# APPLICATION OF ZERO EIGENVALUE FOR SOLVING THE POTENTIAL, HEAT, AND WAVE EQUATIONS USING A SEQUENCE OF SPECIAL FUNCTIONS 

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In the solution of boundary value problems, usually zero eigenvalue is ignored. This case also happens in calculating the eigenvalues of matrices, so that we would often like to find the nonzero solutions of the linear system $A X=\lambda X$ when $\lambda \neq 0$. But $\lambda=0$ implies that $\operatorname{det} A=0$ for $X \neq 0$ and then the rank of matrix $A$ is reduced at least one degree. This comment can similarly be stated for boundary value problems. In other words, if at least one of the eigens of equations related to the main problem is considered zero, then one of the solutions will be specified in advance. By using this note, first we study a class of special functions and then apply it for the potential, heat, and wave equations in spherical coordinate. In this way, some practical examples are also given.

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## 1. Introduction

Let us consider the following sequences:

$$
\begin{align*}
& C_{n}(z ; a(z))=\frac{(a(z))^{n}+(a(z))^{-n}}{2}=\operatorname{Ch}(n \ln (a(z))), \\
& S_{n}(z ; a(z))=\frac{(a(z))^{n}-(a(z))^{-n}}{2}=\operatorname{Sh}(n \ln (a(z))) \tag{1.1}
\end{align*}
$$

in which $a(z)$ can be a complex (or real) function and $n$ is a positive integer number.
It is not difficult to verify that both of these sequences satisfy a unique second-order differential equation in the form

$$
\begin{equation*}
a^{2}(z) a^{\prime}(z) y^{\prime \prime}+\left(a(z)\left(a^{\prime}(z)\right)^{2}-a^{2}(z) a^{\prime \prime}(z)\right) y^{\prime}-n^{2}\left(a^{\prime}(z)\right)^{3} y=0 \tag{1.2}
\end{equation*}
$$

provided that $a^{2}(z) a^{\prime}(z) \neq 0$. Hence, we face only one class of special functions, which is in fact the solution of (1.2). The functions $C_{n}(z ; a(z))$ and $S_{n}(z ; a(z))$ have several subcases that can be useful to study. The first subcase is the Chebyshev polynomials if one takes $a(z)=\exp (i \operatorname{Arccos} z)$ and uses the well-known Euler relation. In this case, the following sequences will be derived:

$$
\begin{align*}
C_{n}(z ; \exp (i \operatorname{Arccos} z)) & =\operatorname{Cos}(n \operatorname{Arccos} z)=T_{n}(z), \\
S_{n}(z ; \exp (i \operatorname{Arccos} z)) & =i \operatorname{Sin}(n \operatorname{Arccos} z)=i \sqrt{1-z^{2}} U_{n-1}(z), \tag{1.3}
\end{align*}
$$

where $T_{n}(z)$ and $U_{n}(z)$ are, respectively, the first and second kinds of Chebyshev polynomials [1]. Moreover, if the selected $a(z)$ is replaced in (1.2), the differential equation of the first-kind Chebyshev polynomials

$$
\begin{equation*}
\left(1-z^{2}\right) y^{\prime \prime}-z y^{\prime}+n^{2} y=0 \tag{1.4}
\end{equation*}
$$

will be obtained. Rational Chebyshev functions are the second subcase that can be generated by choosing $a(z)=\exp (i \operatorname{Arccot} g z)$. So, for this case we get

$$
\begin{align*}
C_{n}(z ; \exp (i \operatorname{Arccot} g z)) & =\operatorname{Cos}(n \operatorname{Arccot} g z) \\
S_{n}(z ; \exp (i \operatorname{Arccot} g z)) & =i \operatorname{Sin}(n \operatorname{Arccot} g z) . \tag{1.5}
\end{align*}
$$

But replacing the assigned $a(z)$ in (1.2) yields

$$
\begin{align*}
& a^{2}(z) a^{\prime}(z)=-\frac{i \exp (3 i \operatorname{Arccot} g z)}{1+z^{2}}, \quad a(z)\left(a^{\prime}(z)\right)^{2}=-\frac{\exp (3 i \operatorname{Arccot} g z)}{\left(1+z^{2}\right)^{2}}, \\
& a^{2}(z) a^{\prime \prime}(z)=\frac{(2 i z-1) \exp (3 i \operatorname{Arccot} g z)}{\left(1+z^{2}\right)^{2}}, \quad\left(a^{\prime}(z)\right)^{3}=\frac{i \exp (3 i \operatorname{Arccot} g z)}{\left(1+z^{2}\right)^{3}} . \tag{1.6}
\end{align*}
$$

Therefore the functions (1.5) eventually satisfy the equation

$$
\begin{equation*}
\left(1+z^{2}\right)^{2} y^{\prime \prime}+2 z\left(1+z^{2}\right) y^{\prime}+n^{2} y=0 \tag{1.7}
\end{equation*}
$$

Note that the explicit forms of the real functions $C_{n}(z ; \exp (i \operatorname{Arc} \cot g z))$ and $-i S_{n}(z ; \exp$ $(i \operatorname{Arc} \cot g z))$ can be extracted by the Moivre's formula directly. To reach this purpose, let us substitute $\theta=\operatorname{Arccot} g x$ in the mentioned formula to get

$$
\begin{equation*}
\frac{(x+i)^{n}}{\left(\sqrt{1+x^{2}}\right)^{n}}=\operatorname{Cos}(n \operatorname{Arccot} g x)+i \operatorname{Sin}(n \operatorname{Arccot} g x) \tag{1.8}
\end{equation*}
$$

Consequently we have

$$
\begin{align*}
C_{n}(x ; \exp (i \operatorname{Arccot} g x)) & =\frac{\sum_{k=0}^{[n / 2]}(-1)^{k}\binom{n}{2 k} x^{n-2 k}}{\left(\sqrt{1+x^{2}}\right)^{n}}=T_{n}^{*}(x), \\
-i S_{n+1}(x ; \exp (i \operatorname{Arccot} g x)) & =\frac{\sum_{k=0}^{[n / 2]}(-1)^{k}\binom{n+1}{2 k+1} x^{n-2 k}}{\left(\sqrt{1+x^{2}}\right)^{n+1}}=U_{n}^{*}(x) . \tag{1.9}
\end{align*}
$$

Here one can observe that the Chebyshev rational functions $T_{n}^{*}(x)$ and $U_{n}^{*}(x)$ are orthogonal with respect to the weight function $W(x)=1 /\left(1+x^{2}\right)$ on $(-\infty, \infty)$ and have the following orthogonality properties, respectively,

$$
\begin{align*}
& \int_{-\infty}^{\infty} \frac{T_{n}^{*}(x) T_{n}^{*}(x)}{1+x^{2}} d x= \begin{cases}0 & \text { if } m \neq n, \\
\pi & \text { if } m=n, \\
2 \pi & \text { if } m=n=0,\end{cases}  \tag{1.10}\\
& \int_{-\infty}^{\infty} \frac{U_{n}^{*}(x) U_{n}^{*}(x)}{1+x^{2}} d x= \begin{cases}0 & \text { if } m \neq n, \\
\pi & \text { if } m=n .\end{cases}
\end{align*}
$$

Let us point out that Boyd in [2] applied the rational functions $T_{n}^{*}\left(\left(x^{1 / 2}-x^{-1 / 2}\right) / 2\right)$ on a semi-infinite interval $[0, \infty)$ for using in spectral methods, and here we mention that the foresaid functions can be derived only by replacing $a(z)=\exp (2 i \operatorname{Arccot} g \sqrt{z})$ in (1.1). But it is known that the Legendre (or associated Legendre) differential equation [1]

$$
\begin{equation*}
\left(1-x^{2}\right) y^{\prime \prime}(x)-2 x y^{\prime}(x)+\left(p-\frac{q}{1-x^{2}}\right) y(x)=0 \tag{1.11}
\end{equation*}
$$

has been solved in the three following cases in the Cartesian coordinate:
(a) $p \neq 0, q \neq 0$ that generates the associated Legendre functions;
(b) $p \neq 0, q=0$ that generates the Legendre polynomials;
(c) $p=0, q=0$ that is reduced to the simple equation $\left(1-x^{2}\right) y^{\prime \prime}(x)-2 x y^{\prime}(x)=0$, which has the solution $y(x)=c_{1} \operatorname{Ln}((1+x) /(1-x))+c_{2}$.
Hence, a fourth case $p=0, q \neq 0$ remains, which is different from the three above-mentioned cases and should be solved. To reach the solution, let us substitute $a(z)=((1-z) /$ $(1+z))^{1 / 2}$ in (1.2). This leads to arrive at the differential equation:

$$
\begin{equation*}
\left(1-z^{2}\right) y^{\prime \prime}-2 z y^{\prime}-\frac{n^{2}}{1-z^{2}} y=0 \tag{1.12}
\end{equation*}
$$

which is a particular case of (1.11) for $p=0, q=n^{2}$. According to (1.1), the solutions of this equation are, respectively,

$$
\begin{align*}
& C_{n}\left(z ;\left(\frac{1-z}{1+z}\right)^{1 / 2}\right)=\frac{1}{2}\left(\left(\frac{1-z}{1+z}\right)^{n / 2}+\left(\frac{1-z}{1+z}\right)^{-n / 2}\right), \\
& S_{n}\left(z ;\left(\frac{1-z}{1+z}\right)^{1 / 2}\right)=\frac{1}{2}\left(\left(\frac{1-z}{1+z}\right)^{n / 2}-\left(\frac{1-z}{1+z}\right)^{-n / 2}\right) \tag{1.13}
\end{align*}
$$

Now, we would like to recall that these functions will be useful if they are considered in the Helmholtz equation in spherical coordinate. In other words, if the equation $\nabla^{2} U(r, \theta$, $\Phi)=k^{2} U(r, \theta, \Phi)$ is separated to ordinary equations, then one of the separate equations takes the form

$$
\begin{equation*}
\frac{1}{\operatorname{Sin} \theta} \frac{d}{d \theta}\left(\operatorname{Sin} \theta \frac{d y}{d \theta}\right)+\left(m(m+1)-\frac{n^{2}}{\operatorname{Sin}^{2} \theta}\right) y(\theta)=0 \tag{1.14}
\end{equation*}
$$

which is the same as (1.11) for $x=\operatorname{Cos} \theta$. Thus, if $z=\operatorname{Cos} \theta$ is considered in (1.12) or equivalently $a(z)=\operatorname{tg}(z / 2)$ in (1.2), then the special case of (1.14) for $m=0$, that is,

$$
\begin{equation*}
y^{\prime \prime}+(\operatorname{Cot} g z) y^{\prime}-\frac{n^{2}}{\operatorname{Sin}^{2} z} y=0 \tag{1.15}
\end{equation*}
$$

has the following solutions:

$$
\begin{align*}
& C_{n}\left(z ; \operatorname{tg} \frac{z}{2}\right)=\frac{1}{2}\left(\left(\operatorname{tg} \frac{z}{2}\right)^{n}+\left(\operatorname{tg} \frac{z}{2}\right)^{-n}\right) \\
& S_{n}\left(z ; \operatorname{tg} \frac{z}{2}\right)=\frac{1}{2}\left(\left(\operatorname{tg} \frac{z}{2}\right)^{n}-\left(\operatorname{tg} \frac{z}{2}\right)^{-n}\right) \tag{1.16}
\end{align*}
$$

These sequences will be used in the given problems of the next section.

## 2. Application of functions (1.16) for the potential, heat, and wave equations in spherical coordinate

Usually most of the boundary value problems related to the wave, heat, and potential equations in spherical coordinate are reduced to the Helmholtz partial differential equation [1], which is mentioned in the form $\nabla^{2} U(r, \theta, \Phi)=k^{2} U(r, \theta, \Phi)$. But $k=0$ in this relation implies

$$
\begin{equation*}
\frac{\partial^{2} U}{\partial r^{2}}+\frac{2}{r} \frac{\partial U}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} U}{\partial \theta^{2}}+\frac{\cot g \theta}{r^{2}} \frac{\partial U}{\partial \theta}+\frac{1}{r^{2} \sin ^{2} \theta} \frac{\partial^{2} U}{\partial \Phi^{2}}=0 \tag{2.1}
\end{equation*}
$$

to be the potential (Laplace) equation in spherical coordinate [3]. Now, if the related variables are separated as $U(r, \theta, \Phi)=R(r) A(\theta) B(\Phi)$, then the ordinary differential equations of (2.1) will appear as follows:

$$
\begin{gather*}
r^{2} R^{\prime \prime}+2 r R^{\prime}-\lambda_{1} R=0 \\
B^{\prime \prime}-\lambda_{2} B=0  \tag{2.2}\\
A^{\prime \prime}+\operatorname{Cot} g \theta A^{\prime}+\left(\lambda_{1}+\frac{\lambda_{2}}{\sin ^{2} \theta}\right) A=0
\end{gather*}
$$

As it is known, the solution of Laplace equation is generally determined when the boundary conditions are known. Nevertheless, if in (2.2) $\lambda_{1}=0$ and $\lambda_{2}=-k^{2} \neq 0$ are supposed,
then for the variable $r$ we must have $R(r)=-c_{1}^{2}(1 / r)+c_{2}\left(c_{1}, c_{2}\right.$ are constant) and for the third equation the same form as (1.15). Hence, the general solution corresponding to the third equation would be

$$
\begin{equation*}
A(\theta)=A_{1} C_{n}\left(\theta ; \operatorname{tg} \frac{\theta}{2}\right)+A_{2} S_{n}\left(\theta ; \operatorname{tg} \frac{\theta}{2}\right)=a_{1} \operatorname{tg}^{k}\left(\frac{\theta}{2}\right)+a_{2} \operatorname{tg}^{-k}\left(\frac{\theta}{2}\right), \tag{2.3}
\end{equation*}
$$

where $A_{1}$ and $A_{2}$, and $a_{1}$ and $a_{2}$ are all constant values. (2.3) implies to have the general solution of the potential equation $\nabla^{2} U(r, \theta, \Phi)=0$, with predetermined condition $R(r)=-c_{1}^{2}(1 / r)+c_{2}$, as

$$
\begin{equation*}
U(r, \theta, \Phi)=\left(-\frac{c_{1}^{2}}{r}+c_{2}\right)\left(b_{1} \operatorname{Cos} k \Phi+b_{2} \operatorname{Sin} k \Phi\right)\left(a_{1} \operatorname{tg}^{k}\left(\frac{\theta}{2}\right)+a_{2} \operatorname{tg}^{-k}\left(\frac{\theta}{2}\right)\right) \tag{2.4}
\end{equation*}
$$

The solution (2.4) shows the sensitivity of the potential equation with respect to the variable $r$, so that we have $\operatorname{Lim}_{r \rightarrow 0} U(r, \theta, \Phi)=\infty$.

As an example, let us consider the Laplace equation $\nabla^{2} U(r, \theta, \Phi)=0,0<r<a$ (in spherical coordinate) when the variable $r$ takes the preassigned form $R(r)=-c_{1}^{2} / r+c_{2}$ and the following initial and boundary conditions hold:

$$
\begin{align*}
\operatorname{Lim}_{r \rightarrow 0} U(r, \theta, \Phi) & =\infty  \tag{2.5a}\\
U\left(\frac{a}{2}, \theta, \Phi\right) & =0  \tag{2.5b}\\
U\left(r, \frac{\pi}{2}, \Phi\right) & =0  \tag{2.5c}\\
U\left(a, \frac{\pi}{3}, \Phi\right) & =\Phi \tag{2.5d}
\end{align*}
$$

The general solution of this problem, according to the given conditions and assuming $A_{k}=a_{1} b_{1} c_{2}, B_{k}=a_{2} b_{1} c_{2}$ would be finally

$$
\begin{gather*}
U(r, \theta, \Phi)=\sum_{k} U_{k}(r, \theta, \Phi)  \tag{2.6}\\
U_{k}(r, \theta, \Phi)=\left(1-\frac{a}{2 r}\right)\left(\operatorname{tg}^{k}\left(\frac{\theta}{2}\right)-\operatorname{tg}^{-k}\left(\frac{\theta}{2}\right)\right)\left(A_{k} \operatorname{Cos} k \Phi+B_{k} \operatorname{Sin} k \Phi\right)
\end{gather*}
$$

On the other hand, putting the last condition (2.5d) in the above relation yields

$$
\begin{equation*}
2 \Phi=\sum_{k}\left(\sqrt{3}^{-k}-\sqrt{3}^{+k}\right)\left(A_{k} \operatorname{Cos} k \Phi+B_{k} \operatorname{Sin} k \Phi\right), \tag{2.7}
\end{equation*}
$$

where $A_{k}$ and $B_{k}$ are calculated by

$$
\begin{align*}
& \left(\sqrt{3}^{-k}-\sqrt{3}^{+k}\right) A_{k}=\frac{4}{\pi} \int_{0}^{\pi} \Phi \operatorname{Cos} k \Phi d \Phi=\frac{4\left((-1)^{k}-1\right)}{\pi k^{2}}  \tag{2.8}\\
& \left(\sqrt{3}^{-k}-\sqrt{3}^{+k}\right) B_{k}=\frac{4}{\pi} \int_{0}^{\pi} \Phi \operatorname{Sin} k \Phi d \Phi=\frac{4(-1)^{k}}{k}
\end{align*}
$$

Thus, the specific solution of the potential equation under the given conditions is

$$
\begin{align*}
U(r, \theta, \Phi)= & \left(1-\frac{a}{2 r}\right) \sum_{k=1}^{\infty}\left(\frac{4\left((-1)^{k}-1\right) k^{-2}}{\pi\left(3^{-k / 2}-3^{k / 2}\right)} \operatorname{Cos} k \Phi+\frac{4(-1)^{k} k^{-1}}{\left(3^{-k / 2}-3^{k / 2}\right)} \sin k \Phi\right)  \tag{2.9}\\
& \times\left(\operatorname{tg}^{k} \frac{\theta}{2}-\operatorname{tg}^{-k} \frac{\theta}{2}\right)
\end{align*}
$$

As we see, a special case of the functions (1.1) appeared in the above solution.
Similarly one can propound application of zero eigenvalue for the heat and wave equations, respectively. For instance, if the heat equation $\nabla^{2} U=\partial U / \partial t$ is considered, then separating the variables as $U(r, \theta, \Phi, t)=S(r, \theta, \Phi) T(t)$, where $S(r, \theta, \Phi)=R(r) A(\theta) B(\Phi)$, yields

$$
\begin{align*}
\nabla^{2} U & =\nabla^{2}(S T)=T \nabla^{2} S  \tag{2.10}\\
\frac{\partial U}{\partial t} & =\frac{\partial(S T)}{\partial t}=S \frac{\partial T}{\partial t}
\end{align*} \Longrightarrow T \nabla^{2} S=T^{\prime}(t) S \Longrightarrow \quad \begin{array}{r}
\nabla^{2} S-\alpha S=0 \\
T^{\prime}-\alpha T=0
\end{array}
$$

which gives the ordinary differential equations:

$$
\begin{gather*}
T^{\prime}-\alpha T=0, \\
r^{2} R^{\prime \prime}+2 r R^{\prime}-\left(\alpha r^{2}+\lambda_{1}\right) R=0, \\
B^{\prime \prime}-\lambda_{2} B=0  \tag{2.11}\\
A^{\prime \prime}+\operatorname{Cot} g \theta A^{\prime}+\left(\lambda_{1}+\frac{\lambda_{2}}{\operatorname{Sin}^{2} \theta}\right) A=0
\end{gather*}
$$

Again, if $\lambda_{1}=0, \lambda_{2}=-n^{2} \neq 0$, and $\alpha=-k^{2} \neq 0$ are assumed in (2.11), then the general solution, when $R(r)=\left(c_{1} J_{1 / 2}(k r)+c_{2} J_{-1 / 2}(k r)\right) / \sqrt{k r}$ is preassigned, takes the form

$$
\begin{align*}
U(r, \theta, \Phi, t)= & \frac{e^{-k^{2} t}}{\sqrt{k r}}\left(c_{1} J_{1 / 2}(k r)+c_{2} J_{-1 / 2}(k r)\right)\left(b_{1} \operatorname{tg}^{n}\left(\frac{\theta}{2}\right)+b_{2} \operatorname{tg}^{-n}\left(\frac{\theta}{2}\right)\right)  \tag{2.12}\\
& \times\left(a_{1} \operatorname{Cos} n \Phi+a_{2} \operatorname{Sin} n \Phi\right)
\end{align*}
$$

in which $J_{1 / 2}(x)$ and $J_{-1 / 2}(x)$ are two particular cases of the Bessel functions $J_{p}(x)$ [3]. Here is a good position to consider the heat equation $\nabla^{2} U(r, \theta, \Phi, t)=\partial U / \partial t, 0<r<a$ (in spherical coordinate) when the variable $r$ takes the preassigned form $R(r)=\left(c_{1} J_{1 / 2}(k r)\right.$ $\left.+c_{2} J_{-1 / 2}(k r)\right) / \sqrt{k r}$ and the following conditions hold:

$$
\begin{gather*}
\operatorname{Lim}_{r \rightarrow 0} U(r, \theta, \Phi, t)<M  \tag{2.13a}\\
U\left(r, \frac{\pi}{2}, \Phi, t\right)=0  \tag{2.13b}\\
U(r, \theta, 0, t)=0  \tag{2.13c}\\
U\left(r, \frac{\pi}{3}, \Phi, 0\right)=f(r, \Phi), \quad \text { arbitrary. } \tag{2.13d}
\end{gather*}
$$

By referring to (2.12) and using the given conditions we get $c_{2}=a_{1}=0$ and $b_{1}+b_{2}=0$. Hence, if $A_{k, n}=a_{2} b_{1} c_{1}$ is taken, then

$$
\begin{gather*}
U(r, \theta, \Phi, t)=\sum_{k} \sum_{n} U_{k, n}(r, \theta, \Phi, t), \\
U_{k, n}(r, \theta, \Phi, t)=A_{k, n} \frac{e^{-k^{2} t}}{\sqrt{k r}} J_{1 / 2}(k r) \cdot\left(\operatorname{tg}^{n} \frac{\theta}{2}-\operatorname{tg}^{-n} \frac{\theta}{2}\right) \cdot \sin n \Phi \tag{2.14}
\end{gather*}
$$

would be the general solution. On the other hand, since the orthogonality relation of Bessel functions $J_{p}(x)$ is represented as [3]

$$
\begin{equation*}
\int_{0}^{a} J_{p}\left(Z_{(p, m)} \frac{x}{a}\right) J_{p}\left(Z_{(p, n)} \frac{x}{a}\right) x d x=\frac{a^{2}}{2} J_{p+1}^{2}\left(Z_{(p, m)}\right) \delta_{n, m} \tag{2.15}
\end{equation*}
$$

where $Z_{(p, m)}$ is $m$ th zero of $J_{p}(x)$ (i.e., $J_{p}\left(Z_{(p, m)}\right)=0$ ), so it is better for the eigenvalues $k$ to be considered as $k=Z_{(1 / 2, m)} / a, p=1 / 2$. Therefore, the general solution of the problem becomes

$$
\begin{gather*}
U(r, \theta, \Phi, t)=\sum_{n} \sum_{m} A_{n, m}^{*} \frac{\exp \left(-\left(Z_{(1 / 2, m)} / a\right)^{2} t\right)}{\sqrt{Z_{(1 / 2, m)} r / a}} J_{1 / 2}\left(Z_{(1 / 2, m)} r / a\right)  \tag{2.16}\\
\times\left(\operatorname{tg}^{n} \frac{\theta}{2}-\operatorname{tg}^{-n} \frac{\theta}{2}\right) \sin n \Phi
\end{gather*}
$$

in which $A_{n, m}^{*}=A_{n, Z_{(1 / 2, m)} / a}$. Now, it is sufficient to compute the coefficients $A_{n, m}^{*}$. To do this, substituting the last condition (2.13d) in the above relation yields

$$
\begin{equation*}
\sqrt{\frac{r}{a}} f(r, \Phi)=\sum_{n} \sum_{m} \frac{A_{n, m}^{*}\left(3^{-n / 2}-3^{n / 2}\right)}{\sqrt{Z_{(1 / 2, m)}}} J_{1 / 2}\left(Z_{(1 / 2, m)} \frac{r}{a}\right) \operatorname{Sin}(n \Phi) . \tag{2.17}
\end{equation*}
$$

So, by applying the orthogonality relation of Bessel functions and using the orthogonality property of the sequence $\{\operatorname{Sin}(n \Phi)\}_{n=1}^{\infty}$ on $[0, \pi], A_{n, m}^{*}$ are obtained as

$$
\begin{equation*}
A_{n, m}^{*}=\frac{4 \sqrt{Z_{(1 / 2, m)}} \int_{0}^{a} \int_{0}^{\pi} f(r, \Phi) J_{1 / 2}\left(Z_{(1 / 2, m)}(r / a)\right) \operatorname{Sin}(n \Phi) r^{3 / 2} d r d \Phi}{\pi\left(3^{-n / 2}-3^{n / 2}\right) a^{3 / 2} J_{3 / 2}^{2}\left(Z_{(1 / 2, m)}\right)} \tag{2.18}
\end{equation*}
$$

Finally, the problem can be stated for the wave equation $\nabla^{2} U=\partial^{2} U / \partial t^{2}$ according to the following stages. First we have

$$
\begin{align*}
& \nabla^{2} U=\nabla^{2}(S T)=T \nabla^{2} S  \tag{2.19}\\
& \frac{\partial^{2} U}{\partial t^{2}}=\frac{\partial^{2}(S T)}{\partial t^{2}}=S \frac{\partial^{2} T}{\partial t^{2}} \quad \Longrightarrow T \nabla^{2} S=T^{\prime \prime}(t) S \Longrightarrow \quad \begin{array}{l}
\nabla^{2} S-\alpha S=0 \\
T^{\prime \prime}-\alpha T=0
\end{array} ~
\end{align*}
$$

which results in the ordinary equations:

$$
\begin{gather*}
T^{\prime \prime}-\alpha T=0, \\
r^{2} R^{\prime \prime}+2 r R^{\prime}-\left(\alpha r^{2}+\lambda_{1}\right) R=0, \\
B^{\prime \prime}-\lambda_{2} B=0  \tag{2.20}\\
A^{\prime \prime}+\operatorname{Cot} g \theta A^{\prime}+\left(\lambda_{1}+\frac{\lambda_{2}}{\operatorname{Sin}^{2} \theta}\right) A=0
\end{gather*}
$$

Now, if $\lambda_{1}=0, \lambda_{2}=-k^{2} \neq 0$ and $\alpha=-n^{2} \neq 0$ are assumed in (2.20), then the general solution of the wave equation when $R(r)=\left(c_{1} J_{1 / 2}(n r)+c_{2} J_{-1 / 2}(n r)\right) / \sqrt{n r}$ would be as follows:

$$
\begin{align*}
U(r, \theta, \Phi, t)= & \left(d_{1} \operatorname{Cos} n t+d_{2} \operatorname{Sin} n t\right)\left(b_{1} \operatorname{Cos} k \Phi+b_{2} \operatorname{Sin} k \Phi\right) \\
& \times\left(a_{1} \operatorname{tg}^{k} \frac{\theta}{2}+a_{2} \operatorname{tg}^{-k} \frac{\theta}{2}\right)\left(\frac{c_{1} J_{1 / 2}(n r)+c_{2} J_{-1 / 2}(n r)}{\sqrt{n r}}\right) . \tag{2.21}
\end{align*}
$$

Here let us consider a sample problem regarding the wave equation in spherical coordinate when the variable $r$ has the form $R(r)=\left(c_{1} J_{1 / 2}(n r)+c_{2} J_{-1 / 2}(n r)\right) / \sqrt{n r}$ and the conditions

$$
\begin{gather*}
\operatorname{Lim} U_{r \rightarrow 0}(r, \theta, \Phi, t)<M \\
U\left(r, \frac{\pi}{2}, \Phi, t\right)=0 \\
U(r, \theta, 0, t)=0  \tag{2.22}\\
U(r, \theta, \Phi, 0)=0 \\
U\left(r, \frac{\pi}{3}, \Phi, q\right)=g(r, \Phi), \quad \text { arbitrary }
\end{gather*}
$$

are established. To solve this problem, replacing the given conditions in general solution (2.21) gives $c_{2}=b_{1}=d_{1}=0$ and $a_{1}+a_{2}=0$. If $B_{k, n}=a_{1} b_{2} c_{1} d_{2}$ is supposed, then (2.21) becomes

$$
\begin{gather*}
U(r, \theta, \Phi, t)=\sum_{k} \sum_{n} U_{k, n}(r, \theta, \Phi, t),  \tag{2.23}\\
U_{k, n}(r, \theta, \Phi, t)=B_{k, n} \operatorname{Sin}(n t) \frac{J_{1 / 2}(n r)}{\sqrt{n r}}\left(\operatorname{tg}^{k} \frac{\theta}{2}-\operatorname{tg}^{-k} \frac{\theta}{2}\right) \cdot \operatorname{Sin} k \Phi .
\end{gather*}
$$

But similar to the previous problem, if $n=Z_{(1 / 2, m)} / a$ is taken, then

$$
\begin{equation*}
U(r, \theta, \Phi, t)=\sum_{k} \sum_{m} B_{k, m}^{*} \sin \left(Z_{(1 / 2, m)} \frac{t}{a}\right) \frac{J_{1 / 2}\left(Z_{(1 / 2, m)} r / a\right)}{\sqrt{Z_{(1 / 2, m)} r / a}}\left(\operatorname{tg}^{k} \frac{\theta}{2}-\operatorname{tg}^{-k} \frac{\theta}{2}\right) \sin k \Phi \tag{2.24}
\end{equation*}
$$

where $B_{k, m}^{*}=B_{k, Z_{(1 / 2, m)} / a}$. By substituting the last condition of the problem in the above relation, that is,

$$
\begin{equation*}
\sqrt{\frac{r}{a}} g(r, \Phi)=\sum_{k} \sum_{m} \frac{B_{k, m}^{*}\left(3^{-k / 2}-3^{k / 2}\right) \operatorname{Sin}\left(Z_{(1 / 2, m)}(q / a)\right)}{\sqrt{Z_{(1 / 2, m)}}} J_{1 / 2}\left(Z_{(1 / 2, m)}(r / a)\right) \operatorname{Sin}(k \Phi) \tag{2.25}
\end{equation*}
$$

and using the orthogonality relation of Bessel functions $J_{1 / 2}\left(Z_{(1 / 2, m)}(r / a)\right)$ on $[0, a]$, the unknown coefficients $B_{k, m}^{*}$ will be derived as follows:

$$
\begin{equation*}
B_{k, m}^{*}=\frac{4 \sqrt{Z_{(1 / 2, m)}} \int_{0}^{a} \int_{0}^{\pi} g(r, \Phi) J_{1 / 2}\left(Z_{(1 / 2, m)}(r / a)\right) \operatorname{Sin}(k \Phi) r^{3 / 2} d r d \Phi}{\pi \operatorname{Sin}\left(Z_{(1 / 2, m)}(q / a)\right)\left(3^{-k / 2}-3^{k / 2}\right) a^{3 / 2} J_{3 / 2}^{2}\left(Z_{(1 / 2, m)}\right)} . \tag{2.26}
\end{equation*}
$$

This will give the final solution of the given problem straightforwardly.
Finally we mention that the defined functions $C_{n}(z ; a(z))$ and $S_{n}(z ; a(z))$ in this paper are special cases of a main class of special functions having several important subclasses and a general differential equation of second order. This comment is in preparation.

## References

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