APPROXIMATE TRANSFORMATIONS FOR VAN DER POL-TYPE EQUATIONS

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Classification of van der Pol-type equations with respect to admitted approximate transformation groups transforming a small parameter is given. It is shown that approximate symmetries transforming the small parameter as well as the usual approximate symmetries can be used for approximate integration (e.g., by method of successive reduction of order) of ordinary differential equations with a small parameter.

1. Introduction

It is well known that the knowledge of a point symmetry of an ordinary differential equation allows us to lower its order by one. Hence, if a second-order ordinary differential has two point symmetries it can then be reduced to quadrature. This method, known as the method of successive reduction of order, was developed by the Norwegian mathematician Sophus Lie and is based on his theory of transformation groups (for details of Lie theory and its applications to ordinary differential equations see, e.g., [5] or [8]).

Recently, Baikov et al. [2] developed the theory of approximate transformation groups with the aim of investigating symmetry properties of differential equations with a small parameter (the detailed discussion on this theory and its algorithms can be found in [3]). It was shown by Ibragimov [7] that if an ordinary differential equation of the second-order admits two parameter point approximate transformation group in the sense of [2], then the method of successive reduction of order can be applied to the equation.

Among ordinary differential equations of the second order with a small parameter, the van der Pol equation

$$u'' + u = \varepsilon(1 - u^2)u', \quad ' = \frac{d}{dt}, \tag{1.1}$$

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occupies a special place in the literature, and the main methods of perturbation theory are illustrated by this equation (see, e.g., [6, 9]). However, investigation of approximate point symmetries for (1.1) has shown that it does not admit nontrivial approximate point symmetries, but it has nontrivial approximate *contact* symmetries [1]. Moreover, in [4] it was shown that all symmetries of oscillation equation

$$u'' + u = 0 \tag{1.2}$$

are *stable* with respect to arbitrary perturbation $\varepsilon F(t,u,u')$ when we consider "correction term" of approximate symmetry in the form of Lie-Bäcklund operator. It means that for every symmetry $X_{(0)}$ of (1.2), the equation

$$u'' + u = \varepsilon F(t,u,u') \tag{1.3}$$

with arbitrary function F approximately admits the operator

$$X = X_{(0)} + \varepsilon X_{(1)}, \quad \text{where } X_{(1)} = f(t,u,u')\frac{\partial}{\partial u} + \cdots, \tag{1.4}$$

with appropriately chosen function $f(t,u,u')$. From this point of view we see that the van der Pol equation is not separated from equations of the form $u'' + u = \varepsilon F(t,u,u')$.

In this paper we will consider only *point* approximate transformations which transform not only independent and dependent variables, but also the small parameter ε. One example of such transformations was considered in [3] for construction of approximately invariant solutions of nonlinear diffusion equation with a small dissipation. In this paper, for the first time, we give algorithm of calculation of such transformations admitted by an equation with a small parameter and use them for approximate integration of ordinary differential equations with small parameter. The reason for considering such transformations is to illustrate their simplicity in comparison with Lie-Bäcklund symmetries and their usefulness for classification and the construction of solutions. We give group classification with respect to these transformations of equations of the form

$$u'' + u = \varepsilon f(u)u' \tag{1.5}$$

with arbitrary function f of u. It is shown that a wide class of equations which includes the van der Pol equation admits nontrivial approximate operators generating approximate transformation transforming the small parameter. Also we illustrate that these operators can be used for reduction of order of equations of this class and for its approximate integration by methods similar to Lie's methods of integration of second-order equations.

For completeness sake we present here some definitions, notation, and algorithms for calculating approximate symmetries as given in [3].

A *perturbed* equation

$$\mathcal{F} \equiv \mathcal{F}_{(0)} + \varepsilon\mathcal{F}_{(1)} = o(\varepsilon) \tag{1.6}$$

is said to be *invariant* with respect to *approximate operator*

$$X \equiv X_{(0)} + \varepsilon X_{(1)} \tag{1.7}$$

if the approximate symmetry condition

$$X\mathcal{F}|_{\mathcal{F}=o(\varepsilon)} \approx 0 \tag{1.8}$$

is satisfied. Here $\theta(\varepsilon) \approx 0$ means that $\theta(\varepsilon) = o(\varepsilon)$. Equation (1.8) is also called *approximate determining equation*.

To construct approximate symmetries for the given perturbed (1.6), one should solve approximate determining equation (1.8) by using the following algorithm.

Step 1. Find the exact symmetry generators $X_{(0)}$ of the *unperturbed equation* $\mathcal{F}_{(0)} = 0$, that is, by solving the determining equation for exact symmetries:

$$X_{(0)}\mathcal{F}_{(0)}|_{\mathcal{F}_{(0)}=0} = 0. \tag{1.9}$$

Step 2. Given $X_{(0)}$ and a perturbation $\varepsilon\mathcal{F}_{(1)}$, calculate the *auxiliary function* H given by

$$H \approx \frac{1}{\varepsilon}X_{(0)}\left(\mathcal{F}_{(0)} + \varepsilon\mathcal{F}_{(1)}\right)|_{\mathcal{F}_{(0)}+\varepsilon\mathcal{F}_{(1)}=0}. \tag{1.10}$$

Step 3. Find the *correction term* (i.e., the operator $X_{(1)}$) from the *determining equation for the correction term*:

$$X_{(1)}\mathcal{F}_{(0)}|_{\mathcal{F}_{(0)}=0} + H = 0. \tag{1.11}$$

If the approximate symmetry $X = X_{(0)} + \varepsilon X_{(1)}$ is such that the operator $X_{(0)} \neq 0$, we say that the symmetry $X_{(0)}$ of the unperturbed equation is *stable* with respect to the perturbation considered.

It is easy to see from approximate determining equation (1.8) that if $X_{(0)}$ is a symmetry of the unperturbed equation then the operator $\varepsilon X_{(0)}$ is approximate symmetry of the perturbed equation. Therefore in what follows we will call symmetries of the form $\varepsilon X_{(0)}$ trivial approximate symmetries.

2. Approximate equivalence transformations

The first step of the group classification of (1.5) with respect to approximate point symmetries is to calculate the approximate equivalence transformations. An approximate equivalence transformation for (1.5) is a nondegenerate (at $\varepsilon = 0$) change of variables t and u of the form

$$\overline{t} \approx \varphi_{(0)}(t,u) + \varepsilon\varphi_{(1)}(t,u), \qquad \overline{u} \approx \psi_{(0)}(t,u) + \varepsilon\psi_{(1)}(t,u), \tag{2.1}$$

which leaves invariant the structure of (1.5). This means that (1.5) is mapped by approximate equivalence transformations into equations of the same form (up to the terms of first order in ε) but, generally speaking, with different function $f(u)$.

We use Ovsiannikov's infinitesimal approach [10] for calculating equivalence transformations in its modified form to calculate approximate equivalence transformations (details of such modification can be seen in [3]). We rewrite (1.5) as a system

$$u'' + u = \varepsilon f u', \qquad \varepsilon f_t = 0, \tag{2.2}$$

where u is considered as a dependent variable of t, and f as a dependent variable of t and u. The generators of the group of approximate equivalence transformations are sought in the form

$$E \equiv E_{(0)} + \varepsilon E_{(1)} = \left(\xi_{(0)} + \varepsilon\xi_{(1)}\right)\frac{\partial}{\partial t} + \left(\eta_{(0)} + \varepsilon\eta_{(1)}\right)\frac{\partial}{\partial u} + \mu\frac{\partial}{\partial f}, \tag{2.3}$$

where $\xi_{(\nu)}$ and $\eta_{(\nu)}$ $(\nu = 0,1)$ are functions of t and u, and μ is a function of t, u, and f. Approximate symmetry condition (1.8) now takes the form

$$\widetilde{E}(u'' + u - \varepsilon f u')|_{(2.2)} = o(\varepsilon), \qquad \widetilde{E}(\varepsilon f_t)|_{(2.2)} = o(\varepsilon), \tag{2.4}$$

where the operator $\widetilde{E}$ is obtained from the operator E by prolonging the necessary derivatives. According to the algorithm for solving approximate determining equations (see Section 1) as the first step we obtain the generator $E_{(0)}$ of point symmetries of unperturbed equation $u'' + u = 0$. It's coefficients $\xi_{(0)}$ and $\eta_{(0)}$ are well known (see, e.g., [3]) and are of the form

$$\begin{gathered}\xi_{(0)} = \left(C_1 \sin t + C_2 \cos t\right)u + C_6 \sin 2t + C_7 \cos 2t + C_5,\\ \eta_{(0)} = \left(C_1 \cos t - C_2 \sin t\right)u^2 + \left(C_6 \cos 2t - C_7 \sin 2t + C_8\right)u + C_3 \sin t + C_4 \cos t,\end{gathered} \tag{2.5}$$

where $C_1,\dots,C_8$ are arbitrary constants.

The second equation of system (2.4) gives us $\mu_t = 0$ and $(\eta_{(0)})_t = 0$, and by using the expression for $\eta_{(0)}$ from (2.5), we obtain

$$C_1 = C_2 = C_3 = C_4 = C_6 = C_7 = 0. \tag{2.6}$$

The auxiliary function H has the form $H = -\mu u'$ and the operator $X_{(1)}$ is found from the determining equation

$$X_{(1)}(u'' + u)\big|_{u''+u=0} - \mu u' = 0. \tag{2.7}$$

The solution of this equation is

$$\begin{gathered}\xi_{(1)} = \left(A_1 \sin t + A_2 \cos t + A_{10}\right)u + A_6 \sin 2t + A_7 \cos 2t + A_5,\\ \eta_{(1)} = \left(A_1 \cos t - A_2 \sin t\right)u^2 + \left(A_6 \cos 2t - A_7 \sin 2tu + A_8 t + A_9\right)u + A_3 \sin t + A_4 \cos t,\\ \mu = 2A_8 + 3uA_{10},\end{gathered} \tag{2.8}$$

with arbitrary constants $A_1,\dots,A_{10}$. Hence (1.5) admits twelve-dimensional approximate Lie algebra L_{12} of operators generating approximate equivalence transformation group. But only transformations generated by operators from three-dimensional subalgebra of L_{12} spanned by the operators

$$E_1 = u\frac{\partial}{\partial u}, \qquad E_2 = \varepsilon t u\frac{\partial}{\partial u} + 2\frac{\partial}{\partial f}, \qquad E_3 = \varepsilon u\frac{\partial}{\partial t} + 3u\frac{\partial}{\partial f} \tag{2.9}$$

change the form of the function $f(u)$ in (1.5) and are therefore essential for our classification. These transformations have the form

$$\begin{gathered}\bar{t} \approx t + \varepsilon a_3 u, \\ \bar{u} \approx a_1 u(1+\varepsilon t a_2), \qquad \bar{f} \approx f + 2a_2 + 3a_3 u,\end{gathered} \tag{2.10}$$

where a_1, a_2, and a_3 are arbitrary constants, $a_1 \neq 0$.

By using these equivalence transformations, any equation of the form (1.5) is transformed (up to terms of the order $o(\varepsilon)$) to an equivalent equation of the same form (1.5) with function $\bar{f}(\bar{u})$ which is obtained from $f(u)$ by the formula

$$\bar{f}(\bar{u}) = f\left(\frac{\bar{u}}{a_1}\right) + 2a_2 + 3\frac{a_3}{a_1}\bar{u}. \tag{2.11}$$

For example, the van der Pol equation (1.1) is equivalent to the equation $u'' + u = -\varepsilon u^2 u'$.

We will now carry out the classification up to the transformations (2.10).

3. Result of classification

For our classification, we will consider approximate transformations which will transform the small parameter, that is, we will consider transformations,

$$\begin{gathered}\bar{t} \approx \varphi_{(0)}(t,u,a) + \varepsilon\varphi_{(1)}(t,u,a) + o(\varepsilon), \\ \bar{u} \approx \psi_{(0)}(t,u,a) + \varepsilon\psi_{(1)}(t,u,a) + o(\varepsilon), \\ \bar{\varepsilon} \approx \varepsilon\theta_{(0)}(a) + o(\varepsilon),\end{gathered} \tag{3.1}$$

of variables t, u and the small parameter ε such that (1.5) have the same form (up to $o(\varepsilon)$) in the new variables $\bar{t}$, $\bar{u}$, and $\bar{\varepsilon}$. The set of all such transformations forms an approximate transformation group with respect to the parameter a, and therefore similar to the Lie's theory for approximate groups, the construction of the approximate group is equivalent to the determination of its approximate operator

$$X = \left(\xi_{(0)}(t,u) + \varepsilon\xi_{(1)}(t,u)\right)\frac{\partial}{\partial t} + \left(\eta_{(0)}(t,u) + \varepsilon\eta_{(1)}(t,u)\right)\frac{\partial}{\partial u} + \varepsilon k\frac{\partial}{\partial \varepsilon}, \tag{3.2}$$

where

$$\begin{gathered}\xi_{(\nu)}(t,u) = \left.\frac{\partial\varphi_{(\nu)}(t,u,a)}{\partial a}\right|_{a=0}, \quad \eta_{(\nu)}(t,u) = \left.\frac{\partial\psi_{(\nu)}(t,u,a)}{\partial a}\right|_{a=0}, \quad \nu = 0,1, \\ k = \left.\frac{d\theta_{(0)}(a)}{da}\right|_{a=0} = \text{const}.\end{gathered} \tag{3.3}$$

The calculation of generators (3.2) of approximate transformation group (3.1) admitted by perturbed equation (1.6) is done by the same algorithm described in Section 1 but with a difference in Step 3. Now this step has the following form.

Step 3. Find the *correction terms* (i.e., the operator $X_{(1)} = \xi_{(1)}(t,u)(\partial/\partial t) + \eta_{(1)}(t,u)(\partial/\partial u)$ and the constant k) from the *determining equation for the correction terms*:

$$X_{(1)}F_{(0)}\big|_{F_{(0)}=0} + kF_{(1)} + H = 0. \tag{3.4}$$

Application of this algorithm to (1.5) gives us the following result: the coefficients $\xi_{(0)}(t,u)$ and $\eta_{(0)}(t,u)$ of the symmetry operator $X_{(0)} = \xi_{(0)}(t,u)(\partial/\partial t) + \eta_{(0)}(t,u)(\partial/\partial u)$ of the unperturbed equation $u'' + u = 0$ are given by formula (2.5); the auxiliary function H has the form

$$\begin{aligned} H = {} & f\left(u^2(C_1\sin t + C_2\cos t) - uu'(C_1\cos t - C_2\sin t) - 2u'^2(C_1\sin t + C_2\cos t)\right. \\ & \left. - C_3\cos t + C_4\sin t - 2u'(C_6\cos 2t - C_7\sin 2t) + 2u(C_6\sin 2t + C_7\cos 2t)\right) \\ & - u'f'\left(u^2(C_1\cos t - C_2\sin t) + u(C_6\cos 2t - C_7\sin 2t + C_8) + C_3\sin t + C_4\cos t\right). \end{aligned} \tag{3.5}$$

Hence, approximate determining equation (3.4) for the correction terms is reduced to solving the following system of equations:

$$(\xi_{(1)})_{uu} = 0, \tag{3.6}$$

$$(\eta_{(1)})_{uu} - 2(\xi_{(1)})_{tu} - 2f(C_2\cos t + C_1\sin t) = 0, \tag{3.7}$$

$$\begin{aligned} & 2(\eta_{(1)})_{tu} + 3u(\xi_{(1)})_u - (\xi_{(1)})_{tt} - kf - uf(C_1\cos t - C_2\sin t) \\ & \quad - 2f(C_6\cos 2t - C_7\sin 2t) - u^2f'(C_1\cos t - C_2\sin t) \\ & \quad - uf'(C_6\cos 2t - C_7\sin 2t + C_8u) - f'(C_3\sin t + C_4\cos t) = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} & (\eta_{(1)})_{tt} - u(\eta_{(1)})_u + 2u(\xi_{(1)})_t + \eta_{(1)} + u^2f(C_2\cos t + C_1\sin t) \\ & \quad + 2uf(C_6\sin 2t + C_7\cos 2t) + f(-C_3\cos t + C_4\sin t) = 0. \end{aligned} \tag{3.9}$$

It follows from (3.6)–(3.9) that if $f(u)$ is an arbitrary function then $C_i = 0$, for all $i \neq 5$ and $k = 0$ and the functions $\xi_{(1)}$ and $\eta_{(1)}$ are given by the same expressions (2.5) with change of coefficients C_i to new coefficients D_i. Hence, the principal Lie algebra $L_{\mathcal{P}}$ (i.e., the symmetry Lie algebra admitted by the equation for arbitrary function $f(u)$) is nine-dimensional and is generated by one nontrivial operator

$$X_1 = \frac{\partial}{\partial t}, \tag{3.10}$$

and eight trivial operators

$$\begin{gathered} Y_1 = \varepsilon\frac{\partial}{\partial t}, \qquad Y_2 = \varepsilon\left(u\cos t\frac{\partial}{\partial t} - u^2\sin t\frac{\partial}{\partial u}\right), \qquad Y_3 = \varepsilon\left(u\sin t\frac{\partial}{\partial t} + u^2\cos t\frac{\partial}{\partial u}\right), \\ Y_4 = \varepsilon\sin t\frac{\partial}{\partial u}, \qquad Y_5 = \varepsilon\cos t\frac{\partial}{\partial u}, \qquad Y_6 = \varepsilon\left(\sin 2t\frac{\partial}{\partial t} + u\cos 2t\frac{\partial}{\partial u}\right), \\ Y_7 = \varepsilon\left(\cos 2t\frac{\partial}{\partial t} - u\sin 2t\frac{\partial}{\partial u}\right), \qquad Y_8 = \varepsilon u\frac{\partial}{\partial u}. \end{gathered} \tag{3.11}$$

To find an extension of the principal Lie algebra $L_{\mathcal{P}}$, we now analyze the compatibility conditions for (3.7)–(3.9) considering them as equations for $\eta_{(1)}$. After some calculations we obtain the following system of the *classifying relations*:

$$C_1 f'' = 0, \qquad C_2 f'' = 0, \qquad C_6 f'' = 0, \qquad C_7 f'' = 0, \tag{3.12}$$

$$C_3 f''' = 0, \qquad C_4 f''' = 0, \qquad 2f'' C_8 + u f''' C_8 + k f'' = 0. \tag{3.13}$$

It follows from (3.12) that we need to consider two different cases: $f'' = 0$ and $f'' \neq 0$.

(I) If $f'' \neq 0$, then from (3.12) we have $C_1 = C_2 = C_6 = C_7 = 0$. The first two equations of (3.13) give us two subcases.

(I.1) If $f''' \neq 0$, then $C_3 = C_4 = 0$. The third equation of (3.13) can be written as $C_8 u f'''/f'' + 2C_8 + k = 0$, whence after differentiating with respect to u we obtain the classifying relation

$$C_8 \left(\frac{u f'''}{f''} \right)' = 0. \tag{3.14}$$

(I.1.1) If $u f'''/f'' = \text{const}$, then we obtain three nonequivalent (up to equivalence transformations (2.10)) forms of $f(u)$: $f(u) = \delta u^{\sigma}$, $\sigma \neq 0, 1, 2$; $\delta = \pm 1$, $f(u) = a \ln u$, $a \neq 0$ is arbitrary constant, and $f(u) = u \ln u$.

(I.1.2) If $u f'''/f'' \neq \text{const}$, then $C_8 = 0$ and $k = 0$ (from the third equation of (3.13)) and we do not obtain an extension of $L_{\mathcal{P}}$.

(I.2) If $f''' = 0$, then up to equivalence transformations (2.10) we obtain $f = \delta u^2$, $\delta = \pm 1$.

(II) All functions $f(u)$ satisfying $f'' = 0$ are equivalent to $f(u) = 0$, that is, this case gives us unperturbed equation (1.2).

After substituting these functions $f(u)$ in system (3.6)–(3.9) and solving the corresponding equations we obtain the following cases of extension of $L_{\mathcal{P}}$.

(1) $f(u) = \delta u^{\sigma}$, $\sigma \neq 0, 1$, $\delta = \pm 1$,

$$X_2 = u \frac{\partial}{\partial u} - \sigma \varepsilon \frac{\partial}{\partial \varepsilon}. \tag{3.15}$$

(2) $f(u) = u \ln u$,

$$X_2 = \frac{1}{3} \varepsilon u \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} - \varepsilon \frac{\partial}{\partial \varepsilon}. \tag{3.16}$$

(3) $f(u) = a \ln u$, $a \neq 0$,

$$X_2 = \left(u + \frac{1}{2} \varepsilon a u t \right) \frac{\partial}{\partial u}. \tag{3.17}$$

From the above classification we see that nonequivalent classes (1)–(3) of equations have only one additional symmetry and for the cases (1) and (2) this symmetry generates an approximate transformation transforming the small parameter. By application of approximate equivalence transformations (2.10) to equations of cases (1) and (2) we obtain a

large number of equations which have the symmetry transforming the small parameter. For example, for the van der Pol equation (1.1) which is included in case (1) when $\delta = -1$ and $\sigma = 2$, additional approximate symmetry has the form

$$X_2 = (u - \varepsilon t u)\frac{\partial}{\partial u} - 2\varepsilon\frac{\partial}{\partial \varepsilon} \tag{3.18}$$

and it is obtained from the corresponding operator X_2 by the equivalence transformation $\overline{t} = t$, $\overline{u} \approx u + \varepsilon t u/2$.

Remark 3.1. For equations which are equivalent to the unperturbed equation $u'' + u = 0$, approximate symmetries transforming the small parameter can be obtained from the operator $X = \varepsilon(\partial/\partial\varepsilon)$ by the equivalence transformation. For example, we can obtain the operator

$$X = \frac{1}{3}\varepsilon u\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \varepsilon} \tag{3.19}$$

approximately admitted by the equation $u'' + u = \varepsilon u u'$ using the equivalence transformation $\overline{t} \approx t + \varepsilon u/3$, $\overline{u} = u$.

4. Approximate integration by using a symmetry transforming the small parameter

It is known that an exact or usual approximate symmetry of a first-order ordinary differential equation can be used to reduce the equation to quadrature (see, e.g., [5] or [8]). In this section we will demonstrate by giving an example that point symmetry transforming the small parameter can also be used for reducing the first-order ordinary differential equation with a small parameter to quadratures. In this case the procedure of integration which we propose below involves two quadratures.

We consider

$$u'' + u = \varepsilon\delta u^{\sigma} u', \quad \sigma \neq 0, 1;\ \delta = \pm 1, \tag{4.1}$$

which admits two nontrivial operators

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = u\frac{\partial}{\partial u} - \sigma\varepsilon\frac{\partial}{\partial \varepsilon}. \tag{4.2}$$

The operator X_1 generates the usual translation along t-axis and the operator X_2 generates approximate transformation transforming the small parameter. These operators span a two-dimensional Abelian Lie algebra, that is, $[X_1, X_2] = 0$ and can be used for approximate integration of (4.1) by the methods of successive reduction of order. Since the operator X_1 has canonical form, by using the standard substitution

$$u' = p(u), \tag{4.3}$$

we can reduce (4.1) to the form

$$p' + \frac{u}{p} - \varepsilon\delta u^{\sigma} = 0. \tag{4.4}$$

In the new variables u and p, the operator X_2 has the form

$$X_2 = u\frac{\partial}{\partial u} + p\frac{\partial}{\partial p} - \sigma\varepsilon\frac{\partial}{\partial \varepsilon}. \tag{4.5}$$

We now introduce new variables s and v such that in these variables the operator X_2 takes the form

$$X_2 = \frac{\partial}{\partial v} - \sigma\varepsilon\frac{\partial}{\partial \varepsilon}. \tag{4.6}$$

By solving the equations $X_2 s(u,p,\varepsilon) \approx 0$, $X_2 v(u,p,\varepsilon) \approx 1$, we obtain

$$s = \frac{u}{p}, \qquad v = \ln u. \tag{4.7}$$

In the variables s and v, (4.4) takes the form

$$v' = \frac{1}{s+s^3} + \varepsilon\frac{\delta s^2}{(s+s^3)^2}\exp(\sigma v). \tag{4.8}$$

If $\varepsilon = 0$, we obtain the equation which can be solved by one quadrature and the solution has the form $v = \ln(K_1 s/\sqrt{1+s^2})$. Hence the approximate solution of (4.8) can be found in the form

$$v \approx \ln\frac{K_1 s}{\sqrt{1+s^2}} + \varepsilon w(s), \tag{4.9}$$

and the equation for $w(s)$ is obtained by the substitution of $v(t,\varepsilon)$ in (4.8). We have

$$w' = \frac{K_1^\sigma \delta s^\sigma}{(1+s^2)^{2+\sigma/2}}, \tag{4.10}$$

and its solution can be obtained by one quadrature.

Remark 4.1. In general a first-order ordinary differential equation which is invariant with respect to operator (4.6) has the form $v' = \phi_{(0)}(s) + \varepsilon\phi_{(1)}(s)e^{\sigma v}$ and its solution can be found by two quadratures in the way described above and is given by

$$v \approx \int \phi_{(0)}(s)ds + \varepsilon\int \phi_{(1)}(s)\exp\left(\sigma\int \phi_{(0)}(r)dr\right)ds. \tag{4.11}$$

We now consider a particular case of (4.10) when $\sigma = 2$. Then the solution of (4.8) is given by

$$v = \ln\frac{K_1 s}{\sqrt{1+s^2}} + \varepsilon\delta K_1^2\left(-\frac{s}{4(1+s^2)^2} + \frac{s}{8(1+s^2)} + \frac{1}{8}\arctan s\right), \quad K_1 = \text{const}. \tag{4.12}$$

After returning to the variables u and p, we obtain

$$p = \sqrt{K_1^2 - u^2} + \varepsilon\delta\left(\frac{u^3}{4} - \frac{u}{8}K_1^2 + \frac{K_1^4}{8\sqrt{K_1^2-u^2}}\arctan\frac{u}{\sqrt{K_1^2-u^2}}\right). \tag{4.13}$$

Whence, by virtue of (4.3) we have the differential equation

$$\frac{du}{dt} = \sqrt{K_1^2 - u^2} + \varepsilon\delta\left(\frac{u^3}{4} - \frac{u}{8}K_1^2 + \frac{K_1^4}{8\sqrt{K_1^2 - u^2}}\arctan\frac{u}{\sqrt{K_1^2 - u^2}}\right) \tag{4.14}$$

which solves by one quadrature and gives us the solution

$$u \approx K_1 \sin(t + K_2) + \varepsilon\delta\left(\frac{K_1^3}{8}\cos^3(t + K_2) + \frac{K_1^3}{8}t\sin(t + K_2)\right) \tag{4.15}$$

of (4.1) for $\sigma = 2$.

As noted earlier, the van der Pol equation can be obtained from (4.1) with $\delta = -1$ and $\sigma = 2$ by an equivalence transformation, and therefore its solution can also be derived from the above solution by the same equivalence transformation.

5. Conclusions

In this paper we carried out the group classification of a class of second-order ordinary differential equations with a small parameter which includes the van der Pol equation with respect to admitted approximate transformation groups transforming a small parameter. We obtained wide classes of equations having point approximate symmetries transforming the small parameter. By using these symmetries we gave here the procedure of approximate integration of these equations.

The methods used in this paper can be applied to other types of second-order as well as to higher-order ordinary differential equations with a small parameter. However, it is not always true that we will get approximate symmetries transforming a small parameter. For instance, Rayleigh-type equations of the form $v'' + v = \varepsilon F(v')$, which are connected to van der Pol-type equations by the differential substitution $v' = u$, do not have *point* approximate symmetries transforming a small parameter. But we think that it would be possible to obtain other type of symmetries transforming a small parameter and this will be studied in our future work.

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