with  $py' \in C[0, \infty)$  and  $0 \leq y(t) \leq r_0$  for  $t \in [0, \infty)$ .

*Proof* Now Theorem 1.1 (with  $\alpha = 0$  and  $\beta = r_0$ ) guarantees that

$$\left\{ \begin{array}{l} \frac{1}{p}(py')' = q(t)f(t, y), \quad 0 < t < \infty \\ -a_0y(0) + b_0 \lim_{t \rightarrow 0^+} p(t)y'(t) = c_0 \\ y(t) \text{ bounded on } [0, \infty) \end{array} \right. \quad (2.12)$$

has a solution  $y \in C[0, \infty) \cap C^2(0, \infty)$ ,  $py' \in C[0, \infty)$  and  $0 \leq y(t) \leq r_0$  for  $t \in [0, \infty)$ . Let  $g(x) = q(x)f(x, 0)$  and notice that

$$\begin{aligned} w(t) &= \exp\left(-m \int_0^t \frac{ds}{p(s)}\right) \left[ \frac{(-c_0)}{a_0 + b_0 m} \right. \\ &\quad \left. + \frac{(a_0 - b_0 m)}{2m(a_0 + b_0 m)} \int_0^\infty p(x) \exp\left(-m \int_0^x \frac{ds}{p(s)}\right) g(x) dx \right] \\ &\quad - \frac{1}{2m} \exp\left(m \int_0^t \frac{ds}{p(s)}\right) \int_t^\infty p(x) \exp\left(-m \int_0^x \frac{ds}{p(s)}\right) g(x) dx \\ &\quad - \frac{1}{2m} \exp\left(-m \int_0^t \frac{ds}{p(s)}\right) \int_0^t p(x) \exp\left(m \int_0^x \frac{ds}{p(s)}\right) g(x) dx \\ &= \exp\left(-m \int_0^t \frac{ds}{p(s)}\right) \left[ \frac{(-c_0)}{a_0 + b_0 m} - \frac{b_0}{a_0 + b_0 m} \int_0^\infty p(x) \exp\left(-m \int_0^x \frac{ds}{p(s)}\right) g(x) dx \right] \\ &\quad - \int_0^t \frac{1}{p(\zeta)} \exp\left(-m \int_\zeta^t \frac{ds}{p(s)}\right) \left( \int_\zeta^\infty p(x) \exp\left(-m \int_\zeta^x \frac{ds}{p(s)}\right) g(x) dx \right) d\zeta \end{aligned}$$

is a nonnegative solution of

$$\begin{cases} \frac{1}{p}(pw')' - \frac{m^2}{p^2(t)}w = g(t), & 0 < t < \infty \\ -a_0w(0) + b_0 \lim_{t \rightarrow 0^+} p(t)w'(t) = c_0 \\ \lim_{t \rightarrow \infty} w(t) = 0. \end{cases} \tag{2.13}$$

Notice (2.10) and l'Hopital's rule guarantees that  $w(\infty) = 0$ .

Now let

$$r(t) = y(t) - w(t).$$

We first show  $r$  cannot have a local positive maximum on  $[0, \infty)$ . Suppose  $r$  has a local positive maximum at  $t_0 \in [0, \infty)$ .

Case (i)  $t_0 \in [0, \infty)$ .

For  $t > 0$  notice from assumption (2.8) that

$$\frac{1}{p}(pr')'(t) = q(t)[f(t, y(t)) - f(t, 0)] - \frac{m^2}{p^2(t)}w(t) \geq \frac{m^2}{p^2(t)}[y(t) - w(t)]. \tag{2.14}$$

We also have  $r'(t_0) = 0$  and  $r''(t_0) \leq 0$ . However (2.14) yields

$$r''(t_0) = \frac{1}{p(t_0)}(pr')'(t_0) \geq \frac{m^2}{p^2(t_0)}[y(t_0) - w(t_0)] > 0,$$

a contradiction.

Case (ii)  $t_0 = 0$ .

Of course if  $b_0 = 0$  we have a contradiction immediately. So suppose  $b_0 \neq 0$ . Then

$$\lim_{t \rightarrow 0^+} p(t)r'(t) = \frac{a_0}{b_0}[y(0) - w(0)]. \tag{2.15}$$

Now since  $y(0) - w(0) > 0$  there exists  $\delta > 0$  with  $y(t) - w(t) > 0$  for  $t \in (0, \delta)$ . Then (2.14) implies  $(pr')' > 0$  on  $(0, \delta)$  and this together with (2.15) (i.e.  $\lim_{t \rightarrow 0^+} p(t)y'(t) > 0$ ) implies  $pr' > 0$  on  $(0, \delta)$ , a contradiction.

Thus  $r(t)$  cannot have a local positive maximum on  $[0, \delta)$ . We now claim that  $r(t) \leq 0$  on  $[0, \infty)$ . If  $r(t) \not\leq 0$  on  $[0, \infty)$  then there exists a  $c_1 > 0$  with  $r(c_1) > 0$ . Now since  $r(t)$  cannot have a positive local maximum on  $[0, \infty)$  it follows that  $r(t_2) > r(t_1)$  for all  $t_2 > t_1 \geq c_1$ ; otherwise  $r(t)$  would have a local positive maximum on  $[0, t_2]$ . Thus  $r(t)$  is strictly increasing for  $t \geq c_1$ . Since both  $y(t)$  and  $w(t)$  are bounded on  $[0, \infty)$  and  $\lim_{t \rightarrow \infty} w(t) = 0$  then

$$\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} [y(t) - w(t)] = \kappa \in (0, r_0]. \tag{2.16}$$

Now there exists  $c_2 \geq c_1$  with  $y(t) \geq \kappa/2$  for  $t \geq c_2$ . The differential equation and (2.8) imply that for  $t > 0$  that we have

$$\begin{aligned} (p(t)y'(t))' &= p(t)q(t)f(t, y(t)) = p(t)q(t)[f(t, y(t)) - f(t, 0)] + p(t)q(t)f(t, 0) \\ &\geq \frac{m^2}{p(t)}y(t) + p(t)q(t)f(t, 0). \end{aligned}$$

Consequently for  $t \geq c_2$  we have

$$(py')'(t) \geq \frac{m^2\kappa}{2p(t)} + p(t)q(t)f(t, 0) = \frac{1}{p(t)} \left[ \frac{m^2\kappa}{2} + p^2(t)q(t)f(t, 0) \right].$$

Assumption (2.10) implies that there is a constant  $c_3 \geq c_2$  with

$$(py')'(t) \geq \frac{m^2\kappa}{4p(t)} \quad \text{for } t \geq c_3.$$

Two integrations together with the fact that  $y \geq 0$  on  $[0, \infty)$  yields

$$y(t) \geq p(c_3)y'(c_3) \int_{c_3}^t \frac{ds}{p(s)} + \frac{m^2\kappa}{4} \int_{c_3}^t \frac{1}{p(s)} \int_{c_3}^s \frac{1}{p(x)} dx ds$$

(not also from Theorem 1.1 that there exist constants  $A_0$  and  $A_1$  with  $|p(t)y'(t)| \leq A_0 + A_1 \int_0^t p(s)q(s) ds$  for  $t \in (0, \infty)$ ). Now assumption (2.11) implies that  $y$  is unbounded on  $[0, \infty)$ , a contradiction. Thus  $r(t) \leq 0$  on  $[0, \infty)$  and the result follows. ■

Notice in Theorem 3.1 that the solution  $y$  of (2.1) satisfies  $r(t) \leq 0$  for  $t \in [0, \infty)$ , and so  $y(t) \leq w(t)$  for  $t \in [0, \infty)$ .

**COROLLARY 2.2** *Let  $f: [0, \infty) \times \mathbf{R} \rightarrow \mathbf{R}$  be continuous and suppose (2.2)–(2.11) hold. Then (2.1) has a solution  $y \in C[0, \infty) \cap C^2(0, \infty)$  with  $py' \in C[0, \infty)$  and  $0 \leq y(t) \leq w(t)$  for  $t \in [0, \infty)$ , with  $w$  given in Theorem 2.1.*

The colloid [4, 7] example motivates our next result.

**THEOREM 2.3** *Let  $f: [0, \infty) \times \mathbf{R} \rightarrow \mathbf{R}$  be continuous and suppose (2.2)–(2.11) hold. In addition assume the following conditions hold:*

$$f(t, u) \geq 0 \text{ for } t \in [0, \infty) \quad \text{and} \quad u \in [0, w(t)]; \text{ here } w \text{ is as in Theorem 2.1} \tag{2.17}$$

and

$$\lim_{t \rightarrow \infty} p(t) \in (0, \infty]. \tag{2.18}$$

Then (2.1) has a solution  $y \in C[0, \infty) \cap C^2(0, \infty)$  with  $py' \in C[0, \infty)$ ,  $0 \leq y(t) \leq w(t)$  for  $t \in [0, \infty)$  and  $\lim_{t \rightarrow \infty} y'(t) = 0$ .

*Proof* From Corollary 2.2 we know that there exists a solution  $y \in C[0, \infty) \cap C^2(0, \infty)$ ,  $py' \in C[0, \infty)$  and  $0 \leq y(t) \leq w(t)$  for  $t \in [0, \infty)$ , to (2.1). Also (2.17) and the differential equation yields

$$(py')'(t) = p(t)q(t)f(t, y(t)) \geq 0 \quad \text{for } t > 0, \tag{2.19}$$

so  $py'$  is nondecreasing on  $(0, \infty)$ , and  $\lim_{t \rightarrow \infty} p(t)y'(t) \in [-\infty, \infty]$ .

Suppose there exists  $t_1 \in [0, \infty)$  with  $p(t_1)y'(t_1) > 0$ . Then

$$p(t)y'(t) \geq a_0 \equiv p(t_1)y'(t_1) \quad \text{for } t \geq t_1,$$

and so

$$y(t) \geq y(t_1) + a_0 \int_{t_1}^t \frac{ds}{p(s)} \quad \text{for } t \geq t_1. \tag{2.20}$$

That is

$$y(t) \geq a_0 \int_{t_1}^t \frac{ds}{p(s)} \quad \text{for } t \geq t_1 \tag{2.21}$$

(notice (2.3) implies that the right hand side of (2.21) goes to  $\infty$  as  $t \rightarrow \infty$ ). This contradicts  $0 \leq y(t) \leq r_0$  for  $t \in [0, \infty)$ . Thus  $p(t)y'(t) \leq 0$  for  $t \in (0, \infty)$ , and so

$$\lim_{t \rightarrow \infty} p(t)y'(t) = \kappa \in [-\infty, 0] \quad \text{and} \quad \lim_{t \rightarrow \infty} y'(t) \in [-\infty, 0]. \tag{2.22}$$

In fact  $\kappa \in (-\infty, 0]$  from (2.19). Finally if  $\kappa < 0$  then there exists  $t_2 > 0$  with  $p(t)y'(t) \leq \kappa/2$  for  $t \geq t_2$ . Integrate from  $t_2$  to  $t$  ( $t \geq t_2$ ) to get

$$y(t) \leq y(t_2) + \frac{\kappa}{2} \int_{t_2}^t \frac{ds}{p(s)} \leq r_0 + \frac{\kappa}{2} \int_{t_2}^t \frac{ds}{p(s)}. \tag{2.23}$$

Now (2.23) together with (2.3) contradicts  $y \geq 0$  on  $[0, \infty)$ . Consequently  $\lim_{t \rightarrow \infty} p(t)y'(t) = 0$ , and this together with (2.18) gives  $\lim_{t \rightarrow \infty} y'(t) = \lim_{t \rightarrow \infty} p(t)y'(t)/p(t) = 0$ . ■

*Example 2.1* (Colloid problem [4, 7]).

The boundary value problem

$$\begin{cases} y'' = 2 \sinh y, & 0 < t < \infty \\ y(0) = c > 0 \\ \lim_{t \rightarrow \infty} y(t) = 0 \end{cases} \tag{2.24}$$

has a solution  $y \in C[0, \infty) \cap C^2(0, \infty)$  with

$$0 \leq y(t) \leq ce^{-t} \quad \text{for } t \in [0, \infty). \quad (2.25)$$

To see this we will apply Corollary 2.2 with

$$p = 1, \quad q = 1, \quad a_0 = 1, \quad c_0 = -c, \quad b_0 = 0 \quad \text{and} \quad r_0 = c.$$

Clearly (2.1)–(2.7), (2.8) since  $f(t, u) - f(t, 0) = \sinh u \geq u$  for  $u \geq 0$ , (2.9)–(2.11) hold. Corollary 2.2 guarantees that (2.24) has a solution  $y \in C[0, \infty) \cap C^2(0, \infty)$  with  $0 \leq y(t) \leq w(t)$  for  $t \in [0, \infty)$ . It is immediate from (2.13) (since  $g = 0$ ) that

$$w(t) = ce^{-t} \quad \text{for } t \in [0, \infty).$$

Finally we remark that the solution  $y$  satisfies  $\lim_{t \rightarrow \infty} y'(t) = 0$ . To see this we need only check that (2.17)–(2.18) hold, but these are immediate.

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