

# Stability of Time-delay Systems via Lyapunov Functions

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In this paper, a Lyapunov function candidate is introduced for multivariable systems with inner delays, without assuming *a priori* stability for the nondelayed subsystem. By using this Lyapunov function, a controller is deduced. Such a controller utilizes an input–output description of the original system, a circumstance that facilitates practical applications of the proposed approach.

**Key words:** Time-delay; Lyapunov stability

## 1 PROBLEM STATEMENT

Aspects related to stability for delayed systems have been dealt with in a number of papers [2–7]. However, Lyapunov stability for this kind of systems has received less attention during recent years [1]. In the following a stabilization result providing both a Lyapunov function for point-delayed systems as well as a stabilizing feedback controller based on such a Lyapunov function, is proved. This result requires some preconditions on the controller structure, related to that of the plant, but does not assume *a priori* stability for the nondelayed subsystem. A main advantage of the associate controller synthesis method is that only output-feedback is required, so that only physically measurable variables are considered for measurement. Thus, the use of an observer or estimator, which is one of the main obstacles for using state-feedback controllers specially for nonlinear systems [9], is avoided.

Let us consider the following system with  $n$  point delays in its state vector:

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}x(t) + \sum_{i=1}^n \mathbf{A}_i x(t - ih) + \mathbf{B}u(t), \\ y(t) &= \mathbf{C}x(t).\end{aligned}\tag{1.1}$$

where  $x(t) \in \mathbb{R}^n$ , is a state vector,  $x(t) = \varphi(t)$ ,  $-nh \leq t \leq 0$ ,  $x(0) = \varphi(0) = x_0$ , and  $\varphi(t)$  is an absolutely continuous vector function on  $[-nh, 0]$ , with possibly isolated bounded discontinuities,  $h > 0$ , is a fixed scalar (delay),  $u(t) \in \mathbb{R}^\ell$  is a control vector,  $y(t) \in \mathbb{R}^m$ , is an output vector and matrices are of appropriate dimensions;  $\ell, m \leq n$ . Equation (1.1) is the general form for several physical systems, such as the Minorsky ship rolling problem [13],

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certain metal rolling systems, remote control, and pasteurization systems with interconnected heat exchangers [14].

In the third section it will be demonstrated that the Lyapunov candidate

$$V(t) \equiv x^T(t)\mathbf{P}x(t) + \sum_{i=1}^n \int_{t-ih}^t x^T(\theta)x(\theta) d\theta, \quad (1.2)$$

is a Lyapunov function for the closed-loop regulator system (1.1), after generation of appropriate feedback, provided that  $\mathbf{P}$  is a symmetric positive definite matrix fulfilling a set of conditions leading to the synthesis of a stabilizing controller. The rest of the paper is organized as follows: Section 2 introduces some previous results. Section 3 presents the main results. Section 4 introduces the main algorithms to be implemented in order to synthesize the control law. Section 5 presents an illustrative example. Finally, conclusions end the paper.

## 2 PREVIOUS RESULTS

Firstly we will introduce some previous results that will be useful in the sequel.

**LEMMA 2.1** (Banach's Perturbation Lemma) *Let  $\mathbf{A}, \mathbf{B} \in \mathbf{C}^{n \times n}$ . Assume  $\mathbf{A}$  invertible. Then, if  $\|\mathbf{B}\| < 1/\|\mathbf{A}^{-1}\|$ , for any matrix norm, the following facts hold:*

$$(a) \quad \mathbf{A} + \mathbf{B}, \text{ invertible} \quad (2.1)$$

$$(b) \quad \|(\mathbf{A} + \mathbf{B})^{-1}\| \leq \frac{\|\mathbf{A}^{-1}\|}{1 - \|\mathbf{A}^{-1}\|\|\mathbf{B}\|}, \quad (2.2)$$

**LEMMA 2.2** *Let  $\mathbf{A}, \mathbf{B} \in \mathbf{C}^{n \times n}$ . Assume  $\mathbf{A}$  nonsingular. Assume  $\mathbf{A}, \mathbf{B}$ , symmetric. Suppose  $\mathbf{A}$  positive definite. Then, a necessary condition to ensure that  $\mathbf{A} + \mathbf{B}$ , is positive definite is given by the next inequality:*

$$\|\mathbf{B}\|_2 < \frac{1}{\|\mathbf{A}^{-1}\|_2}. \quad (2.3)$$

*Proof* From (2.3), and taking into account that both  $\mathbf{A}$  and  $\mathbf{B}$  are symmetric and  $\mathbf{A}$  is positive definite, which implies their eigenvalues to be real, the following inequality holds

$$|\lambda_{\max}(\mathbf{B})| < \frac{1}{|\lambda_{\max}(\mathbf{A}^{-1})|} = |\lambda_{\min}(\mathbf{A})| = \lambda_{\min}(\mathbf{A}) \Rightarrow |\lambda_{\max}(\mathbf{B})| < \lambda_{\min}(\mathbf{A}). \quad (2.4)$$

Therefore,  $\lambda_i(\mathbf{A} + \mathbf{B}) > 0 \forall i = 1, \dots, n$  positive definite.

*Remark* Note that this last Lemma can be intuitively seen in terms of the translation of eigenvalues in the complex plane. If a symmetric matrix  $\mathbf{A}$  is positive definite then it is nonsingular and all its eigenvalues are positive. Thus, if by a continuous numerical change in some of its parameters it reaches the point of not being invertible, which implies not being positive-definite, then at least one of its eigenvalues becomes zero. This is due to the fact by

which the eigenvalues corresponding to a symmetric matrix are real, and therefore their transition from the right-hand halfplane to the left-hand one is done through the origin (0, 0).

LEMMA 2.3 [8] *If  $A = B \oplus C$ , where  $\oplus$  stands for the direct sum, then  $A$  is positive definite if  $B, C$ , are positive definite.*

### 3 MAIN RESULTS

RESULT 3.1 *System (1) is asymptotically stabilizable by the output feedback controller*

$$u(t) = -Ky(t) - \sum_{i=1}^n K_i y(t - ih), \tag{3.1}$$

*if there exists a symmetric positive definite  $n \times n$  matrix  $P$  such that the following  $n(n + 1) \times n(n + 1)$  matrix  $Q$  is negative definite*

$$Q = \begin{bmatrix} -I_n & \bar{0} & \dots & \bar{0} & A_n^T P - C^T K_n^T B^T P \\ \bar{0} & -I_n & & \bar{0} & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ \bar{0} & \bar{0} & \dots & -I_n & A_1^T P - C^T K_1^T B^T P \\ PA_n - PBK_n C & \dots & \dots & PA_1 - C^T K_1^T B^T P & A^T P - C^T K^T B^T P + PA - PBKC + nI_n \end{bmatrix}. \tag{3.2}$$

where  $\bar{0}$  is a  $n \times n$  block of zeros.

*Proof* Consider the Lyapunov function candidate (1.2). Obviously  $V(t) > 0$  for  $x(t) \neq 0$ . Let us differentiate  $V(t)$  with respect to time  $t$ :

$$\dot{V}(t) = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + \sum_{i=1}^n [x^T(t)x(t) - x^T(t - ih)x(t - ih)]. \tag{3.3}$$

By using the first of Eqs. (1.1) one obtains

$$\begin{aligned} \dot{V}(t) &= x^T(t)A^T Px(t) + \sum_{i=1}^n x^T(t - ih)A_i^T Px(t) + u^T(t)B^T Px(t) + x^T(t)PAx(t) \\ &+ \sum_{i=1}^n x^T(t)PA_i x(t - ih) + x^T(t)PBu(t) + \sum_{i=1}^n [x^T(t)x(t) - x^T(t - ih)x(t - ih)]. \end{aligned} \tag{3.4}$$

Then, by using the second of Eqs. (1.1) together with (3.1) one gets

$$\begin{aligned}
\dot{V}(t) &= x^T(t)A^T Px(t) + \sum_{i=1}^n x^T(t-ih)A_i^T Px(t) - x^T(t)C^T K^T B^T Px(t) \\
&\quad - \sum_{i=1}^n x^T(t-ih)C^T K_i^T B^T Px(t) + x^T(t)PAx(t) + \sum_{i=1}^n x^T(t)PA_i x(t-ih) \\
&\quad - x^T(t)PBK Cx(t) - \sum_{i=1}^n x^T(t)PBK_i Cx(t-ih) \\
&\quad + \sum_{i=1}^n [x^T(t)x(t) - x^T(t-ih)x(t-ih)]. \tag{3.5}
\end{aligned}$$

This leads to

$$\begin{aligned}
\dot{V}(t) &= x^T(t)[A^T P - C^T K^T B^T P + PA - PBK C + nI]x(t) \\
&\quad + \sum_{i=1}^n x^T(t-ih)[A_i^T P - C^T K_i^T B^T P]x(t) + \sum_{i=1}^n x^T(t)[PA_i - PBK_i C]x(t-ih) \\
&\quad + \sum_{i=1}^n x^T(t-ih)(-I)x(t-ih) \\
&= [x^T(t-nh) \quad \dots \quad x^T(t-h) \quad x^T(t)] \cdot Q \cdot [x^T(t-nh) \quad \dots \quad x^T(t-h) \quad x^T(t)]^T. \tag{3.6}
\end{aligned}$$

But, by hypothesis,  $Q$  is negative definite matrix and, therefore,  $\dot{V}(t) \leq -\lambda_{\min}(-Q) \sum_{i=0}^n \|x(t-ih)\|_2^2 < 0$ , over all non-zero trajectories of (1.1). Hence, (1.2) is a Lyapunov function for system (1.1).

**RESULT 3.2** Consider matrix  $Q$  (3.2). Decompose  $Q = -Q_d - Q_{nd}$ , where  $-Q_d = -\text{BlockDiag}(Q_{d_1}, \dots, Q_{d_n})$ , is  $n \times n$  block diagonal, where the  $Q_{d(\cdot)}$  block matrices are positive definite. Let  $\|\cdot\|_2$  be the  $l_2$  matrix norm. Providing that matrix  $Q_d$  is nonsingular, then matrix  $Q$  is negative definite if the following two conditions hold:

$$(i) \quad \|Q_{nd}\|_2 < \frac{1}{\|Q_d^{-1}\|_2}. \tag{3.7}$$

(ii) The associated system  $\dot{x} = -Sx$ , is globally stable, with

$$S = C^T K^T B^T P + PBK C - A^T P - PA - nI_n. \tag{3.8}$$

*Proof* In order to prove that  $Q$  is negative definite, note that, by construction,  $Q$  is symmetric, and therefore  $-Q$  is also symmetric. Decompose  $-Q = Q_d + Q_{nd}$ , being  $Q_d$   $n \times n$  block diagonal, and being  $Q_{nd}$  a matrix with  $n \times n$  zero blocks  $\bar{0}$  in its  $n \times n$  block diagonal. As  $-Q$  and  $Q_d$  are both symmetric,  $Q_{nd} = -(Q + Q_d)$  is also symmetric. Thus, in order to ensure that  $-Q$  is definite positive it is enough to show that  $Q_d$  is positive definite and, by Lemma 2, that  $\|Q_{nd}\|_2 < 1/\|Q_d^{-1}\|_2$ . As this latter inequality is ensured by assumption (3.7), it is only necessary to prove that  $Q_d$  is positive definite. Note that  $Q_d = I_{nn} \oplus S$ . Obviously  $I_{nn}$  is positive definite. And assumption (3.8) implies that all eigenvalues of the symmetric matrix  $-S$  are negative, and therefore  $-S$  is negative definite, which implies  $S$  positive definite.

Then, by Lemma 2.3,  $\mathbf{Q}_d$  is positive definite. More specifically, if  $\lambda_{\min}(\cdot)$  and  $\lambda_{\max}(\cdot)$  denote, respectively, the minimum and maximum eigenvalues of the  $(\cdot)$ -square matrix, note that

$$\begin{aligned}\lambda_{\min}(-\mathbf{Q}) &\geq \lambda_{\min}(\mathbf{Q}_d) + \lambda_{\min}(\mathbf{Q}_{nd}) \geq \lambda_{\min}(\mathbf{Q}_d) - |\lambda_{\max}(\mathbf{Q}_{nd})| \\ &= \lambda_{\min}(\mathbf{Q}_d) - \|\mathbf{Q}_{nd}\|_2 = \frac{1}{\lambda_{\max}(\mathbf{Q}_d^{-1})} - \|\mathbf{Q}_{nd}\|_2 = \frac{1}{\|\mathbf{Q}_d^{-1}\|} - \|\mathbf{Q}_{nd}\|_2 > 0\end{aligned}$$

if (i) holds, since

$$\begin{aligned}\|\mathbf{Q}_{nd}\|_2 &= \lambda_{\max}^{1/2}(\mathbf{Q}_{nd}^T \mathbf{Q}_{nd}) = |\lambda_{\max}(\mathbf{Q}_{nd})|, \text{ and} \\ \lambda_{\min}(\mathbf{Q}_{nd}) &= -\lambda_{\min}(-\mathbf{Q}_{nd}) \geq -\lambda_{\max}(-\mathbf{Q}_{nd}) \geq -|\lambda_{\max}(\mathbf{Q}_{nd})|.\end{aligned}$$

*Discussion* Observe from the hypothesis implying matrix  $\mathbf{S}$  (3.8) to be positive definite that by the cancellation of the coefficient matrix  $\mathbf{K}$  of the output feedback controller (3.1), the corresponding stabilizability condition becomes an extra condition on the open-loop system (*i.e.*, an *a priori* condition on the stability of the free system). If, for instance,  $\mathbf{K} = 0$ , then, from (3.8) it follows that  $-\mathbf{A}^T \mathbf{P} - \mathbf{P} \mathbf{A} - n \mathbf{I}_n$ , is a stable matrix, which is a restriction on the state matrix  $\mathbf{A}$ , taking into account that  $\mathbf{P}$  is positive definite. In a similar way, a proper selection of coefficients of matrices  $\mathbf{K}_1, \dots, \mathbf{K}_n$  in (3.2) could cause matrix  $\mathbf{Q}$  to be negative definite, although  $\mathbf{Q}$  is not negative definite with parameters of the open-loop system only. Therefore, in general, it is more convenient to choose the output feedback controller (3.1) with all  $\mathbf{K}, \mathbf{K}_i, i = 1, \dots, n$ , nonzero. An appropriate selection of those controlling matrix gains will ensure asymptotic stability for the closed-loop system provided that a symmetric positive-definite matrix  $\mathbf{P}$  exists with  $\mathbf{Q}$  being negative definite.

#### 4 ALGORITHMS FOR CONTROL SYNTHESIS

In this section, the main algorithms to be implemented in order to synthesize the control law (3.1) for a given system (1.1) are provided at a first level of abstraction. Note that Algorithm 4.2 can be implemented by using the structure of Result 3.2.

##### ALGORITHM 4.1 Control\_Synthesis( $\mathbf{A}, \mathbf{i}, \mathbf{A}_i, \mathbf{b}, \mathbf{C}, \mathbf{h}$ )

```

Read system (1)
Positive_definite(-Q)
If pd = 0 then no solution
Otherwise
  Select (P)
  Select K(P)
  Select K1 to Kn
end

```

##### ALGORITHM 4.2 Positive\_definite(-Q)

```

Check if matrix -Q is positive definite. In case -Q is defined parametrically, this
algorithm gives back ranges of parameters of -Q for which -Q is positive definite.
pd := 1 if there is solution, pd := 0 otherwise
If pd := 1 then
  store parametrical ranges in the structured field PR(-Q)
end

```

**ALGORITHM 4.3** Select **(P)**

Select matrix **P** given by Positive\_Definite( $-\mathbf{Q}$ ) in the field  $\text{PR}(-\mathbf{Q})$  so that system (14) has its poles as far as possible from the right-hand complex halfplane  
end

**ALGORITHM 4.4** Select **K(P)**

Select parameters of **K** for a given **P** positive definite fulfilling **Q** negative definite so that the poles of the associated system (14) are as far as possible from the right-hand complex halfplane  
end

**ALGORITHM 4.5** Select **K<sub>1</sub> to K<sub>n</sub>**

From  $\text{PR}(-\mathbf{Q})$  select those values for **K<sub>1</sub>, ..., K<sub>n</sub>** which cause  $\det[-\mathbf{Q}]$  to be positive and maximum, as a measure for **Q** to be “more” negative definite  
end

**5 ILLUSTRATIVE EXAMPLES**

State feedback control law can be used on a linear system to place its closed loop poles in any desired configuration [10]. This, however, cannot be extended in general to nonlinear cases, including linear delayed systems, which many authors consider a kind of nonlinearity. In this section, a comparison will be established between the traditional approach for the stabilization of delayed systems – *i.e.*, state feedback control – and our approach – *i.e.*, deducing an output feedback control law by using a Lyapunov functional.

*Example 5.1* Consider the following delay-differential system [11]:

$$\dot{x} = \mathbf{A}x(t) + \mathbf{A}_1x(t - \tau) + \mathbf{B}u(t), \quad (5.1a)$$

$$y = \mathbf{C}x. \quad (5.1b)$$

where  $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ;  $\mathbf{A}_1 = \begin{bmatrix} 2/3 & 2/3 \\ 2/3 & 0 \end{bmatrix}$ ;  $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Assume that  $\mathbf{C} = [1 \ 1]$  and  $\tau = 0.2$ . According to the state-feedback approach used in [11], a stabilizing control law is given by  $u = -3\mathbf{B}'x$ . Although mathematically such a law can stabilize system (5.1), when going into practicalities such an approach might become inapplicable, because the state vector, in general, is not directly measurable. Thus an observer (also known as estimator) must be designed in order to reconstruct the instantaneous state by using the output and a model of the system. Although the original feedback controller is a static state feedback, in practice the inclusion of the estimator introduces dynamics in the controller. For example, consider the problem of the error dynamics associated to the state feedback [15]. This procedure can lead to new kinds of unstabilities in the closed system. For instance, assume that the following observer is constructed for system (5.1):

$$\dot{\hat{x}} = (\mathbf{A}^* + \mathbf{M}\mathbf{C}^*)\hat{x}(t) + \mathbf{A}_1^*\hat{x}(t - \tau) + \mathbf{B}^*u(t) - \mathbf{M}y, \quad (5.2)$$

where  $\hat{x}(t)$ , stands for the estimated state, which, in general, does not coincide with the exact state. In addition, matrices  $\mathbf{A}^*$ ,  $\mathbf{C}^*$ ,  $\mathbf{A}_1^*$ ,  $\mathbf{B}^*$ , are a model of the plant, and therefore do not contain its exact parameters. By using observer (5.2), one can interconnect the control law

with the reconstructed state  $\hat{x}(t)$ , to obtain  $u = -3\mathbf{B}'x$ . Figure 5.1 shows the interconnection of the plant, the observer and the control law.

This scheme can give rise to two main new problems regarding the stability of the closed-loop systems. The first one is the open-loop stability of the observer itself, which is not assured in principle. Differential systems with delayed state have – in general – infinite poles because their characteristic equation is transcendental and can have infinite solutions [4]; the observer reproduces an equation of similar nature than that of the plant, and is not simple to guarantee its stability without a specific controller for the observer, which again would require a second observer to reconstruct the state of the first observer, and so on. Thus, the stability of the observer should be checked independently of the stability of the plant. The second problem concerns the establishment of an extra loop, as can be observed in Figure 5.1. The control law designed by using state feedback does not consider the problem of this extra loop, but this is a great problem because in general one does not know which error is being introduced by using the matrices corresponding to the model instead of the exact plant.

Let us now apply our approach to design a stabilizing control law for system (5.1). Such a control law should be of the form  $u = -Ky - K_1y(t - 0.2)$ . Let us postulate  $\mathbf{P} = \mathbf{I}_2$ . Therefore one obtains

$$\mathbf{Q} = \begin{bmatrix} -1 & 0 & \frac{2}{3} - K_1 & \frac{2}{3} \\ 0 & -1 & \frac{2}{3} - K_1 & 0 \\ \frac{2}{3} - K_1 & \frac{2}{3} - K_1 & 4 - 2K & -K \\ \frac{2}{3} & 0 & -K & -4 \end{bmatrix}, \tag{5.3}$$

which is negative definite for many values of (in this example scalar) gains  $K$  and  $K_1$ , provided that they satisfy the following inequalities:

$$\frac{44}{9} - 2K - \frac{8K_1}{3} + 2K_1^2 < 0, \tag{5.4}$$

$$-\frac{424}{81} + 8K - K^2 + \frac{272K_1}{27} - \frac{4KK_1}{3} - \frac{68K_1^2}{9} < 0. \tag{5.5}$$

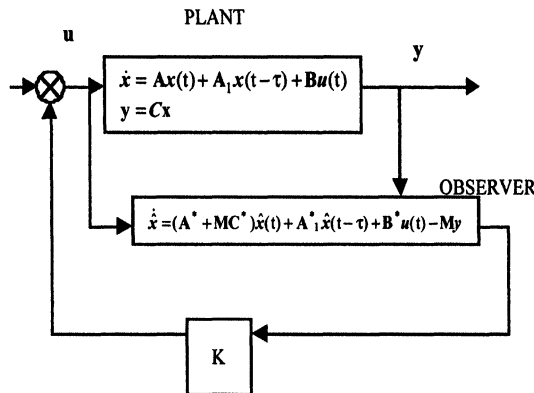


FIGURE 5.1 Interconnection of the observer and control law with the plant.

For instance,  $K = 100$  and  $K_1 = 1$  stabilize the system. Our approach avoids the use of an observer or estimation stage, and therefore it avoids the stability problems which are inherent to that method.

*Example 5.2* The present method may be extended to the case of non linear actuators, if specific extra conditions are added to the output feedback controller (3.1). For instance, let us consider a time-delay system containing a saturating actuator described by the following differential equation

$$\begin{aligned}\dot{x}(t) &= \mathbf{A}_0 x(t) + \mathbf{A}_1 x(t-h) + \mathbf{B} \text{ sat } u(t) \\ y &= \mathbf{C}x\end{aligned}\quad (5.6)$$

where “sat” stands for the saturation function [12] in which the operation range is inside the sector  $[w, 1]$ . In [12] the following state feedback control law is considered

$$u(t) = -\mathbf{F}_0 x(t) - \mathbf{F}_1 x(t-h), \quad (5.7)$$

and in the mentioned work it is demonstrated that the controlled saturating time-delay system (5.6) with control law (5.7) is stable if the following relation holds

$$\frac{k\{\|\tilde{\mathbf{A}}_1\| + (1-w)\|\mathbf{B}\| \cdot \|\mathbf{F}_1\|/2\}}{\eta} < \left\{ \frac{1 - k(1-w)\|\mathbf{B}\| \cdot \|\mathbf{F}_0\|}{2\eta} \right\}, \quad (5.8)$$

where  $k \geq 1$ ,  $\eta > 0$ , are such that

$$\|\exp(\tilde{\mathbf{A}}_0 t)\| \leq k \exp(-\eta t), \quad t \geq 0, \quad (5.9)$$

being

$$\tilde{\mathbf{A}}_0 = \mathbf{A}_0 - \frac{(1+w)\mathbf{B}\mathbf{F}_0}{2}, \quad \tilde{\mathbf{A}}_1 = \mathbf{A}_1 - \frac{(1+w)\mathbf{B}\mathbf{F}_1}{2} \quad (5.10)$$

Let us rewrite the state feedback control law (5.7) as an output feedback control law

$$u(t) = -\mathbf{F}_0 x(t) - \mathbf{F}_1 x(t-h) = -\mathbf{F}_0 \mathbf{C}^{-1} y(t) - \mathbf{F}_1 \mathbf{C}^{-1} y(t-h), \quad (5.11)$$

which is formally identical to (3.1) if one identifies

$$\mathbf{K} = -\mathbf{F}_0 \mathbf{C}^{-1}; \mathbf{K}_1 = -\mathbf{F}_1 \mathbf{C}^{-1}. \quad (5.12)$$

Therefore, introducing substitutions (5.12) into condition (5.8) would lead to an extra condition for control law (3.1). This extra condition should be taken into account together with the conditions stated in Result 3.1, in the presence of saturating actuators. In such case, the utility of our method is that it can assure stability even for time-delay systems with more than one time-delay in the state vector, while in [12] only one time-delay in the state vector is considered.



## 6 CONCLUSIONS

This paper has introduced some results which provide sufficient conditions for the stabilizability of a time-delayed system with multiple delays in its state vector, by using a Lyapunov-candidate function. The proposed approach uses an input–output description of the open-loop system in order to synthesize the control signal, a fact which overcomes the need for an estimation stage.

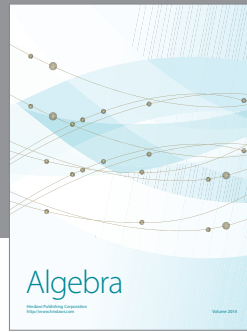
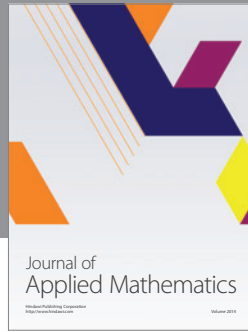
Extra conditions on the control law deduced accordingly to the specific nonlinearities present in the actuator may extend the method to the non linear case, as was illustrated in the last section.

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