# THE RELATIONSHIP BETWEEN THE LOCAL TEMPERATURE AND THE LOCAL HEAT FLUX WITHIN A ONE-DIMENSIONAL SEMI-INFINITE DOMAIN OF HEAT WAVE PROPAGATION

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The relationship between the local temperature and the local heat flux has been established for the homogeneous hyperbolic heat equation. This relationship has been written in the form of a convolution integral involving the modified Bessel functions. The scale analysis of the hyperbolic energy equation has been performed and the dimensionless criterion for the mode of energy transport, similar to the Reynolds criterion for the flow regimes, has been proposed. Finally, the integral equation, relating the local temperature and the local heat flux, has been solved numerically for those processes of surface heating whose time scale is of the order of picoseconds.

## 1. Introduction

The direction of development of nowadays technology is towards the smaller scales. Many commonly used devices of today already operate on nanoseconds time scale with energy transport happening between parts whose linear size is of the size of a single atom. This modus operandi can be seen, for instance, in many electronic devices (personal computers, cellular phones, etc.) involving microelements. A deeper penetration and the use of the most elementary (fundamental) natural scales require a better understanding and a finer analysis of those laws that govern physical processes on those scales. Indeed, at those levels where the classical assumptions made for energy transport (e.g., Fourier's law) become no longer applicable due to the fact that the continuum hypothesis fails at those scales, the mathematical description of the physical laws governing the process of energy transport also assumes different forms. In particular, one has to take into account that the speed of the thermal energy transport cannot be considered infinite (the intrinsic assumption hidden behind Fourier's law). Therefore, it becomes necessary to account for the time lag between the temperature gradient and the heat flux induced by it, that is, $\mathbf{q}'' + \tau(\partial \mathbf{q}''/\partial t) = -k\nabla T$, which substitutes Fourier's law in the case of a finite speed of heat propagation. Although the conservation law remains valid at all scales, being combined with the new constitutive relation, it no longer leads to the classical parabolic heat


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equation, but to the energy equation that is mathematically identical to the wave equation with damping due to the energy diffusion (see [5] for details). Thus, the energy equation is no more parabolic and becomes hyperbolic at those time or spatial scales where Fourier's law is not applicable. It is therefore clear that the solutions to the heat transfer problems at those scales will be by the very nature of the energy equation different from parabolic solutions of the classical heat equation, albeit the geometry, initial and boundary conditions might be the same.

The present study focuses on the one-dimensional homogeneous hyperbolic energy equation, restricting its analysis to the heat transfer problem in a semi-infinite domain. This choice is well motivated since, for many processes whose characteristic time is as short as considered below, the domain of heat propagation can be considered as semi-infinite with a very high level of accuracy.

## 2. Mathematical model

Consider the one-dimensional thermal wave equation (hyperbolic diffusion equation)

$$\tau\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}, \tag{2.1}$$

where $\alpha$ is the thermal diffusivity and $\tau$ is the relaxation time in phonon collisions, defined as

$$\tau=\frac{3\alpha}{c^2}, \tag{2.2}$$

where $c$ is the speed of sound [5].

Equation (2.1) is now applied to solve an initial value problem in a semi-infinite domain under the condition of initial thermal equilibrium of the domain ($T=T_0$ everywhere at time $t=0$ and $\partial T/\partial t|_{t=0}=0$).

By introducing the new variable $\xi=x/\alpha^{1/2}$, the new timelike variable $\vartheta=t/\tau^{1/2}$, and the excess temperature $\theta=T-T_0$, (2.1) becomes

$$\frac{\partial^2\theta}{\partial\vartheta^2}+\tau^{-1/2}\frac{\partial\theta}{\partial\vartheta}=\frac{\partial^2\theta}{\partial\xi^2} \tag{2.3}$$

with the initial conditions $\theta=0$ at $\vartheta=0$ and $\partial\theta/\partial\vartheta|_{\vartheta=0}=0$.

Upon taking the Laplace transform of (2.3) and rearranging the terms, one obtains

$$\frac{d^2\Theta}{d\xi^2}-s\left(s+\tau^{-1/2}\right)\Theta=0, \tag{2.4}$$

where $\Theta$ is the Laplace transform of the excess temperature.

Equation (2.4) has a general solution

$$\Theta(\xi;s)=C_1(s)\exp\left\{-\xi\left[s\left(s+\tau^{-1/2}\right)\right]^{1/2}\right\}+C_2(s)\exp\left\{\xi\left[s\left(s+\tau^{-1/2}\right)\right]^{1/2}\right\}. \tag{2.5}$$

The physics requires this solution to be bounded as $\xi \to \infty$ and, therefore, $C_2(s)$ must be identically zero. Denoting $C(s) \equiv C_1(s)$, one gets

$$\Theta(\xi;s) = C(s)\exp\left\{-\xi\left[s\left(s+\tau^{-1/2}\right)\right]^{1/2}\right\}. \tag{2.6}$$

It is now possible to eliminate the arbitrary "constant" $C(s)$ in exactly the same way as it has been done in [3], that is, by taking the derivative of (2.6) with respect to $\xi$.

Indeed,

$$\frac{d\Theta}{d\xi} = -\left[s\left(s+\tau^{-1/2}\right)\right]^{1/2}C(s)\exp\left\{-\xi\left[s\left(s+\tau^{-1/2}\right)\right]^{1/2}\right\} = -\left[s\left(s+\tau^{-1/2}\right)\right]^{1/2}\Theta, \tag{2.7}$$

which can be rewritten as

$$-\Theta = \left[s\left(s+\tau^{-1/2}\right)\right]^{-1/2}\frac{d\Theta}{d\xi}. \tag{2.8}$$

The inverse Laplace transform of $[s(s+\tau^{-1/2})]^{-1/2}$ is simply $I_0(\vartheta/2\tau^{1/2})\exp(-\vartheta/2\tau^{1/2})$ (see [1, #29.3.49, page 1024]), where $I_0(z)$ is the modified Bessel function (see [1, pages 374–379]). Thus, taking the inverse Laplace transform of (2.8) and applying the convolution theorem, one obtains

$$\theta = -\int_0^{t/\tau^{1/2}} \frac{\partial\theta}{\partial\xi} I_0\left(\frac{\vartheta-\zeta}{2\sqrt{\tau}}\right)\exp\left(-\frac{\vartheta-\zeta}{2\sqrt{\tau}}\right)d\zeta. \tag{2.9}$$

After fully restoring the original variables and rearranging the terms, (2.9) transforms into

$$T(x,t) = T_0 - \left(\frac{\alpha}{\tau}\right)^{1/2}\int_0^t \frac{\partial T}{\partial x} I_0\left(\frac{t-t^*}{2\tau}\right)\exp\left(-\frac{t-t^*}{2\tau}\right)dt^* \tag{2.10}$$

which gives the relationship between the temperature and its spatial derivative at any moment in time and at any location in the domain in question.

It is necessary to emphasize here that in the case of a finite relaxation time $\tau$, the speed of the thermal wave propagation cannot be considered infinite and, therefore, the Fourier law is not applicable. In this case, it is necessary to use the constitutive relationship which takes into account the lagging behavior of the thermal wave due to the finite value of the relaxation time (the speed of the thermal wave). As pointed out in [5], such a relationship between the temperature $T$ and the heat flux $q''$ is

$$q''(x,t) + \tau\frac{\partial q''}{\partial t}(x,t) \cong -k\frac{\partial T}{\partial x}(x,t), \tag{2.11}$$

where $k$ is the thermal conductivity of the medium. Note that (2.11) holds if $\tau \ll t$, that is, in the case when the relaxation time is much shorter than the characteristic time of the transient process.

Substituting (2.11) into (2.10), one obtains

$$T(x,t) = T_0 + (k\rho c_p \tau)^{-1/2} \int_0^t \left[ q''(x,t^*) + \tau \frac{\partial q''}{\partial \zeta}(x,t^*) \right] I_0\left(\frac{t-t^*}{2\tau}\right) \exp\left(-\frac{t-t^*}{2\tau}\right) dt^* \tag{2.12}$$

which relates the temperature $T$ and the heat flux $q''$ at any moment in time and at any location inside the domain.

## 3. Scale analysis of the thermal wave equation: a criterion for solution applicability

We now rewrite (2.1) in terms of characteristic scales of the process, that is,

$$\tau \frac{T}{t_W^2} + \frac{T}{t_D} \sim \alpha \frac{T}{\delta^2}. \tag{3.1}$$

If both terms on the left-hand side are of the same order of magnitude, one obtains expressions for the wave and diffusion time scales, namely, $t_W$ and $t_D$, using the fact that each of these terms has to be of the same order as the term on the right-hand side, that is,

$$t_W \propto \frac{\delta}{C}, \tag{3.2}$$

$$t_D \propto \frac{\delta^2}{\alpha}, \tag{3.3}$$

where $\delta$ is the scale of linear dimension and $C = (\alpha/\tau)^{1/2}$ is the speed of thermal waves.

In fact, the wave component of the energy transport dominates if the term $T/t_D$, responsible for the change due to diffusion, is much smaller than the term $\tau T/t_W^2$, responsible for the wave transport. In other words, the wave transport dominates if $t_W^2/t_D\tau \ll 1$.

To put it in another way, when the intrinsic length of the heat diffusion

$$\lambda_D = (\alpha t)^{1/2} \tag{3.4}$$

is significantly smaller than the intrinsic length scale of the thermal wave

$$\lambda_W = Ct, \tag{3.5}$$

where $C = (\alpha/\tau)^{1/2}$ is the speed of thermal waves, the effect of wave transport can be neglected. Otherwise, this effect must be taken into account.

It is now clear that if the characteristic time of energy transport is smaller than the relaxation time, that is, $t < \tau$, the transport by means of wave must be taken into account; and when $t > \tau$, the diffusion predominates.

Based on the scale analysis of (2.1), one can now introduce the criterion to distinguish between the possible types of transport processes. Such a criterion is the relaxation frequency number $N_r = t/\tau = C^2 t/\alpha$. If one now notices that the length scale of the process

is $\delta = Ct$, one obtains

$$N_r = \frac{\delta C}{\alpha} \begin{array}{ll} < 1, & \text{transport by means of waves,} \\ \sim 1, & \text{transport by means of both waves and diffusion (transition),} \\ > 1, & \text{transport by means of diffusion.} \end{array} \tag{3.6}$$

It is amazing how this criterion resembles the famous Reynolds number. This resemblance, however, is by no means coincidental. As it has been shown in [2], a similar scale analysis performed for a buckling of streams in a fluid flow leads to the definition of the Reynolds number seen as the factor of competition between the buckling waves and viscous diffusion.

From (3.6), one can observe that for metals with $\alpha \sim 10^{-5}\,\mathrm{m^2/s}$ and $C \sim 10^3\,\mathrm{m/s}$, the transport by thermal waves must be taken into account as $\delta \sim 10^{-8}\,\mathrm{m}$ and below. It is equivalent to the time scale of $t \sim 10^{-11}$ second or smaller (see (3.2) or (3.3)).

## 4. Numerical results

Equation (2.12) has been solved numerically, given the representative physical properties of metals, that is, $\alpha = 10^{-5}\,\mathrm{m^2/s}$, $\rho c_p = 10^6\,\mathrm{J/m^3K}$, and $C = 10^3\,\mathrm{m/s}$, in order to compute the surface temperature for a given heat flux at the boundary. The heat flux was represented by the Gaussian, namely,

$$q_b(t) = \exp\left[-\left(\frac{t-b}{\sigma}\right)^2\right] \tag{4.1}$$

with $b = 10$ picoseconds and $\sigma = 5.0$ picosecond which mimic the incidence power flux of a laser. The time evolution of the normalized surface temperature, $\theta = (T_s - T_0)/(T_{\max} - T_0)$, is shown in Figure 4.1.

On the same figure, this solution is shown in comparison with that one obtained by means of the classical (Fourier) assumption [4].

From Figure 4.1, one can see that the solution behaves in a manner similar to that reported in [4] and found by the methods of the fractional calculus for the processes at nanosecond time scale. However, although the behavior of both solutions is qualitatively similar, it is obvious that the relaxation of the surface temperature is much faster in the hyperbolic case than in the parabolic, classical, case. This can be explained by the fact that, in the classical case, there is only one mechanism of energy transport diffusion (heat conduction). In the hyperbolic case, on the other hand, two mechanisms are involved, that is, transport by means of waves is added to the transport by diffusion, making relaxation faster.

Figure 4.2 shows a comparison of the results obtained for different values of the relaxation time. The value of the relaxation time was varied in the range of $\pm 10$ percent with respect to the exact value, that is, $\tau = 0.7438$ picosecond. From the figure, one can see that the solution is stable with respect to small variations of the relaxation time.

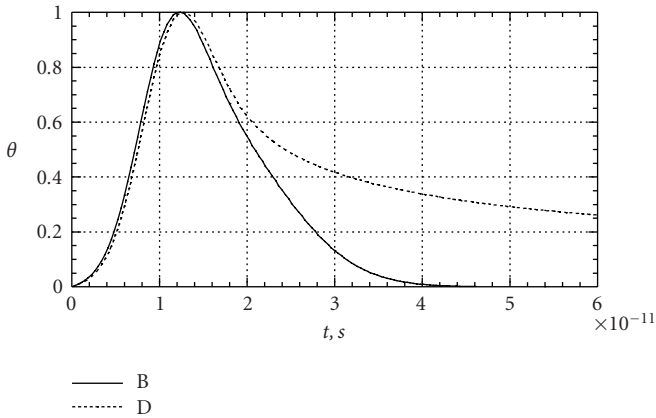


Figure 4.1. Comparison between the solutions obtained for a thermally thick sample of a metal in the case of the hyperbolic (wave) transport (B) and the classical (Fourier) diffusion (D).

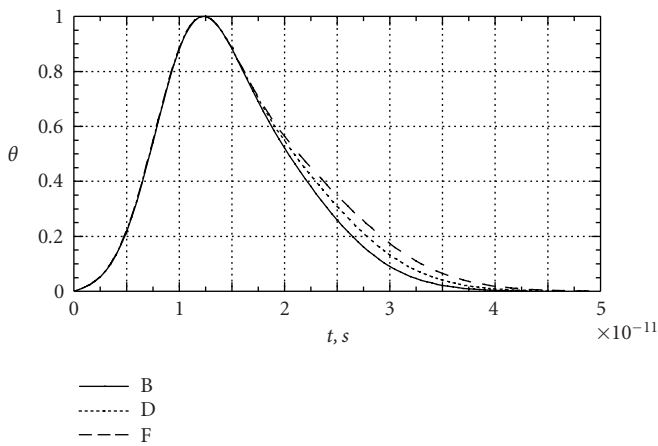


Figure 4.2. Time evolution of the surface temperature for different values of the relaxation time: (D) $\tau = 0.7438\,\text{ps}$, (B) $\tau = 0.6694\,\text{ps}$, (F) $\tau = 0.8182\,\text{ps}$.

## 5. Conclusion

The relationship between the local temperature and the local heat flux has been established for the homogeneous hyperbolic heat equation. This relationship can be written in the form of a convolution integral involving the modified Bessel functions, and is obtained by the same method as described in [4]. The scale analysis of the hyperbolic energy equation leads to the dimensionless criterion for the mode of energy transport. This criterion (*relaxation frequency number*) is similar to the Reynolds criterion (*Reynolds number*) to distinguish between laminar and turbulent flow regimes and identify the transition between both. An amazing explanation of the Reynolds criterion is given by means of the

buckling theory of viscous fluid flows in [2]. The similarity between these two criteria seems to be by no means accidental and reflects *competition between the energy transports by waves and by diffusion* that is dictated by the dual, wave-corpuscular, nature of the matter. Between the two possible ways of transport one is chosen, which appears to be the *most effective* way for a given time scale, in a strict accordance with the least-action principle.

Finally, the integral equation, relating the local temperature and the local heat flux, has been solved numerically for those processes of surface heating whose time scale is of the order of picoseconds. Although experimental results for such a process are not currently available, the authors believe that the obtained result will provide a good explanation of such results in the future, when the development of microscale technologies and the level of experimental tools makes this result necessary and possible.

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