ROBUST STABILIZATION WITH H_∞ PERFORMANCE FOR A CLASS OF LINEAR PARAMETER-DEPENDENT SYSTEMS

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We focus on the issue of robust stabilization with H_∞ performance for a class of linear time-invariant parameter-dependent systems under norm-bounded nonlinear uncertainties. By combining the idea of polynomially parameter-dependent quadratic Lyapunov functions and linear matrix inequalities formulations, some parameter-independent conditions with high precision are given to guarantee robust asymptotic stability and robust disturbance attenuation of the linear time-invariant parameter-dependent system in the presence of norm-bounded nonlinear uncertainties. The parameter-dependent state-feedback control is designed based on the Hamilton-Jacobi-Isaac (HJI) method. The applicability of the proposed design method is illustrated in a simple example.

1. Introduction

Linear time-invariant parameter-dependent (LTIPD) systems have gained a lot of interest as they provide a systematic means of computing gain-scheduled controllers, especially those related to vehicle and aerospace control [1, 8, 13]. Generally speaking, an LTIPD system is a linear system in which the system matrices are fixed functions of a known parameter vector. An LTIPD system can be viewed as a nonlinear system that is linearized along a trajectory determined by the parameter vector. Hence, the parameter vector of an LTIPD system corresponds to the operating point of the nonlinear system. In the LTIPD framework, it is assumed that the parameter vector is measurable for control. In many industrial applications, like flight control and process control, the operating point can indeed be determined from measurement, making the LTIPD approach viable; see for example [12, 14].

Over the last three decades, considerable attention has been paid to robustness analysis and control of linear systems affected by structured real parameters. For LTIPD systems, establishing stability via the use of constant Lyapunov functions is conservative. To

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investigate the stability of LTIPD systems one needs to resort to the use of parameter-dependent Lyapunov functions to achieve necessary and sufficient conditions of system stability; see for instance [2–4, 7, 9, 11, 17] and the references therein. However, Bliman [4] proposed the problem of robust stability for LTIPD systems with scalar parameters. He also developed some conditions for robust stability in terms of solvability of some linear matrix inequalities (LMIs) without conservatism. Moreover, the existence of a polynomially parameter-dependent quadratic Lyapunov function for systems, which are robustly stable, is investigated in [5]. Recently, sufficient conditions for robust stability of the linear state-space models affected by polytopic uncertainty have been provided in [6] using homogeneous polynomially parameter-dependent quadratic Lyapunov functions, which are formulated in terms of LMI feasibility tests.

In this paper, we provide a systematic way for the use of *polynomially parameter-dependent quadratic* (PPDQ) Lyapunov functions in the robust asymptotic stability and robust disturbance attenuation problem of finite-dimensional LTIPD systems in the presence of norm-bounded nonlinear uncertainties. Sufficient conditions of increasing precision for the existence of PPDQ Lyapunov functions are given using LMI formulations. This paper is essentially an extension of the stabilization technique of the parameter-dependent systems presented in [3] to the robust control problem with guaranteed H_∞ performance. This paper makes three specific contributions. First, it suggests a PPDQ Lyapunov function, which can be applied to derive sufficient and necessary stability conditions for LTIPD systems. Second, robust stabilization and disturbance attenuation of such systems are investigated using the Hamiltonian approach. Then, the state feedback gain matrix can be constructed from the positive-definite solution to certain parameter-independent inequalities. The existence proof is constructive, thus yielding a method to compute the gain matrix. Finally, the simulation results show that the obtained state feedback control law can achieve the robust stability and disturbance attenuation, simultaneously.

The notations used throughout the paper are fairly standard. The matrices I_n, 0_n, $0_{n\times p}$ are the identity matrix and the $n\times n$ and $n\times p$ zero matrices, respectively. The symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix. Also, the symbol $\otimes$ denotes Kronecker product, the power of Kronecker products being used with the natural meaning $M^{0\otimes} = I$, $M^{p\otimes} := M^{(p-1)\otimes} \otimes M$. Let $\widehat{J}_k, \widetilde{J}_k \in \Re^{k\times(k+1)}$, and $\vartheta^{[k]}$ be defined by $\widehat{J}_k := [I_k \quad 0_{k\times 1}]$, $\widetilde{J}_k := [0_{k\times 1} \quad I_k]$, and $\vartheta^{[k]} := [1 \quad \vartheta \quad \cdots \quad \vartheta^{k-1}]^T$, respectively, which have essential roles for polynomial manipulations [4]. $C^0(U,V)$ denotes a set of continuous functions from U to V. Finally, given a signal $x(t)$, $\|x(t)\|_2$ denotes the L_2 norm of $x(t)$; that is, $\|x(t)\|_2^2 = \int_0^\infty x(t)^T x(t)dt$.

2. Problem description

In this section, we consider a class of LTIPD systems with norm-bounded nonlinear uncertainties as

$$\dot{x}(t) = A(\rho)x(t) + B_u(\rho)u(t) + B_w(\rho)w(t) + \Delta(x,u;\rho), \qquad z(t) = C_1x(t) + C_2u(t) \tag{2.1}$$

whose dependency of the state-space matrices affinely on the parameter vector $\rho = [\rho_1,$

$\rho_2,\ldots,\rho_m]^T \in \zeta \subset \Re^m$ is shown as

$$\begin{bmatrix} A(\rho) & B_u(\rho) & B_w(\rho) \end{bmatrix} = \begin{bmatrix} A_0 & B_{0u} & B_{0w} \end{bmatrix} + \sum_{i=1}^{m} \rho_i \begin{bmatrix} A_i & B_{iu} & B_{iw} \end{bmatrix}, \tag{2.2}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^l$, $w(t) \in \Re^s$, and $z(t) \in \Re^p$ are the state vector, the control input, the disturbance vector, and the controlled output, respectively. Also, the nonlinear term $\Delta(x,u;\rho)$ is norm-bounded function of uncertainty space, which will be defined in Assumption 2.1. Furthermore, it is known that the vector ρ is contained in a priori given set whereas the actual curve of the vector ρ is unknown but can be measured online for control process. In the sequel, we will make the following assumption and definitions for the system (2.1).

Assumption 2.1. There exist the parameter-dependent matrices $G_1(\rho)$ and $G_2(\rho)$ such that the nonlinear uncertainty $\Delta(x,u;\rho)$ satisfy the following bounded condition:

$$||\Delta(x,u;\rho)||_2^2 \le ||G_1(\rho)x||_2^2 + ||G_2(\rho)u||_2^2, \quad \forall x \in \Re^n,\ \rho \in \zeta, \tag{2.3}$$

where the parameter-dependent matrices $G_1(\rho)$ and $G_2(\rho)$ are defined as follows:

$$\begin{aligned} G_1(\rho) &= G_{01} + \sum_{i=1}^{m} \rho_i G_{i1}, \\ G_2(\rho) &= G_{02} + \sum_{i=1}^{m} \rho_i G_{i2}. \end{aligned} \tag{2.4}$$

Denote the corresponding uncertainty set by

$$\Xi(x,u;\rho) = \left\{ \Delta(x,u;\rho) : ||\Delta(x,u;\rho)||_2^2 \le ||G_1(\rho)x||_2^2 + ||G_2(\rho)u||_2^2 \right\}. \tag{2.5}$$

Definition 2.2. (1) A state feedback $u(t;\rho) = -K(\rho)x(t)$ with $K(\rho) \in C^0(\Re^m, \Re^{l\times n})$ is said to achieve robust global asymptotic stability of the system (2.1) if for $w = 0$ and any $\rho \in \zeta$ and $\Delta(x,u;\rho) \in \Xi(x,u;\rho)$ the closed-loop system

$$\dot{x}(t) = (A(\rho) - B_u(\rho)K(\rho))x(t) + \Delta(x, -K(\rho)x;\rho) \tag{2.6}$$

is globally asymptotically stable in the Lyapunov sense.

(2) A state feedback $u(t;\rho) = -K(\rho)x(t)$ with $K(\rho) \in C^0(\Re^m, \Re^{l\times n})$ is said to achieve robust disturbance attenuation if under zero initial condition there exists $0 \le \gamma < \infty$ for which the performance bound is such that

$$||z(t)||_2 < \gamma ||w(t)||_2, \quad \forall w \in L^2, \qquad \rho \in \zeta, \quad \Delta(x,u;\rho) \in \Xi(x,u;\rho). \tag{2.7}$$

According to Definition 2.2, the main objective of the paper is to design the state feedback control for achieving both the robust global asymptotic stability and the robust disturbance attenuation of the LTIPD system (2.1).

Definition 2.3. A polynomially parameter-dependent quadratic (PPDQ function for short) function is said to be any quadratic function $x^T(t)S(\rho)x(t)$ such that $S(\rho)$ is defined as

$$S(\rho) := \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n\right)^T S_k \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n\right) \tag{2.8}$$

for a certain $S_k \in \Re^{k^m n}$. The integer $k-1$ is called the degree of the PPDQ Lyapunov function of $S(\rho)$ [11].

3. Main results

The main approach employed here is to design the state feedback control in the presence of the disturbance and norm-bounded nonlinear uncertainties based on the standard HJI method. Hence, we define a quadratic energy function in the form

$$E(x(t);\rho) = x^T(t)P_\rho x(t), \tag{3.1}$$

where $P_\rho := P(\rho) := (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n)^T P_k (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n) > 0$, the PPDQ function of degree $k-1$, is to be determined.

Suppose that there exists the following Hamilton-Jacobi-Isaac (HJI) function:

$$H[u,w,\Delta(x,u;\rho)] = \frac{dE(x(t);\rho)}{dt} + z^T(t)z(t) - \gamma^2 w^T(t)w(t), \tag{3.2}$$

where the derivative of $E(x(t);\rho)$ is evaluated along the trajectory of the closed-loop system (2.6). It is well known that a sufficient condition for achieving robust disturbance attenuation is that the inequality $H[u,w,\Delta(x,u;\rho)] < 0$ for every $w \in L^2$, $\rho \in \zeta$, and $\Delta(x,u;\rho) \in \Xi(x,u;\rho)$ results in a function $E(x(t);\rho)$, which is strictly radially unbounded (see, e.g., [15, 18]); $E(x(t);\rho)$ may be regulated as a Lyapunov function for the closed-loop system (2.6). In this paper we will establish conditions under which

$$\operatorname*{Inf}_{u} \sup_{\Delta(x,u;\rho)\in\Xi(x,u;\rho)} \sup_{w\in L^2} H[u,w,\Delta(x,u;\rho)] < 0, \tag{3.3}$$

then for every T, taking the definite integral from 0 to T of both sides of (3.2) gives

$$\int_0^T z^T(t)z(t)dt - \gamma^2 \int_0^T w^T(t)w(t)dt < E(x(0)) - E(x(T)) \le E(x(0)) = 0, \tag{3.4}$$

that is, constraint of (2.7).

Noting to the expressions (3.1) and (3.2), we find

$$\begin{aligned} &H[u,w,\Delta(x,u;\rho)] \\ &\quad = x^T(A^T(\rho)P_\rho + P_\rho A(\rho))x + u^T B_u^T(\rho)P_\rho x + x^T P_\rho B_u(\rho)u \\ &\qquad + w^T B_w^T(\rho)P_\rho x + \Delta^T(x,u;\rho)P_\rho x + x^T P_\rho \Delta(x,u;\rho) \\ &\qquad + x^T P_\rho B_w(\rho)w + x^T P_\rho B_w(\rho)w + (C_1x + C_2u)^T(C_1x + C_2u) - \gamma^2 w^T w. \end{aligned} \tag{3.5}$$

It is easy to show that the worst-case disturbance in (3.5) occurs when

$$w^* = \gamma^{-2} B_w^T(\rho) P_\rho x. \tag{3.6}$$

By substituting (3.6) into (3.5), we obtain

$$\begin{aligned}
&\operatorname*{Sup}_{w\in L^2} H(u, w, \Delta(x,u;\rho)) \\
&\quad = H(u, w^*, \Delta(x,u;\rho)) \\
&\quad = x^T\left(A^T(\rho)P_\rho + P_\rho A(\rho) + C_1^T C_1 + P_\rho\left(\gamma^{-2} B_w(\rho) B_w^T(\rho)\right)P_\rho\right)x \\
&\qquad + u^T\left(B_u^T(\rho)P_\rho + C_2^T C_1\right)x + x^T\left(P_\rho B_u(\rho) + C_1^T C_2\right)u + u^T C_2^T C_2 u \\
&\qquad + \Delta^T(x,u;\rho)P_\rho x + x^T P_\rho \Delta(x,u;\rho).
\end{aligned} \tag{3.7}$$

Lemma 3.1. *For an arbitrary positive scalar $\varepsilon > 0$ and a positive-definite matrix H, the following inequality is satisfied:*

$$\Delta^T(x,u;\rho)Hx(t) + x^T H\Delta(x,u;\rho) \le x^T\left(\varepsilon H^2 + \frac{1}{\varepsilon}G_1^T(\rho)G_1(\rho)\right)x + \frac{1}{\varepsilon}u^T G_2^T(\rho)G_2(\rho)u. \tag{3.8}$$

Proof. By using Assumption 2.1 and Lemma A.2, we can conclude the inequality above.

By utilizing Lemma 3.1 and the norm-bounded condition (2.3), we eliminate the norm-bounded nonlinear uncertainty $\Delta(x,u;\rho)$ in (3.7), and then we have

$$\begin{aligned}
&\operatorname*{Sup}_{\Delta(x,u;\rho)\in\Xi(x,u;\rho)} H(u, \Delta(x,u;\rho)) \\
&\quad \le x^T\left(A^T(\rho)P_\rho + P_\rho A(\rho) + C_1^T C_1 + \frac{1}{\varepsilon}G_1^T(\rho)G_1(\rho) + P_\rho\left(\varepsilon I_n + \gamma^{-2}B_w(\rho)B_w^T(\rho)\right)P_\rho\right)x \\
&\qquad + u^T\left(B_u^T(\rho)P_\rho + C_2^T C_1\right)x + x^T\left(P_\rho B_u(\rho) + C_1^T C_2\right)u + u^T\left(C_2^T C_2 + \frac{1}{\varepsilon}G_2^T(\rho)G_2(\rho)\right)u.
\end{aligned} \tag{3.9}$$

The optimal control law, which minimizes the right-hand side of (3.9), is given by

$$u(t;\rho) = -\left(C_2^T C_2 + \frac{1}{\varepsilon}G_2^T(\rho)G_2(\rho)\right)^{-1}\left(C_2^T C_1 + B_u^T(\rho)P_\rho\right)x(t). \tag{3.10}$$

As a result, we have

$$\operatorname*{Inf}_{u}\ \operatorname*{Sup}_{\Delta(x,u;\rho)\in\Xi(x,u;\rho)} H[u, \Delta(x,u;\rho)] \le x^T M_\rho x, \tag{3.11}$$

where the *parameter-dependent matrix* M_ρ is defined as

$$\begin{aligned} M_\rho := {} & A^T(\rho)P_\rho + P_\rho A(\rho) \\ & - (P_\rho B_u(\rho) + C_1^T C_2)\left(C_2^T C_2 + \frac{1}{\varepsilon}G_2^T(\rho)G_2(\rho)\right)^{-1}(B_u^T(\rho)P_\rho + C_2^T C_1) \\ & + P_\rho(\varepsilon I_n + \gamma^{-2}B_w(\rho)B_w^T(\rho))P_\rho + C_1^T C_1 + \frac{1}{\varepsilon}G_1^T(\rho)G_1(\rho). \end{aligned} \tag{3.12}$$

Consequently, if there exist positive scalars γ and ε and a positive-definite solution P_ρ to the matrix inequality $M_\rho < 0_n$, then we obtain

$$H[u, w, \Delta(x, u;\rho)] < 0, \quad \forall w \in L^2,\ \rho \in \zeta,\ \Delta(x, u;\rho) \in \Xi(x, u;\rho). \tag{3.13}$$

□

Corollary 3.2. *If there exist a positive scalar ε and a positive-definite matrix P_ρ to the following matrix inequality:*

$$A^T(\rho)P_\rho + P_\rho A(\rho) + \varepsilon P_\rho(I_n - B_u(\rho)(G_2^T(\rho)G_2(\rho))^{-1}B_u^T(\rho))P_\rho + \frac{1}{\varepsilon}G_1^T(\rho)G_1(\rho) < 0, \tag{3.14}$$

then the system (2.6) would be stabilized with a parameter-dependent state feedback $u(t;\rho) = -K(\rho)x(t)$, where the stabilizing state feedback $K(\rho) \in C^0(\Re^m, \Re^{l\times n})$ is given by $K(\rho) = \varepsilon(G_2^T(\rho)G_2(\rho))^{-1}B_u^T(\rho)P_\rho$.

In the sequel, we provide the robust global asymptotic stability and robust disturbance attenuation in the sense of (2.6) and (2.7) for the LTIPD system (2.1), respectively. This may be done by converting inequality $M_\rho < 0_n$ into the associated LMI and then we are able to determine the positive-definite solution P_ρ.

Using Schur's complement lemma, the inequality $M_\rho < 0_n$ holds if and only if

$$\begin{bmatrix} \begin{pmatrix} A^T(\rho)P_\rho + P_\rho A(\rho) \\ +\varepsilon^{-1}G_1^T(\rho)G_1(\rho) \end{pmatrix} & P_\rho B_u(\rho) & C_1^T & P_\rho & P_\rho B_w(\rho) \\ * & \varepsilon^{-1}G_2^T(\rho)G_2(\rho) & C_2^T & 0_{l\times n} & 0_{l\times s} \\ * & * & -I_p & 0_{p\times n} & 0_{p\times s} \\ * & * & * & -\varepsilon^{-1}I_n & 0_{n\times s} \\ * & * & * & * & -\gamma^2 I_s \end{bmatrix} < 0_{2n+s+l+p}. \tag{3.15}$$

Lemma 3.3. *Let the degree of the PPDQ Lyapunov function P_ρ be $k-1$. The nonquadratic matrix $P_\rho B_u(\rho)$ can be represented as*

$$P_\rho B_u(\rho) := \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n\right)^T H_k\left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l\right), \tag{3.16}$$

where the matrix $H_k \in \Re^{((k+1)^m n)\times((k+1)^m l)}$ *which depends linearly on the matrix* P_k *is defined as*

$$H_k = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k \left((\hat{J}_k^{m\otimes} \otimes B_{0u}) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_{iu}) \right). \tag{3.17}$$

According to Lemma 3.3, the representation of the matrix $P_\rho B_w(\rho)$ will be

$$P_\rho B_w(\rho) := \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n\right)^T F_k \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_s\right), \tag{3.18}$$

where the matrix $F_k \in \Re^{((k+1)^m n)\times((k+1)^m s)}$ is expressed as

$$F_k = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k \left((\hat{J}_k^{m\otimes} \otimes B_{0w}) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes B_{iw}) \right). \tag{3.19}$$

Remark 3.4. The PPDQ function of degree k for the positive-definite matrix $R_\rho = A^T(\rho)P_\rho + P_\rho A(\rho)$ is defined as

$$R_\rho := \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n\right)^T R_k \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n\right) \tag{3.20}$$

and from Lemma 3.3, the matrix $R_k \in \Re^{(k+1)^m n}$ in (3.20), which depends on the matrix P_k, is obtained as follows:

$$\begin{aligned} R_k = &\left((\hat{J}_k^{m\otimes} \otimes A_0) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i) \right)^T P_k (\hat{J}_k^{m\otimes} \otimes I_n) \\ &+ (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k \left((\hat{J}_k^{m\otimes} \otimes A_0) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i) \right). \end{aligned} \tag{3.21}$$

For quadratic parameter-dependent matrices $G_1^T(\rho)G_1(\rho)$ and $G_2^T(\rho)G_2(\rho)$, the PPDQ function representations of degree k are as follows:

$$\begin{aligned} G_1^T(\rho)G_1(\rho) &:= \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n\right)^T \overline{G}_k \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n\right) \\ &= \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n\right)^T (\hat{J}_k^{m\otimes} \otimes I_n)^T \overline{G}_k (\hat{J}_k^{m\otimes} \otimes I_n) \\ &\quad \times \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n\right), \end{aligned} \tag{3.22}$$

$$\begin{aligned} G_2^T(\rho)G_2(\rho) &:= \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_l\right)^T \tilde{G}_k \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_l\right) \\ &= \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l\right)^T (\hat{J}_k^{m\otimes} \otimes I_l)^T \tilde{G}_k (\hat{J}_k^{m\otimes} \otimes I_l) \\ &\quad \times \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l\right), \end{aligned} \tag{3.23}$$

where the certain matrices $\overline{G}_k$ and $\widetilde{G}_k$ are defined, respectively, as

$$\overline{G}_k := \text{Block diagonal}\left(\begin{bmatrix} G_{01}^T \\ G_{11}^T \\ \vdots \\ G_{m1}^T \end{bmatrix}\begin{bmatrix} G_{01} & G_{11} & \cdots & G_{m1} \end{bmatrix}, \underbrace{0_n,\ldots,0_n}_{(k^m-m-1)\text{-elements}}\right),$$

$$\widetilde{G}_k := \text{Block diagonal}\left(\begin{bmatrix} G_{02}^T \\ G_{12}^T \\ \vdots \\ G_{m2}^T \end{bmatrix}\begin{bmatrix} G_{02} & G_{12} & \cdots & G_{m2} \end{bmatrix}, \underbrace{0_l,\ldots,0_l}_{(k^m-m-1)\text{-elements}}\right). \tag{3.24}$$

Similarly, the parameter-independent matrices C_1, C_2, and $C_2^TC_2$ can be also represented as

$$\begin{aligned} C_1 &:= \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_p\right)^T \overline{C}_k \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n\right) \\ &= \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_p\right)^T (\hat{J}_k^{m\otimes} \otimes I_p)^T \overline{C}_k (\hat{J}_k^{m\otimes} \otimes I_n) \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_n\right), \end{aligned} \tag{3.25}$$

$$\begin{aligned} C_2 &:= \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_p\right)^T \widetilde{C}_k \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_l\right) \\ &= \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_p\right)^T (\hat{J}_k^{m\otimes} \otimes I_p)^T \widetilde{C}_k (\hat{J}_k^{m\otimes} \otimes I_l) \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l\right), \end{aligned} \tag{3.26}$$

$$\begin{aligned} C_2^TC_2 &:= \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_l\right)^T \widetilde{C}_k^T\widetilde{C}_k \left(\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_l\right) \\ &= \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l\right)^T (\hat{J}_k^{m\otimes} \otimes I_l)^T \widetilde{C}_k^T\widetilde{C}_k (\hat{J}_k^{m\otimes} \otimes I_l) \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l\right), \end{aligned} \tag{3.27}$$

where the certain matrices $\overline{C}_k$, $\widetilde{C}_k$ are defined, respectively, as follows:

$$\overline{C}_k := \operatorname{diag}\big(C_1, \underbrace{0_{p\times n},\ldots,0_{p\times n}}_{(k^m-1)\text{-elements}}\big), \qquad \widetilde{C}_k := \operatorname{diag}\big(C_2, \underbrace{0_{p\times l},\ldots,0_{p\times l}}_{(k^m-1)\text{-elements}}\big). \tag{3.28}$$

Remark 3.5. From (3.23) and (3.27), the parameter-dependent matrix $C_2^TC_2 + (1/\varepsilon)G_2^T(\rho)G_2(\rho)$ can be shown as

$$\begin{aligned} C_2^TC_2 + \frac{1}{\varepsilon}G_2^T(\rho)G_2(\rho) &= \left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l\right)^T (\hat{J}_k^{m\otimes} \otimes I_l)^T \left(\widetilde{C}_k^T\widetilde{C}_k + \frac{1}{\varepsilon}\widetilde{G}_k\right) \\ &\quad \times (\hat{J}_k^{m\otimes} \otimes I_l)\left(\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_l\right), \end{aligned} \tag{3.29}$$

therefore, using linear algebra properties and some matrix manipulations, the nonsingularity condition of the matrix $C_2^T C_2 + (1/\varepsilon) G_2^T(\rho) G_2(\rho)$ can be stated as the nonsingularity condition of the following matrix:

$$\tilde{C}_k^T \tilde{C}_k + \frac{1}{\varepsilon} \begin{bmatrix} G_{02}^T \\ G_{12}^T \\ \vdots \\ G_{m2}^T \end{bmatrix} \begin{bmatrix} G_{02} & G_{12} & \cdots & G_{m2} \end{bmatrix}. \tag{3.30}$$

THEOREM 3.6. *Let the positive integer $k-1$ as the degree of the PPDQ functions be given. Consider the LTIPD system (2.1) with norm-bounded nonlinear uncertainties. If there exist positive scalars γ and ε and parameter-independent positive-definite matrix $P_k \in \Re^{k^m n \times k^m n}$, and a set of parameter-independent positive-definite multipliers $(\hat{Q}_{i,k}^{(1)}, \hat{Q}_{i,k}^{(4)}) \in \Re^{k^{m-i+1}(k+1)^{i-1} n \times k^{m-i+1}(k+1)^{i-1} n}$, $\hat{Q}_{i,k}^{(2)} \in \Re^{k^{m-i+1}(k+1)^{i-1} l \times k^{m-i+1}(k+1)^{i-1} l}$, $\hat{Q}_{i,k}^{(3)} \in \Re^{k^{m-i+1}(k+1)^{i-1} p \times k^{m-i+1}(k+1)^{i-1} p}$, and $\hat{Q}_{i,k}^{(5)} \in \Re^{k^{m-i+1}(k+1)^{i-1} s \times k^{m-i+1}(k+1)^{i-1} s}$ for $i = 1, 2, \ldots, m$ to the following parameter-independent* LMI,

$$(\mathrm{LMI}_{m,k}) : \begin{bmatrix} \Sigma_{11} & H_k & (\hat{J}_k^{m\otimes} \otimes I_n)^T \overline{C}_k^T (\hat{J}_k^{m\otimes} \otimes I_p) & (\hat{J}_k^{m\otimes} \otimes I_n)^T P_k (\hat{J}_k^{m\otimes} \otimes I_n) & F_k \\ * & \Sigma_{22} & (\hat{J}_k^{m\otimes} \otimes I_l)^T \tilde{C}_k^T (\hat{J}_k^{m\otimes} \otimes I_p) & 0 & 0 \\ * & * & \Sigma_{33} & 0 & 0 \\ * & * & * & \Sigma_{44} & 0 \\ * & * & * & * & \Sigma_{55} \end{bmatrix} < 0, \tag{3.31}$$

where $\hat{I}_k :=$ *Block diagonal* $(I_p, \underbrace{0_p, \ldots, 0_p}_{(k^m-1)\text{-elements}})$, $\overline{I}_k :=$ *Block diagonal* $(I_n, \underbrace{0_n, \ldots, 0_n}_{(k^m-1)\text{-elements}})$, $\tilde{I}_k :=$ *Block diagonal* $(I_s, \underbrace{0_s, \ldots, 0_s}_{(k^m-1)\text{-elements}})$, *and*

$$\Sigma_{11} = R_k + \varepsilon^{-1} (\hat{J}_k^{m\otimes} \otimes I_n)^T \overline{G}_k (\hat{J}_k^{m\otimes} \otimes I_n) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1} n})^T \hat{Q}_{i,k}^{(1)} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1} n})$$
$$- \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1} n})^T \hat{Q}_{i,k}^{(1)} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1} n}),$$

$$\Sigma_{22} = \varepsilon^{-1} (\hat{J}_k^{m\otimes} \otimes I_l)^T \tilde{G}_k (\hat{J}_k^{m\otimes} \otimes I_l) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1} l})^T \hat{Q}_{i,k}^{(2)} (\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1} l})$$
$$- \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1} l})^T \hat{Q}_{i,k}^{(2)} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1} l}),$$

$$\Sigma_{33} = -\left(\hat{J}_k^{m\otimes} \otimes I_p\right)^T \hat{I}_k \left(\hat{J}_k^{m\otimes} \otimes I_p\right) + \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}p}\right)^T \hat{Q}_{i,k}^{(3)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}p}\right)$$

$$- \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}p}\right)^T \hat{Q}_{i,k}^{(3)} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}p}\right),$$

$$\Sigma_{44} = -\varepsilon^{-1} \left(\hat{J}_k^{m\otimes} \otimes I_n\right)^T \bar{I}_k \left(\hat{J}_k^{m\otimes} \otimes I_n\right) + \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(4)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)$$

$$- \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(4)} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),$$

$$\Sigma_{55} = -\gamma^2 \left(\hat{J}_k^{m\otimes} \otimes I_s\right)^T \tilde{I}_k \left(\hat{J}_k^{m\otimes} \otimes I_s\right) + \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}s}\right)^T \hat{Q}_{i,k}^{(5)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}s}\right)$$

$$- \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}s}\right)^T \hat{Q}_{i,k}^{(5)} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}s}\right), \tag{3.32}$$

then the parameter-dependent state feedback control law

$$u(t;\rho) = -\left(C_2^T C_2 + \frac{1}{\varepsilon} G_2^T(\rho) G_2(\rho)\right)^{-1} \left(C_2^T C_1 + B_u^T(\rho) P_\rho\right) x(t) \tag{3.33}$$

achieves both robust global asymptotic stability and robust disturbance attenuation with the attenuation bound γ in the sense of Definition 2.2.

Proof. By substituting the relations (3.18)–(3.30) into the parameter-dependent inequality (3.15), one parameter-dependent matrix inequality is obtained which includes left- and right-multiplication of the $(\mathrm{LMI}_{m,k})$ by

$$\begin{bmatrix} (Z \otimes I_n)^T & 0 & 0 & 0 & 0 \\ * & (Z \otimes I_l)^T & 0 & 0 & 0 \\ * & * & (Z \otimes I_p)^T & 0 & 0 \\ * & * & * & (Z \otimes I_n)^T & 0 \\ * & * & * & * & (Z \otimes I_s)^T \end{bmatrix}, \tag{3.34}$$

and its transpose, where $Z = \rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]}$. Then, it can be concluded that the $(\mathrm{LMI}_{m,k})$, which included the positive-definite multipliers $\hat{Q}_{i,k}^{(1)}, \ldots, \hat{Q}_{i,k}^{(2r+3)}$ for $i = 1, 2, \ldots, m$ (refer to [10]), are *sufficient conditions* to fulfil the parameter-dependent matrix inequality (3.15) for any $\rho \in \zeta$. □

Remark 3.7. Notice that the conditions of Theorem 3.6 are sufficient to both asymptotic stability and H_∞ performance of the LTIPD system (2.1) in the sense of Definition 2.2. Moreover, the theorem gives a suboptimal solution to the robust H_∞ control and this result can be reformulated as an optimal H_∞ control by solving the following convex optimization problem:

$$\underset{\text{subject to } (\mathrm{LMI}_{m,k}) \text{ where } \delta:=\gamma^2}{\mathrm{Min}} \delta. \tag{3.35}$$

Remark 3.8. It is essential in this result that P_k is calculated independently from the parameter vector ρ and after that P_ρ and the control law are found analytically. It is also observed that the $(\mathrm{LMI}_{m,k})$ is linear in P_k, $\hat{Q}_{i,k}^{(1)}$, $\hat{Q}_{i,k}^{(2)}$, $\hat{Q}_{i,k}^{(3)}$, $\hat{Q}_{i,k}^{(4)}$, $\hat{Q}_{i,k}^{(5)}$, thus the standard LMI techniques can be exploited to find the positive-definite solutions [10].

Remark 3.9. A new set of matrices verifying $(\mathrm{LMI}_{m,k+1})$ can be generated, with index $k+1$ instead of k. In this case, the solvability of $(\mathrm{LMI}_{m,k})$ implies the same property for the larger values of the index k; in other words, one deduces in particular that, for any positive integer k,

$$(\mathrm{LMI}_{m,k}) \text{ is solvable } = (\mathrm{LMI}_{m,k'}) \text{ is solvable} \quad \text{for } k' \geq k. \tag{3.36}$$

Remark 3.10. The result of Theorem 3.6 may be conservative due to the use of Lemmas 3.1 and A.2. However, such conservativeness can be significantly reduced by appropriate choice of the parameter ε in a matrix norm sense. The relevant discussion and corresponding numerical algorithm can be found in [16] and the references therein.

4. Simulation results

In this section, we illustrate the proposed methodology on a simple flexible mechanical system. The parameter-dependent plant in the state-space form is given by

$$\begin{aligned}\dot{x}(t) = {} & \left(\begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix} + \rho_1 \begin{bmatrix} 0.1 & 1 \\ 0.1 & 0.1 \end{bmatrix}\right) x(t) + \left(\begin{bmatrix} 0 \\ -4\xi\omega \end{bmatrix} + \rho_1 \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}\right) u(t) \\ & + \left(\begin{bmatrix} 0.01 \\ -1 \end{bmatrix} + \rho_1 \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}\right) w(t) + \begin{bmatrix} 0.9\sin\left(x_1(t)+x_2(t)\right) \\ 0.8\cos\left(u(t)\right) \end{bmatrix}, \\ z(t) = {} & [0 \quad 1]x(t) + 0.1u(t)\end{aligned} \tag{4.1}$$

with initial condition $x(0) = [0 \quad 1]^T$, natural frequency $\omega = 1$ (rad/s), damping $\xi = 0.1$, and $\rho_1 \in \zeta = (-1,1)$ is a real parameter. The norm-bounded uncertainty set is considered as

$$\Xi(x,u;\rho_1) = \left\{\Delta(x,u;\rho_1) : \|\Delta(x,u;\rho_1)\|_2^2 \leq \|(G_{01}+\rho_1 G_{11})x\|_2^2 + \|(G_{02}+\rho_1 G_{12})u\|_2^2\right\}, \tag{4.2}$$

where $G_{01} = I_2$, $G_{11} = 0.1 \times I_2$, $G_{02} = 1$, and $G_{12} = 0.1$.

It is clear that the single parameter ρ_1 appears in the dynamic matrix. Solving the matrix inequality $(\mathrm{LMI}_{m,k})$, using the toolbox Lmitool of the Matlab software [10], gives

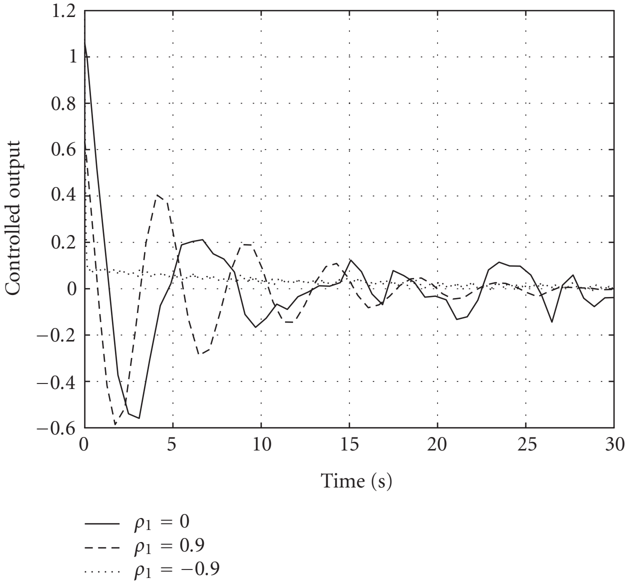

Figure 4.1. Robust stability of the controlled output.

the positive-definite matrix $P_k > 0$ for $m = 1$ and $k = 3$ as

$$P_3 = \begin{bmatrix} 1.0674 & -0.0771 & -0.2551 & -0.8016 & 0.0971 & -0.0575 \\ -0.0771 & 0.3692 & 0.0538 & 0.1192 & 0.0011 & 0.0107 \\ -0.2551 & 0.0538 & 1.7732 & 0.1120 & -0.2757 & -0.8472 \\ -0.8016 & 0.1192 & 0.1120 & 1.5052 & 0.1185 & 0.4343 \\ 0.0971 & 0.0011 & -0.2757 & 0.1185 & 1.4041 & 0.1513 \\ -0.0575 & 0.0107 & -0.8472 & 0.4343 & 0.1513 & 1.4310 \end{bmatrix}, \tag{4.3}$$

with $\gamma = 0.9173$ and $\varepsilon = 15.1150$.

Robust stability and disturbance attenuation of the controlled output in the presence of disturbance has been depicted for three different values of parameter ρ_1 in Figure 4.1. Therefore, we conclude that system (4.1) can be stabilized by the parameter-dependent state feedback control law (3.33), which has been depicted for three different values of ρ_1 in Figure 4.2. And also the correctness of the disturbance attenuation on the controlled output, that is, $\|z(t)\|_2^2 - \gamma^2 \|w(t)\|_2^2 < 0$, has been depicted in Figure 4.3.

5. Conclusion

The issue of robust disturbance attenuation and robust asymptotic stability problem for a class of LTIPD systems with norm-bounded nonlinear uncertainties was considered in this paper. By combining the idea of PPDQ Lyapunov functions and LMIs formulations, sufficient conditions with high precision were given to guarantee robust stabilization of

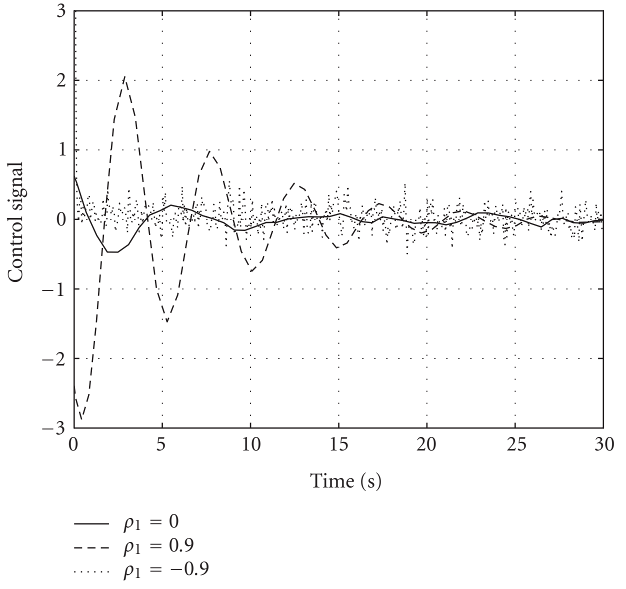

Figure 4.2. Parameter-dependent state feedback control.

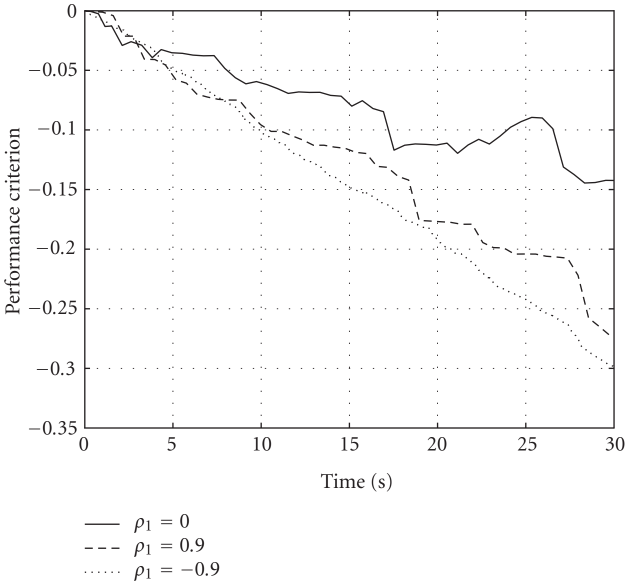

Figure 4.3. The plot of $\|z(t)\|_2^2 - \gamma^2\|w(t)\|_2^2$.

the LTIPD system. Based on the HJI method, the parameter-dependent state feedback control was designed. Finally, the applicability of the proposed method was illustrated in a simple example.

Appendix

LEMMA A.1 (Schur complement lemma). *Given constant matrices* Ψ_1, Ψ_2, *and* Ψ_3 *where* $\Psi_1 = \Psi_1^T$ *and* $\Psi_2 = \Psi_2^T > 0$, *then* $\Psi_1 + \Psi_3^T \Psi_2^{-1} \Psi_3 < 0$ *if and only if*

$$\begin{bmatrix} \Psi_1 & \Psi_3^T \\ \Psi_3 & -\Psi_2 \end{bmatrix} < 0 \quad \text{or equivalently,} \quad \begin{bmatrix} -\Psi_2 & \Psi_3 \\ \Psi_3^T & \Psi_1 \end{bmatrix} < 0. \tag{A.1}$$

LEMMA A.2 [18]. *For any matrices X and Y with appropriate dimensions and for any constant* $\eta > 0$, *the following inequality holds:*

$$X^T Y + Y^T X \le \eta X^T X + \frac{1}{\eta} Y^T Y. \tag{A.2}$$

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