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Global Regular Solutions for The Nonhomogeneous Carrier Equation

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We study in a $n+1$-dimensional cylinder Q global solvability of the mixed problem for the nonhomogeneous Carrier equation

$$u_{tt} - M(x, t, \|u(t)\|^2)\Delta u + g(x, t, u_t) = f(x, t)$$

without restrictions on a size of initial data and $f(x, t)$. For any natural n, we prove existence, uniqueness and the exponential decay of the energy for global generalized solutions. When n = 2, we prove $C^\infty(Q)$-regularity of solutions.

Key words: Carrier equation; Global smooth solutions; Existence

1 INTRODUCTION

G. Carrier in [2] derived the equation which models vibrations of an elastic string when changes in tension are not small

$$\rho u_{tt} - \left(1 + \frac{EA}{LT_0}\int_0^L u^2\,\mathrm{d}x\right)u_{xx} = 0. \tag{1.1}$$

Here $u(x,t)$ is the x-derivative of the deformation, T_0 is the tension in the rest position, E is the Young's modulus, A is the cross-section of a string, L is the length of a string and ρ is the density of a material. Clearly, if properties of a material vary with x and t, then we have a hyperbolic quasilinear equation of the type

$$u_{tt} - M(x, t, \|u(t)\|^2)\Delta u = 0.$$

Moreover, a material of a string has internal friction that can be a nonlinear function of u_t and can also depend on x,t. In the present paper, we study the existence of global solutions for the nonhomogeneous Carrier equation,

$$u_{tt} - M(x, t, \|u(t)\|^2)\Delta u + g(x, t, u_t) = f(x, t). \tag{1.2}$$

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We must say that dependence of $M(x, t, \|u(t)\|^2)$ on x,t brings essential difficulties in proving global a priori estimates; and there are not published results on the existence of global solutions for the nonhomogeneous Kirchhoff-Carrier equation without restriction on the size of solutions. Of course, there are results on local in time solutions for general Kirchhoff-Carrier type equations, see [4]. When functions $M(\lambda)$ and $g(u_t)$ do not depend on x,t, we proved in [3] the existence of global solutions to the mixed problem for the nonhomogeneous damped Carrier equation without restrictions on a size of initial data.

Here we consider for (1.2) the mixed problem with the Dirichlet boundary conditions and prove existence and uniqueness of global solutions without restrictions on a size of initial data and $f(x, t)$. Moreover, we prove the exponential decay of the energy.

Existence of global regular solutions for quasilinear hyperbolic equations with a nonlinear damping, when initial data are large, was proved in [5]. Our paper consists of 3 main chapters. In chapter 2, we prove the existence of global strong solutions for any space dimension n and for large initial data. In chapter 3, using ideas of Nakao [8], under suitable restrictions for n, $g(u_t)$, $M(x, t, \|u(t)\|^2)$, we prove the exponential decay of the energy. From the physical point of a view, this stability result corresponds to the case of thin strings when the ratio of the cross-section of a string to its length is sufficiently small.

In chapter 4, we prove that for $n = 2$ strong solutions are smooth even for large initial data and $f(x, t)$. In chapter 5, we explain assumptions that we impose to prove our results and give some simple examples of the Carrier type equations in order to illustrate the practicality of them.

2 STRONG SOLUTIONS

Let $x \in D \subset R^n$, where D is a bounded domain with a sufficiently smooth boundary Γ. In $Q = D \times (0, T)$ we consider the following problem,

$$u_{tt} - M(x, t, \|u(t)\|^2)\Delta u + g(x, t, u_t) = f(x, t), \tag{2.1}$$

$$u|_{\Gamma \times (0,T) = S} = 0, \tag{2.2}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x). \tag{2.3}$$

Here and in the sequel, we use standard functional spaces [6], otherwise necessary definitions will be given such as

$$(u, v)(t) = \int_D u(x, t)v(x, t)\,\mathrm{d}x, \quad \|u(t)\|^2 = (u, u)(t),$$

M_t, M_{x_i} denote partial derivatives of the function $M(x, t, \|u(t)\|^2)$ with respect to corresponding parameters and $M_\lambda = (\partial/\partial\lambda)M(x, t, \lambda)$.

ASSUMPTIONS 2

(2.1) $M \in C^1(Q \times R^+) \cap C(\bar{Q} \times R^+), R^+ = [0, \infty)$.
(2.2) There exists a continuous function $\phi(\lambda)$ such that for all $(x, t) \in \bar{Q}$, $0 < m_0 \leq \phi(\lambda) \leq M(x, t, \|u(t)\|^2) \leq C_1\phi(\lambda)$.
(2.3) $|M_\lambda \lambda^{1/2}| \leq k_0 M$.
(2.4) $|M_t(x, t, \lambda)| \leq C_2 M(x, t, \lambda)$.

(2.5) $g(x,t,u_t) \in C^1(Q \times R^1)$, $|g| + |g_t| + \sum_{i=1}^n |g_{x_i}| \le C_3(1 + |u_t|^{\rho+1})$.
(2.6) $g_\lambda(x,t,\lambda) \ge C_4|\lambda|^\rho$, $g(x,t,0) = 0$.

Here $k_0, C_1, \dots, C_4, m_0$ are positive constants, $\rho > 1$.

THEOREM 2.1 *Let assumptions* 2 *hold*; $u_0 \in H^2(D) \cap H_0^1(D)$, $u_1 \in H_0^1(D) \cap L^{2\rho+2}(D)$. *Then for any* $f \in H^1(0,T;L^2(D))$ *there exists a unique strong solution to* (2.1)–(2.3) *from the class*:

$$u \in L^\infty(0,T;H^2(D) \cap H_0^1(D)),$$

$$u_t \in L^\infty(0,T;H_0^1(D)) \cap L^{\rho+2}(Q),$$

$$u_{tt} \in L^\infty(0,T:L^2(D)).$$

Proof Uniqueness of a strong solution can be proved by the standard way [3]. Here we prove only the existence part of theorem 2.1 constructing approximate solutions to (2.1)–(2.3) in the form,

$$u^N(x,t) = \sum_{j=1}^N g_j^N(t) w_j(x),$$

$$\Delta w_j + \lambda_j w_j = 0 \text{ in } D, \quad w_j|_\Gamma = 0, \tag{2.4}$$

where $g_j^N(t)$ are solutions to the following Cauchy problem,

$$\begin{aligned}(Lu^N, w_j)(t) = &\left(\frac{u_{tt}^N}{M(x,t,\|u_{(t)}^N\|^2)}, w_j\right)(t) + (\nabla u^N, \nabla w_j)(t)\\ &+\left(\frac{g(x,t,u_t^N)}{M(x,t,\|u_{(t)}^N\|^2)}, w_j\right)(t) = \left(\frac{f}{M(x,t,\|u_{(t)}^N\|^2)}, w_j\right)(t),\end{aligned}$$

$$g_j^N(0) = (u_0, w_j)(0), \quad g_{jt}^N(0) = (u_1, w_j)(0), \quad j = 1, \dots, N. \tag{2.5}$$

Due to assumptions 2, problem (2.5) has solutions on some interval $(0, T_N)$; and we need a priori estimates in order to prove that $T_N = T$ and to pass to the limit in (2.5) as $N \to \infty$. In the sequel, we will omit the index N remembering that we work with approximations (2.4), (2.5). Also, we will not write independent variables (x,t) in functions

$$M(x,t,\|u(t)\|^2), \quad g(x,t,u_t).$$

First Estimate

Substituting in (2.5) $w_j = 2u_t$, and omitting the index N we obtain

$$\begin{aligned}&\frac{d}{dt}[(u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2] + 2(g, u_t M^{-1})(t) + 2(M_\lambda M^{-2}(u,u_t)(t), u_t^2)(t)\\ &\quad + (M_t M^{-2}, u_t^2)(t) = 2(f, u_t M^{-1})(t).\end{aligned} \tag{2.6}$$

Using assumption 2.3, we get

$$I_1 = 2|(M_\lambda M^{-2}(u, u_t)(t), u_t^2)(t)| \le 2k_0\|u_t(t)\|(u_t^2, M^{-1})(t).$$

Now assumption 2.2 and the Young inequality imply

$$I_1 \le \frac{2k_0}{\phi(\|u(t)\|^2)}\|u_t(t)\|^3 \le \frac{\epsilon\|u_t(t)\|^{\rho+2}_{L^{\rho+2}(D)} + C(\epsilon)}{\phi(\|u(t)\|^2)},$$

where ϵ is an arbitrary positive number.

The same arguments and assumption 2.6 give

$$I_2 = 2(g, M^{-1}u_t) \ge \frac{2C_4}{c_1\phi(\|u(t)\|^2)}\|u_t(t)\|^{\rho+2}_{L^{\rho+2}(D)}.$$

Substituting I_1, I_2 into (2.6), choosing $\epsilon > 0$ sufficiently small and taking into account assumption 2.4, we come to the inequality,

$$\begin{aligned}\frac{d}{dt}&[(u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2] + \|u_t(t)\|^{\rho+2}_{L^{\rho+2}(D)}\phi^{-1}(\|u(t)\|^2)\\ &\le C(1 + (u_t^2, M^{-1})(t) + \|f(t)\|^2),\end{aligned} \tag{2.7}$$

where the constant C does not depend on t, N, but can depend on T. By Gronwall's lemma,

$$((u_t^N)^2, M^{-1})(t) + \|\nabla u^N(t)\|^2 \le C \quad \forall t \in (0, T).$$

From here, $\|u^N(t)\| \le C$ and

$$m_0 \le M(x, t, \|u^N(t)\|^2) \le M_1 < \infty. \tag{2.8}$$

Returning to (2.7), we have

$$\|u_t^N(t)\|^2 + \|\nabla u^N(t)\|^2 + \int_0^t \|u_s^N(s)\|^{\rho+2}_{L^{\rho+2}(D)}\, ds \le C \quad \forall t \in (0, T). \tag{2.9}$$

Second Estimate

Derivating (2.5) and substituting $w_j = u_{tt}^N$, we obtain

$$\begin{aligned}\frac{d}{dt}&[(u_{tt}^2, M^{-1})(t) + \|\nabla u_t(t)\|^2] - 2(M_\lambda M^{-2}(u, u_t)(t), u_{tt}^2)(t)\\ &- (M_t, M^{-2}u_{tt}^2)(t) + 2(g_{u_t}, M^{-1}u_{tt}^2)(t) + 2(g_t, M^{-1}u_{tt})(t)\\ &= 2(g, u_{tt}M^{-2}(M_t + 2M_\lambda(u, u_t)(t)))(t)\\ &+ 2([f_tM^{-1} - M_tM^{-2}f - 2M_\lambda M^{-2}(u, u_t)(t)f], u_{tt})(t).\end{aligned}$$

Taking into account assumptions 2.5, 2.6 and estimate (2.9), we transform this equality to the inequality,

$$\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \le C\left(1 + \int_0^t [\|u_{ss}(s)\|^2 + \|f(s)\|^2 + \|f_s(s)\|^2]\, \mathrm{d}s\right),$$

whence,

$$\|u_{tt}^N(t)\| + \|\nabla u_t^N(t)\| \le C, \tag{2.10}$$

where the constant C does not depend on N, t.

Third Estimate

Due to (2.4), we rewrite (2.5) in the form,

$$\begin{aligned} -(Lu, \Delta u_t)(t) = (\nabla[u_{tt}M^{-1}], \nabla u_t)(t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u(t)\|^2 \\ + (\nabla[gM^{-1}], \nabla u_t)(t) = -(fM^{-1}, \Delta u_t)(t). \end{aligned} \tag{2.11}$$

The first term in (2.11) transform as follows,

$$\begin{aligned} I_1 = (\nabla[u_{tt}M^{-1}], \nabla u_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u_t|^2, M^{-1})(t) \\ + \frac{1}{2}(|\nabla u_t|^2, M^{-2}[M_t + M_\lambda 2(u, u_t)(t)])(t) \\ - \left(u_{tt}, \sum_{i=1}^n M_{x_i}M^{-2}, \nabla u_t\right)(t). \end{aligned}$$

Taking into account estimates (2.9), (2.10), we get

$$I_1 \ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u_t|^2, M^{-1})(t) - C. \tag{2.12}$$

On the other hand, assumptions 2.5, 2.6 and estimates (2.9), (2.10) imply

$$\begin{aligned} I_2 = (\nabla[gM^{-1}], \nabla u_t) = \left(\sum_{i=1}^n g_{x_i}M^{-1}, \nabla u_t\right) + (g_{u_t}M^{-1}, |\nabla u_t|^2) - \left(\sum_{i=1}^n M_{x_i}M^{-2}g, \nabla u_t\right) \\ \ge C_4(|u_t|^\rho|\nabla u_t|^2, M^{-1}) - C(1 + |u_t|^\rho|\nabla u_t|, M^{-1}). \end{aligned}$$

Using Young's inequality, and (2.9), we obtain

$$I_2 \ge \frac{1}{2}C_4(|u_t|^\rho|\nabla u_t|^2, M^{-1}) - C([1 + |\nabla u_t|^2 + |u_t|^{\rho+2}], M^{-1}). \tag{2.13}$$

Here all the constants do not depend on N, t.

The last term in (2.11) we treat in the manner,

$$I_3 = -(fM^{-1}, \Delta u_t)(t) = -\frac{\mathrm{d}}{\mathrm{d}t}(fM^{-1}, \Delta u)(t) + (f_t M^{-1}, \Delta u)(t) - (fM^{-2}[M_t + 2(u, u_t)(t)M_\lambda], \Delta u)(t).$$

With the help of (2.9) we reduce it to the inequality,

$$I_3 \le -\frac{\mathrm{d}}{\mathrm{d}t}(fM^{-1}, \Delta u)(t) + C[\|\Delta u(t)\|^2 + \|f(t)\|^2 + \|f_t(t)\|^2]. \quad (2.14)$$

Substituting (2.12)–(2.14) into (2.11), we come to the inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}[(|\nabla u_t|^2, M^{-1})(t) + \|\Delta u(t)\|^2] \le -2\frac{\mathrm{d}}{\mathrm{d}t}(fM^{-1}, \Delta u)(t) + C[1 + \|f(t)\|^2 + \|f_t(t)\|^2 + \|\Delta u(t)\|^2 + \|u_t(t)\|^{\rho+2}_{L^{\rho+2}(D)}].$$

Integrating this inequality and exploiting (2.9), (2.10) and Gronwall's lemma, we obtain

$$\|\Delta u(t)\| \le C \quad \forall t \in [0, \infty). \quad (2.15)$$

Remembering that (2.9), (2.10), (2.15) are estimates of Galerkin approximations (2.4), uniform in N, we can pass to the limit in (2.5) as $N \to \infty$, therewith to prove the existence result of Theorem 2.1. ■

3 STABILITY

To prove our stability result, we use the method of Nakao based on the following lemma, see [8].

LEMMA 3.1 *Let $\phi(t)$ be a nonnegative function on $[0, \infty)$ satisfying*

$$\sup_{t \le \tau < t+1} \phi(\tau) \le K_0(\phi(t) - \phi(t+1)), \quad K_0 > 0.$$

Then it follows that

$$\phi(t) \le C\phi(0)\mathrm{e}^{\lambda t}, \quad \lambda > 0.$$

ASSUMPTIONS 3

(3.1) $M(x, t, \lambda) \in C^1(D \times R^+ \times R^+)$,
(3.2) $g(x, t, u_t)u_t \ge \alpha u_t^2 + \beta|u_t|^{\rho+2}$,
(3.3) $|g(x, t, u_t)| \le C_2(|u_t| + |u_t|^{\rho+1})$,
(3.4) There exists a continuous function $\phi(\lambda)$ such that $0 < m_0 \le \phi(\lambda) \le M(x, t, \lambda) \le C_1\phi(\lambda)$,
(3.5) $|M_\lambda|\lambda^{1/2} \le k_0 M$, where $k_0 \le (\alpha\rho/\rho - 1)^{\rho-1}(\beta/(C_1 4^{(2\rho-1)/\rho}(\mu_D)^{(2\rho-1)/2}))$,
(3.6) $M_t(x, t, \lambda) \ge 0$,

where $\rho > 1$; α, β, C_1, C_2 are positive constants; μ_D is the Lebesgue measure of D.

THEOREM 3.1 *Let* $f = 0$, *assumptions* 3 *hold*; $1 < \rho < 4/(n-2)$ *if* $n > 2$, *and* $\rho > 1$ *when* $n = 1, 2$. *Then there exist positive constants* K *and* θ *such that strong solutions to* (2.1)–(2.3) *satisfy the following inequality,*

$$\|u_t(t)\|^2 + \|u(t)\|^2_{H^1_0(D)} \leq K\,\mathrm{e}^{-\theta t}.$$

Proof Obviously, conditions of theorem 3.1 guarantee the existence of strong solutions to (2.1)–(2.3) for all finite $t > 0$. From the identity,

$$(u_{tt}M^{-1}, u_t)(t) + (\nabla u, \nabla u_t)(t) + (gM^{-1}, u_t)(t) = 0,$$

integrating by parts and taking into account assumption 3.3, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&[(u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2] + (u_t^2, M_tM^{-2})(t) + 2\alpha(u_t^2, M^{-1})(t)\\ &+ 2\beta(|u_t|^{\rho+2}, M^{-1})(t) + 4(u_t^2, M_\lambda M^{-2})(t)(u, u_t)(t) \leq 0.\end{aligned} \tag{3.1}$$

As in section 2, using assumptions 3.4, 3.5 and Young's inequality, we get

$$\begin{aligned}I_1 &= 4|(u_t^2, M_\lambda M^{-2})(t)(u, u_t)(t)| \leq \frac{4k_0}{\phi(\|u(t)\|^2)}\|u_t(t)\|^3\\ &\leq \frac{4\epsilon\mu_D^{1/2\rho}}{\rho\phi(\|u(t)\|^2)}\|u_t(t)\|^{\rho+2}_{L^{\rho+2}(D)}\\ &\quad+\frac{4k_0^{\rho/\rho-1}(\rho-1)}{\epsilon^{\rho/\rho-1}\rho\phi(\|u(t)\|^2)}\|u_t(t)\|^2 \quad \forall\epsilon > 0.\end{aligned}$$

On the other hand, assumption 3.4 yields

$$\begin{aligned}I_2 &= 2\alpha(u_t^2, M^{-1})(t) \geq \frac{2\alpha}{C_1\phi(\|u(t)\|^2)}\|u_t(t)\|^2,\\ I_3 &= 2\beta(|u_t|^{\rho+2}, M^{-1})(t) \geq \frac{2\beta}{C_1\phi(\|u(t)\|^2)}\|u_t(t)\|^{\rho+2}_{L^{\rho+2}(D)}.\end{aligned}$$

Substituting $I_1 - I_3$ into (3.1) and taking into account assumption 3.6, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&[(u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2] + \left(\frac{2\beta}{C_1} - \frac{4\epsilon\mu_D^{1/2\rho}}{\rho}\right)\frac{\|u_t(t)\|^{\rho+2}_{L^{\rho+2}(D)}}{\phi(\|u(t)\|^2)}\\ &+\left(\frac{2\alpha}{C_1} - \frac{4(\rho-1)k_0^{\rho/\rho-1}}{\rho\epsilon^{\rho/\rho-1}}\right)\frac{\|u_t(t)\|^2}{\phi(\|u(t)\|^2)} \leq 0.\end{aligned} \tag{3.2}$$

Putting $\epsilon = (\rho\beta/4C_1\mu_D^{\rho/2})$ and using assumption 3.5, we come to the inequality,

$$\frac{d}{dt}[(u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2] + \frac{\beta\|u_t(t)\|_{L^{\rho+2}(D)}^{\rho+2}}{C_1\phi(\|u(t)\|^2)} + \frac{\alpha\|u_t(t)\|^2}{C_1\phi(\|u(t)\|^2)} \leq 0. \tag{3.3}$$

Denoting

$$\psi(t) = (u_t^2, M^{-1})(t) + \|\nabla u(t)\|^2,$$

we conclude from (3.3) that $\psi(t)$ is a nonincreasing positive function and

$$\psi(t) + \frac{\beta}{C_1}\int_0^t \frac{\|u_s(s)\|_{L^{\rho+2}(D)}^{\rho+2}}{\phi(\|u(s)\|^2)}\, ds + \frac{\alpha}{C_1}\int_0^t \frac{\|u_s(s)\|^2}{\phi(\|u(s)\|^2)}\, ds \leq \psi(0). \tag{3.4}$$

Hence, by Poincare's inequality,

$$\|\nabla u(t)\| \leq \psi^{1/2}(t) \leq \psi^{1/2}(0),$$

$$\|u(t)\| \leq C_D\psi^{1/2}(0) \quad \forall t \geq 0, \tag{3.5}$$

where the constant C_D depends only on D. Due to assumption 3.4.

$$\phi(\|u(t)\|^2) \leq C_\phi < \infty \tag{3.6}$$

and it follows from (3.4) that

$$\int_t^{t+1} \|u_s(s)\|_{L^{\rho+2}(D)}^{\rho+2}\, ds + \int_t^{t+1} \|u_s(s)\|^2\, ds \leq C_3F^2(t), \tag{3.7}$$

where C_3 is a positive constant depending on α, β, C_1, C_2, C_ϕ and

$$F^2(t) = \psi(t) - \psi(t+1). \tag{3.8}$$

By the Mean Value Theorem for integrals, there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$|u_t(t_i)| \leq C_4F(t), \quad i = 1, 2. \tag{3.9}$$

Multiplying the identity

$$Lu = \frac{u_{tt}}{M} - \Delta u + \frac{g}{M} = 0$$

by u and integrating over $D \times (t_1, t_2)$, we obtain

$$\int_{t_1}^{t_2} \|\nabla u(s)\|^2 \, ds = (u_t, uM^{-1})(t_1) - (u_t, uM^{-1})(t_2) + \int_{t_1}^{t_2} (u_s^2, M^{-1})(s) \, ds - \int_{t_1}^{t_2} (u, u_s M_s M^{-2})(s) \, ds - 2\int_{t_1}^{t_2} (u_s, u)(s)(u_s, uM_\lambda M^{-2})(s) \, ds - \int_{t_1}^{t_2} (g, uM^{-1})(s) \, ds.$$

Taking into account assumption 3.4, estimates (3.5)–(3.7), (3.9), we transform this equality to the following inequality,

$$\int_{t_1}^{t_2} \|\nabla u(s)\|^2 \, ds \le \epsilon \sup_{t \le s \le t+1} \psi(s) + C_5(\epsilon) F^2(t) + C_6 \int_{t_1}^{t_2} |(g, u)(s)| \, ds \ \ \forall \epsilon > 0. \tag{3.10}$$

Using assumption 3.3; (3.9) and embedding $H_0^1(D) \subsetneq L^{\rho+2}(D)$, we obtain

$$\begin{aligned} C_6 \int_{t_1}^{t_2} |(g, u)(s)| \, ds &\le C_7 \int_{t_1}^{t_2} |([\,|u_s(s)| + |u_s(s)|^{\rho+1}], u)(s)| \, ds \\ &\le C_8 \sup_{t \le s \le t+1} \|u(s)\|_{H_0^1(D)} \int_{t_1}^{t_2} [\|u_s(s)\| + \|u_s(s)\|_{L^{\rho+2}(D)}^{\rho+1}] \, ds \\ &\le C_9 \sup_{t \le s \le t+1} \|\nabla u(s)\| [F^{2(\rho+1)/(\rho+2)}(t) + F(t)] \\ &\le \epsilon \sup_{t \le s \le t+1} \psi(s) + C_{10}(\epsilon)[F^2(t) + F^{4(\rho+1)/(\rho+2)}(t)], \end{aligned}$$

where ϵ is the same as in (3.10). Substituting this expression into (3.10), we get

$$\int_{t_1}^{t_2} \|\nabla u(s)\|^2 \, ds \le 2\epsilon \sup_{t \le s \le t+1} \psi(s) + C_{11}[F^2(t) + F^{4(\rho+1)/(\rho+2)}(t)],$$

whence, by the definition of $\psi(t)$ and (3.7),

$$\int_{t_1}^{t_2} \psi(s) \, ds \le \epsilon C_{12} \sup_{t \le s \le t+1} \psi(s) + C_{13}(\epsilon)[F^2(t) + F^{4(\rho+1)/(\rho+2)}(t)].$$

By the Mean Value Theorem,

$$\psi(\bar{t}) \le 2\epsilon C_{12} \sup_{t \le s \le t+1} \psi(s) + 2C_{13}(\epsilon)[F^2(t) + F^{4(\rho+1)/(\rho+2)}(t)],$$

where $\bar{t} \in (t_1, t_2)$.

Integrating over $D \times (\bar{t}, t)$ the following identity,

$$(u_{tt} M^{-1} - \Delta u + gM^{-1}) u_t = 0,$$

and acting as above, we obtain

$$\psi(t) \leq \epsilon C_{14} \sup_{t \leq s \leq t+1} \psi(s) + C_{15}(\epsilon)[F^2(t) + F^{4(\rho+1)/(\rho+2)}(t)].$$

Putting ($\epsilon = 1/2C_{14}$) and recalling that $\mathrm{d}/\mathrm{d}t\,\psi(t) \leq 0$, we come to the inequality,

$$\sup_{t \leq s \leq t+1} \psi(s) \leq CF^2(t)[1 + F^{2\rho/(\rho+2)}(t)].$$

Exploiting definition of $F^2(t)$, its boundness and lemma 3.1, we get

$$\psi(t) \leq K\,\mathrm{e}^{-\theta t},$$

where K and θ are positive constants.

This proves theorem 3.1. ■

Remark 3.1 Assumption 3.5 means that k_0 in (1.1) is small. Recalling that $k_0 = (EA/LT_0)$, where A is the cross-section area of a string and L its length, we conclude that for a small ratio A/L (the case of a thin string), k_0 is also small.

4 SMOOTH SOLUTIONS

In this section we prove that for $n = 2$, under some restrictions on the growth of $g(u_t)$, global strong solutions to (2.1)–(2.3) are smooth solutions provided functions $M(x, t, \|u(t)\|^2)$, $g(x, t, u_t)$, $f(x, t)$ and u_0, u_1 are smooth. Obviously, it is sufficient to prove that $u \in H^k(Q)$ where $k \geq 3$ is an arbitrary natural number.

ASSUMPTIONS 4

(4.1) $M \in C^k(\bar{Q} \times R^+)$, $g \in C^k(\bar{Q} \times R^1)$, $k \geq 3$.
(4.2) There exists a continuous function $\phi(\lambda)$ such that for all $x, t, \in \bar{Q}$, $0 < m_0 \leq \phi(\lambda) \leq M(x, t, \lambda) \leq C_1\phi(\lambda)$.
(4.3) $|M_\lambda(x, t, \lambda)||\lambda|^{1/2} \leq k_0 M(x, t, \lambda)$, $|M_t(x, t, \lambda)| \leq C_2 M(x, t, \lambda)$.
(4.4) $|g(x, t, u_t)| + |g_t(x, t, u_t)| + |g_{tt}(x, t, u_t)| \leq C_3(1 + |u_t|^{\rho+1})$.
(4.5) $g(x, t, 0) = 0$.
(4.6) $g_\lambda(x, t, \lambda) \geq C_4|\lambda|^\rho$, $g^2_{\lambda\lambda} \leq C_5(1 + g_\lambda)$; $|g_{\lambda t}| \leq C_6 g_\lambda$. Here $m_0, k_0, C_1, \ldots, C_6$ are positive constants, $\rho > 1$.
(4.7) Compatibility conditions: initial data u_0, u_1 and Eq. (2.1) satisfy at $t = 0$, ∂D the following conditions,

$$\partial_t^j[f + M\Delta u - g] = 0, \quad j = 0, \ldots, k-3.$$

THEOREM 4.1 *Let $n = 2$, $k \geq 3$ be a natural number and assumptions 4 hold. Given $u_0 \in H^k(D) \cap H^1_0(D)$, $u_1 \in H^{k-1}(D) \cap H^1_0(D)$ when k is even and $u_0 \in H^k(D) \cap H^1_0(D)$, $u_1 \in H^k(D) \cap H^1_0(D)$ when k is odd. If $f \in H^{k-1}(Q)$, then there exists a unique solution to (4.1)–(4.3), $u \in H^k(Q)$.*

Proof To prove this theorem we use again approximations 2.4. Actually, under assumptions 4, theorem 2.1 provides a priori estimates (2.9), (2.10), (2.15). More necessary estimates will be proved in the form of lemmas 4.1, 4.2.

LEMMA 4.1 *Under assumptions* 4, *the following inequalities hold*:

$$\|u^N_{ttt}(t)\| + \|\Delta u^N_t(t)\| \leq C(\|f\|_{H^2(0,T;L^2(D))} + \|u_1\|_{H^2(D)} + \|u_0\|_{H^2(D)}), \ \sup_{\bar{Q}} |u^N_t(x,t)| \leq C.$$

Proof Considering the scalar product,

$$((Lu^N)_{tt}, u^N_{ttt})(t) = (\partial^2_t(fM^{-1}(x,t,\|u^N(t)\|^2)), u^N_{ttt})(t),$$

we come, omitting N, to the equality,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[\|\nabla u_{tt}(t)\|^2 + (u^2_{ttt}, M^{-1})(t)] + 3(u^2_{ttt}, \partial_t M^{-1})(t) \\ &\quad + 2(u_{ttt}u_{tt}, \partial^2_t M^{-1})(t) + 2(\partial^2_t g, u_{ttt}M^{-1})(t) \\ &\quad + 4(\partial_t g, \partial_t M^{-1}, u_{ttt})(t) + 2(g\partial^2_t M^{-1}, u_{ttt})(t) \\ &= 2([f_{tt}M^{-1} + 2f_t\partial_t M^{-1} + f\partial^2_t M^{-1}], u_{ttt})(t). \end{aligned} \tag{4.1}$$

In the sequel, we will need estimates for the following expressions:

$$\partial_t M^{-1} = -[M_t(x,t,\|u(t)\|^2) + M_\lambda(x,t,\|u(t)\|^2)2(u,u_t)(t)]M^{-2},$$

where $M_\lambda = M_\lambda(x,t,\lambda(t))$;

$$\begin{aligned} \partial^2_t M^{-1} &= \partial_t(\partial_t M^{-1}) = -[M_{tt} + 4M_{\lambda t}(u,u_t)(t) \\ &\quad + 4M_{\lambda\lambda}(u,u_t)^2(t)]M^{-2} - 2M_\lambda[\|u(t)\|^2 + (u,u_{tt})(t)]M^{-2} \\ &\quad + 2[M_t + 2M_\lambda(u,u_t)(t)]^2 M^{-3}. \end{aligned}$$

Estimates (2.9), (2.10) yield

$$|\partial_t M^{-1}| + |\partial^2_t M^{-1}| \leq C. \tag{4.2}$$

We have

$$I_1 = \partial_t g(x,t,u_t) = g_t + g^{(1)}u_{tt}, \quad g^{(i)} = \partial^i_\eta g(x,t,\eta).$$

$$I_2 = \partial^2_t g(x,t,u_t) = g_{tt} + 2g^{(1)}_t u_{tt} + g^{(2)}u^2_{tt} + g^{(1)}u_{ttt}.$$

We treat the 4th term in (4.1) as follows,

$$I_3 = 2(\partial_t^2 g(x,t,u_t), u_{ttt}M^{-1}) = 2(g_{tt}M^{-1}, u_{ttt}) \\ + 4(g_t^{(1)}u_{tt}M^{-1}, u_{ttt}) + (2(g^{(2)}u_{tt}^2M^{-1}, u_{ttt}) + 2(g^{(1)}M^{-1}, u_{ttt}^2)).$$

Taking into account (4.2) and assumption 4.6, we obtain

$$I_{31} = 2(g^{(2)}u_{tt}^2M^{-1}, u_{ttt}) + 2(g^{(1)}M^{-1}, u_{ttt}^2) \\ \geq (2-\epsilon)(g^{(1)}M^{-1}, u_{ttt}^2) - C(\epsilon)[(M^{-1}, u_{ttt}^2) + \|u_{tt}\|_{L^4(D)}^4],$$

where ϵ is an arbitrary positive number.

Analogously,

$$I_{32} = 4(g_t^{(1)}u_{tt}M^{-1}, u_{ttt}) \leq \epsilon(g^{(1)}M^{-1}, u_{ttt}^2) + C(\epsilon)(g^{(1)}, u_{tt}^2);$$

$$I_{33} = 2(g_{tt}M^{-1}, u_{ttt}) \leq (M^{-1}, u_{ttt}^2) + (M^{-1}, g_{tt}^2).$$

Due to assumption 4.4 and estimates (2.9), (2.10),

$$I_3 \geq (2-3\epsilon)(g^{(1)}M^{-1}, u_{ttt}^2) - C(\epsilon)[1 + (M^{-1}, u_{ttt}^2) + \|u_{tt}\|_{L^4(D)}^4].$$

When $n = 2$, the Gagliardo–Nirenberg inequality and (2.10) imply

$$\|u_{tt}\|^4 \leq C\|u_{tt}\|^2\|\nabla u_{tt}\|^2 \leq C\|\nabla u_{tt}\|^2,$$

hence, putting $\epsilon = 1/3$, we obtain

$$I_3 \geq (g^{(1)}M^{-1}, u_{ttt}^2) - C[1 + (M^{-1}, u_{ttt}^2) + \|\nabla u_{tt}\|^2]. \tag{4.3}$$

Substituting (4.2), (4.3) into (4.1), we reduce it to the inequality,

$$\frac{d}{dt}[(M^{-1}, u_{ttt}^2)(t) + \|\nabla u_{tt}\|^2] \leq C[\|f(t)\|^2 + \|f_t(t)\|^2 \\ + \|f_{tt}(t)\|^2 + \|\nabla u_{tt}(t)\|^2 + (M^{-1}, u_{ttt}^2)(t) + 1].$$

Integrating this inequality and making use of Gronwall's lemma, we get

$$\|u_{ttt}(t)\|^2 + \|\nabla u_{tt}(t)\|^2 \leq C \quad \forall t \in (0, T). \tag{4.4}$$

Rewriting the system,

$$((Lu^N)_t, w_j)(t) = ((fM^{-1})_t, w_j)(t), \quad j = 1, \ldots, n$$

in the form.

$$(\nabla u_t^N, \nabla w_j)(t) = ([u_{ttt}^N - u_{tt}^N \partial_t M^{-1} - M^{-1}\partial_t g - g\partial_t M^{-1} + \partial_t(fM^{-1}]w_j)(t),$$

we may conclude that

$$-\Delta(P_N u_t(t)) = F(t) \in L^2(D),$$

where P_N is a projectional operator: $L^2(D) \to [w_1, \dots, w_N]$. Hence

$$u_t^N \in H^2(D) \cap H_0^1(D). \tag{4.5}$$

In the case $n = 2$, it yields

$$u_t^N \in C(\bar{D}) \tag{4.6}$$

uniformly in $N, t \in (0, T)$. ■

LEMMA 4.2 *Let for any natural $l \geq 3$ the following inequality holds,*

$$\|\partial_t^l u^N(t)\| + \|\nabla \partial_t^{l-1} u^N(t)\| \leq C. \tag{4.7}$$

Then

$$\|\partial_t^{l+1} u^N(t)\| + \|\nabla \partial_t^l u^N(t)\| \leq C.$$

Proof We consider the equality,

$$(\partial_t^l(Lu^N), \partial_t^{l+1}u^N)(t) = (\partial_t^l(fM^{-1}), \partial_t^{l+1}u^N)(t).$$

Omitting the index N, we write it in the form,

$$\begin{aligned}(\partial_t^l(u_{tt}M^{-1}), \partial_t^{l+1}u)(t) + (\nabla\partial_t^l u, \nabla\partial_t^{l+1}u)(t)\\ + (\partial_t^l(gM^{-1}), \partial_t^{l+1}u)(t) = (\partial_t^l(fM^{-1}), \partial_t^{l+1}u)(t).\end{aligned} \tag{4.8}$$

Using the chain rule, we obtain

$$\partial_t^l(fg) = \sum_{j=0}^{l} C_j^l \partial_t^j f \partial_t^{l-j} g, \tag{4.9}$$

where C_j^l are binomial coefficients. On the other hand,

$$\partial_t^l g(t, u_t(t)) = \sum_{j=0}^{l} C_j^l \bar{\partial}_t^{l-j} g D_t^j g(u_t). \tag{4.10}$$

Here

$$\bar{\partial}_t^s g = \frac{\partial^s}{\partial t^s} g(t, u_t),$$

$$D_t^j g = \sum_{\rho=1}^{j} C_j^\rho g^{(\rho)}(t, u_t)(\partial_t u_t)^{s_1}(\partial_t^2 u_t)^{s_2} \cdots (\partial_t^j u_t)^{s_j}, \tag{4.11}$$

$$g^{(\rho)}(t, u_t) = \partial_{u_t}^\rho g(t, u_t); \quad s_1 + \cdots + s_j = \rho, \quad 1s_1 + 2s_2 + \cdots + js_j = j, \tag{4.12}$$

C_j^ρ are numerical coefficients. For more details on the chain rule formulas see [1].

Now we apply (4.9)–(4.12) to all of the terms in (4.8):

$$\begin{aligned} I_1 &= (\partial_t^l(u_{tt}M^{-1}), \partial_t^{l+1}u) = \left(\sum_{j=0}^{l} C_j^l(\partial_t^j u_{tt})\partial_t^{l-j}M^{-1}, \partial_t^{l+1}u\right) \\ &= (M^{-1}\partial_t^{l+2}u, \partial_t^{l+1}u) + (|\partial_t^{l+1}u|^2, \partial_t M^{-1}) \\ &\quad + \sum_{j=0}^{l-2} C_j^l(\partial_t^j u_{tt}(\partial_t^{l-j}M^{-1}), \partial_t^{l+1}u) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\partial_t^{l+1}u|^2, M^{-1}) \\ &\quad + \left(l - \frac{1}{2}\right)(|\partial_t^{l+1}u|^2, \partial_t M^{-1}) + \sum_{j=0}^{l-2} C_j^l(\partial_t^{j+2}u, (\partial_t^{l-j}M^{-1})\partial_t^{l+1}u). \end{aligned} \tag{4.13}$$

It can be seen that the highest derivative of $M^{-1}(x, t, \|u(t)\|^2)$ is $\partial_t^l(M^{-1})$ and its highest derivative of $\|u(t)\|$ is $\partial_t^l\|u(t)\|$ which is bounded by (4.7). It means that

$$|\partial_t^j M^{-1}(x, t, \|u(t)\|^2)| \leq C, \quad j = 0, \ldots, l, \tag{4.14}$$

hence

$$I_1 \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(M^{-1}, |\partial_t^{l+1}u|^2)(t) - C[1 + (M^{-1}, |\partial_t^{l+1}u|^2)(t)]. \tag{4.15}$$

We treat in the same manner the term

$$(\partial_t^l(gM^{-1}), \partial_t^{l+1}u) = \left(\sum_{j=0}^{l} C_j^l(\partial_t^j g(x, t, u_t))\partial_t^{l-j}M^{-1}, \partial_t^{l+1}u\right), \tag{4.16}$$

where $\partial_t^j g(x, t, u_t(t))$ is defined by (4.10)–(4.12).

Straight calculations in (4.16) show that for $l \geq 3$ the highest derivatives $\partial_t^{l+1} u$ and $\partial_t^l u$ enter in the expression for $\partial_t^l g(x, t, u_t)$ linearly. It means that we can rewrite (4.16) as follows.

$$\begin{aligned}(\partial_t^l(gM^{-1}), \partial_t^{l+1}u) = (M^{-1}g^{(1)}, |\partial_t^{l+1}u|^2) + (M^{-1}g^{(2)}\partial_t^l u, \partial_t^{l+1}u)\\ + (\Phi(x, t, u_t, \|u(t)\|^2), \partial_t^{l+1}u),\end{aligned} \tag{4.17}$$

where a continuous function $\Phi(x, t, u_t, \|u(t)\|^2)$ contains derivatives up to the order $\partial_t^{l-1}u$.

Lemma 4.1 yields that $u_t \in C(\bar{D})$, hence, the functions $g^{(\rho)}(x, t, u_t), \rho = 1, \ldots, l$ are bounded. Taking into account (4.7) and assumption 4.6, we transform (4.17) to the form.

$$\begin{aligned}(\partial_t^l(M^{-1}), \partial_t^{l+1}u) &\geq C_0(|u_t|^p, |\partial_t^{l+1}u|^2) - C[\|\partial_t^{l+1}u(t)\|^2 + \|\partial_t^l u(t)\|^2 + 1]\\ &\geq C_0(|u_t|^p, |\partial_t^{l+1}u|^2) - C[1 + \|\partial_t^{l+1}u(t)\|^2],\end{aligned}$$

where C_0 is a positive constant.

Using this and (4.15), we reduce (4.8) to the inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}[(M^{-1}, |\partial_t^{l+1}u|^2)(t) + \|\nabla\partial_t^l u(t)|^2] \leq C[1 + \|\partial_t^l f(t)\|^2 + (M^{-1}|\partial_t^{l+1}u(t)|^2)(t)].$$

Integration and the Gronwall lemma give

$$\|\partial_t^{l+1}u^N(t)\| + \|\nabla\partial_t^l u^N(t)\| \leq C,$$

where the constant C does not depend on $N, t \in (0, T)$. ■

Passing to the limit as $N \to \infty$, we prove

LEMMA 4.3 *Let all the conditions of theorem* 4.1 *hold. Then for any* $f \in H^{k-1}(0, T; L^2(D))$ *there exists a unique solution to* (2.1)–(2.3) *from the class,*

$$u, u_t \in L^\infty(0, T; H^2(D) \cap H_0^1(D)),$$

$$\partial_t^j u \in L^\infty(0, T; H_0^1(D)), \quad j = 2, \ldots, k-1,$$

$$\partial_t^k u \in L^\infty(0, T; L^2(D)).$$

Proof The proof is obvious, we omit it. ■

Now, returning to (2.1), we write it in the form.

$$(\nabla\partial_t^j u, \nabla\phi)(t) = (\partial_t^j[M^{-1}(f - g - u_{tt})], \phi)(t), \quad j = 2, \ldots, k-2.$$

where ϕ is an arbitrary function from $H_0^1(D)$. Because from assumption 4.7, $\partial_t^j u|_{\partial D} = 0$, from the theory of elliptic problems it easily follows that

$$\partial_t^{k-j} u \in L^\infty(0, T; H^j(D) \cap H_0^1(D)), \quad j = 1, \dots, k,$$

$$\partial_t^k u \in L^\infty(0, T; L^2(D)).$$

This completes the proof of theorem 4.1. ■

5 COMMENTS

Here we want to explain assumptions that we made and to give simple examples of equations that satisfy these assumptions. We have 3 types of results:

(1) Existence and uniqueness of global strong solutions for all finite $t > 0$.
(2) The exponential energy decay of strong solutions.
(3) Regularity result when solutions belong to $H^k(Q)$ for any $k \in N$.

It is clear that these results are different, hence, the conditions that guarantee them are also different.

ASSUMPTIONS 2 They guarantee the existence of global strong solutions for any finite $T > 0$. The following equation

$$u_{tt} - (1 + a(x, t) \int_0^1 u^2(x, t)\, \mathrm{d}x) u_{xx} + b(x, t)|u_t|^\rho u_t = f(x, t),$$

where $a, b \in C^1(D \times [0, \infty))$; $0 < a_0 \leq a(x, t)$; $0 < b_0 < b(x, t)$; $\rho > 1$, satisfy assumptions 2. We have

$$\begin{aligned}
\lambda = \int_0^1 u^2\, \mathrm{d}x; \quad & \phi(\|u(t)\|^2) = 1 + \int_0^1 u^2\, \mathrm{d}x \geq 1, M(x, t, \|u(t)\|^2) \\
&\leq \max\left(1, \max_{\bar{Q}}(a(x, t))\right)\phi(\|u(t)\|^2), \lambda^{1/2} M_\lambda = a(x, t)\lambda^{1/2} \\
&\leq \max_{\bar{Q}} a^{1/2}(x, t) M(x, t, \lambda), g(x, t, u_t)u_t = b(x, t)|u_t|^{\rho+2} \\
&\geq b_0|u_t|^{\rho+2}; g_{u_t}(x, t, u_t) \geq b_0|u_t|^\rho.
\end{aligned}$$

The restriction $\rho > 1$ implies the existence of global solutions when the initial data are not small. The case $\rho = 1$ also can be considered if k_0 from assumption 2.3 is sufficiently small, see for details [3]. It can be easily seen that assumption 2.3 is fulfilled for any polynomial $M(\lambda)$. On the other hand, assumptions 2 are not uniform in $t \in (0, \infty)$, hence, do not guarantee any decay of the energy when $t \to \infty$. For this purpose we have

ASSUMPTIONS 3 which are satisfied by the following equation.

$$u_{tt} - \left(1 + \frac{\delta}{1+e^{-t}} \int_0^1 u^2(x,t)\,dx\right) u_{xx} + au_t + \beta |u_t|^\rho u_t = f(x,t),$$

where α, β, δ are positive numbers such that

$$\delta < \left(\frac{\rho}{\rho-1}\right)^{2(\rho-1)} \frac{\alpha^{2(\rho-1)}\beta^2}{2^{(\delta\rho-4)/\rho}C_1}, \quad \rho > 1.$$

The nonlinear damping, $\beta > 0$, implies the existence of global solutions and due to the linear damping, $\alpha > 0$, we prove the exponential decay of the energy.

ASSUMPTIONS 4 Here we prove the existence of smooth solutions for any finite $T > 0$ when the initial data are not small. Actually, it is an answer to the question of J.-L Lions [7]: what conditions on ρ, and n guarantee the existence of global smooth solutions to the problem

$$u_{tt} - \Delta u + |u_t|^\rho u_t = f(x,t), \Delta = \sum_1^n \frac{\partial^2}{\partial x_i^2}.$$

The answer is given by theorem 4.1 when $\rho = 2$, $n = 2$. The following equation illustrates assumptions 4:

$$u_{tt} - (1 + a(x,t) \int_0^1 u^2(x,t)\,dx)(u_{x_1x_1} + u_{x_2x_2}) + b(x,t)u_t^3 = f(x,t),$$

where $a, b \in C^\infty(Q)$; $a_0 \le a(x,t)$, $b_0 \le b(x,t), f \in H^k(Q)$.

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