BERNOULLI SERIAL PRODUCTION LINES WITH QUALITY CONTROL DEVICES: THEORY AND APPLICATION

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This paper develops the procedure for the analysis of the production systems with quality control devices. The evaluation of the production system requires an expression for the system performance measures as functions of the machine and buffer parameters. This paper presents a method for evaluating these functions and illustrates their practical utility using a case study at a production plant.

1. Introduction

Serial production lines with quality control (*QC*) devices are sets of processing machines, *QC* devices and material handling devices arranged in the consecutive order so as to produce high-quality products. The *QC* devices are installed to check whether the parts being processed meet a required specification or not [8, 13]. By doing this, a high-quality product can be produced to satisfy the customer. Thus, the manufacturing enterprise can be successful and everlasting in the competing market. Therefore, the goal of this paper is to provide an engineering method for estimating the performance of a serial production line with *QC* devices. Such a method can be used to improve an existing production line or to design a new production line so that it can perform in a highly efficient manner and have capability of producing high-quality products.

The processing machines and the *QC* devices are subject to random breakdowns, which lead to unscheduled downtime and loss of production. Two basic models of machines reliability have been discussed in the literature: Markovian [4–7] and Bernoulli [2, 3, 10–12]. Usually, the machine in the Markovian model needs two parameters to describe its behavior: one for the uptime and the other for the downtime. The parameters are conditional probabilities which are used to determine the state of the machine in a cycle time (i.e., the time necessary to process a part), given the condition being the state of the machine in the previous cycle. This is a Markov process, which is why the term "Markovian" is used. In the Bernoulli model, the state of a machine in each cycle is determined

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by the process of Bernoulli trials. Each of the models has its domain of applicability. Markovian models seem to be more appropriate for machining operations where the machine stops due to a technical malfunction and may stay in this state until a repair is carried out. Bernoulli models seem to be appropriate for assembly operations where the machines are interrupted for typically short periods of time, comparable with the cycle time. In this paper, the Bernoulli model is used to describe the machine reliability.

Two reasons for adopting the Bernoulli model are as follows. First, the simplicity of the model allows us to carry out the more complete analysis for the production line. Second, in the picture tube production lines, where we test the applicability of our results, a machine being down is often due to the pallet jam on the operation conveyor. To correct for this problem, a short period of time is required. The duration of the "breakdown" is the order of the cycle, and, therefore, the Bernoulli model is more appropriate.

Performance analysis of production lines with quality control devices was initiated in [9]. Only the simplest case of two-machine one-buffer asymptotically reliable serial production line was addressed. Since in most applications more than two machines are involved in the production line, development of performance analysis of production lines with more than two machines and *QC* devices is desirable. In [1], a model of a long serial production line with quality control devices is introduced and analyzed. However, only highly reliable machines are considered. Therefore, the purpose of this research is to develop performance analysis of a long serial production line with unreliable processing machines and *QC* devices, and demonstrate its applicability using a case study at a production plant.

The model used in this paper is summarized in the following: the machines in the production line operate asynchronously with a fixed cycle time, and machine i fails during any cycle with a very small probability. A nondefective part is made defective by machine i with a very small probability. A defective part can be identified by the quality control device, Q_i, with probability $0 \le d_i \le 1$. The above assumptions with $0 \le d_i \le 1$, for all i, and unreliable machines (not necessary to be asymptotically reliable) are considered in the model of this paper.

The outline of this paper is as follows: in Section 2, system model of a production line with *QC* devices is introduced, and the problem addressed is stated. Section 3 is devoted to the development of numerical approach for the system, while Section 4 is devoted to the development of analytical approach. The case study is described in Section 5. Finally, the conclusions are formulated in Section 6. All of the proofs are given in the appendix.

2. System model

Consider a serial production line with *QC* devices defined by the following assumptions (see Figure 2.1).

(i) The system consists of M machine, m_i, $i = 1,\dots,M$, and $(M-1)$ *QC* devices, Q_i, $i = 1,\dots,M-1$, placed immediately after each machine except the last one, and $(M-1)$ buffers, B_i, $i = 1,\dots,M-1$, separating each consecutive pair of machines and *QC* devices.

(ii) The machines (including *QC* devices) have identical cycle time t_c. The time axis is slotted with the slot duration t. Machines begin operating at the beginning of each time slot, but *QC* devices begin checking at the ending of each time slot.

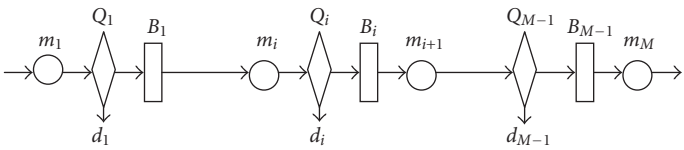

Figure 2.1. Serial production line with QC devices.

(iii) Each buffer is characterized by its capacity, $N_i < \infty$, $i = 1,\ldots,M-1$.

(iv) A nondefective part is made defective by machine m_i with probability d_i, $i = 1,\ldots,M-1$. A defective part can be detected by QC device Q_i and discarded from the system. Parameters d_i, $i = 1,\ldots,M-1$, are referred to as bad part ratios.

(v) Machine m_i is starved during a time slot if buffer B_{i-1} is empty at the beginning of the time slot. QC device Q_i is starved during a time slot if machine m_i fails to produce a part at the beginning of the time slot. Machine m_1 is never starved.

(vi) Machine m_i is blocked during a time slot if QC device Q_i is blocked or is down at the time slot. QC device Q_i is blocked during a time slot with the conditional probability $(1-d_i)$, given that buffer B_i has N_i parts at the beginning of the time slot and machine m_{i+1} fails to take a part during the time slot. Machine m_M is never blocked.

(vii) Machine m_i and QC device Q_i, being neither blocked nor starved during a time slot, produces a part with probability p_i and operates to check a part with probability c_i, respectively, and fails to do so with probability $(1-p_i)$ and $(1-c_i)$, respectively. Parameters p_i and c_i are referred to as the production rate of machine m_i and working rate of QC device Q_i in isolation, respectively.

A few remarks concerning this model are in order.

Remark 2.1. To gain the computation simplicity of the performance evaluation for the production line, we assume that there is no QC device after machine m_M in assumption (i). If there is a QC device installed after the last machine on the factory floor, one virtual large buffer (e.g., $N_M = 20$) and one virtual perfect machine (i.e., $p_{M+1} = 1$) can be added to accommodate this situation. Since the performance of the modified production line is close to the performance of the original production line, the former one can be analyzed by using the method developed in this paper to evaluate the performance of the latter one.

Remark 2.2. If some $c_i = 1$ and $d_i = 0$ in assumptions (i)–(vii), then machine m_i can be viewed as a machine without a QC device after it. That is, there is no quality inspection in the stage i, where machine m_i is involved. In the case of some $p_i = 1$, it means that there is only quality inspection involved in the stage i.

Remark 2.3. In the case of $c_i = 1$ and $d_i = 0$, $i = 1,\ldots,M-1$, in assumptions (iv) and (vii), the model is reduced to the model of a production line without QC devices, as defined in [10]. Therefore, this model is a more general model for Bernoulli production lines.

Manufacturing systems, defined by (i)–(vii), are called serial production lines with QC devices. The performance measures addressed in this paper are as follows.

(a) *Production rate*, PR, the average steady-state number of parts produced by the machine m_M during a cycle time t_c.

(b) *Average steady-state buffer occupancy* $E[b_i]$, $i = 1,\dots,M-1$, where b_i are the steady-state occupancy of buffer B_i.

Remark 2.4. Throughout, symbols with the "—" denote the exact values of the appropriate quantity. The respective approximations, to be introduced below, are denoted by the same symbols but without the "—."

Unfortunately, no tractable exact methods for calculation of $\overline{\mathrm{PR}}$ and $E[\overline{b}_i]$, $i = 1,\dots,M-1$, for line (i)–(vii) are at present known. The goal of this paper is to derive the estimates PR and $E[b_i]$ and to evaluate their accuracy. The problem of such a line, addressed in this paper, can be described by the following characteristics.

Problem 2.5. Under assumptions (i)–(vii), given the parameters p_i, $i = 1,\dots,M-1$, and c_i, d_i, N_i, $i = 1,\dots,M-1$, find the line production rate $\overline{\mathrm{PR}}$ and the average buffer occupancy $E[\overline{b}_i]$, $i = 1,\dots,M-1$.

3. Numerical approach

Before deriving the analytical performance evaluation, the numerical procedure of the problem is developed. Based on the numerical results of the procedure, the accuracy of the analytical performance evaluation can be analyzed.

Consider the line (i)–(vii) with M machines, $(M-1)$ QC devices, and $(M-1)$ buffers. Let $K = [k_1,\dots,k_{M-1}]$ be the state of the system where the consecutive buffers $B_1,\dots,B_i,\dots,B_{M-1}$ contain $k_1,\dots,k_i,\dots,k_{M-1}$ parts, respectively, $0 \le k_i \le N_i$. It follows that $k \in \{\{0,\dots,N_1\},\dots,\{0,\dots,N_{M-1}\}\}$. Let $\overline{Y}(K,j)$ denote the joint probability of buffers B_i, $i = 1,\dots,M-1$, in the state K at time slot j. Since the line can be described by Markov chain, and by total probability theorem, then we write

$$\overline{Y}(K,j+1) = \sum_{K} G(K \mid K')\overline{Y}(K',j), \tag{3.1}$$

where $G(K \mid K')$ denotes the transition probability of the state changing from K' to K, and $K' \in \{\{0,\dots,N_1\},\dots,\{0,\dots,N_{M-1}\}\}$. Let

$$\overline{Y}(j) = \begin{bmatrix} \overline{Y}([0,\dots,0],j) \\ \overline{Y}([0,\dots,1],j) \\ \vdots \\ \overline{Y}([0,\dots,N_{M-1}],j) \\ \overline{Y}([0,1,\dots,0],j) \\ \vdots \\ \overline{Y}([N_1,\dots,N_{M-1}],j) \end{bmatrix}, \qquad \overline{Y}(j+1) = T\overline{Y}(j), \tag{3.2}$$

where

$$T = \begin{bmatrix} G([0,\dots,0] \mid [0,\dots,0]) & G([0,\dots,0] \mid [0,\dots,1]) & \dots & G([0,\dots,0] \mid [N_1,\dots,N_{M-1}]) \\ G([0,\dots,1] \mid [0,\dots,0]) & G([0,\dots,1] \mid [0,\dots,1]) & \dots & G([0,\dots,1] \mid [N_1,\dots,N_{M-1}]) \\ \vdots & \vdots & \vdots & \vdots \\ G([N_1,\dots,N_{M-1}] \mid [0,\dots,0]) & G([N_1,\dots,N_{M-1}] \mid [0,\dots,1]) & \dots & G([N_1,\dots,N_{M-1}] \mid [N_1,\dots,N_{M-1}]) \end{bmatrix}. \tag{3.3}$$

Note that $\overline{Y}(j)$ and $\overline{Y}(j+1)$ are column vectors with $\prod_{i=1}^{M-1}(N_i+1)$ elements, and T is a square matrix with $\prod_{i=1}^{M-1}(N_i+1)^2$ elements. Then, (3.2) can be expressed in terms of the initial value as follows:

$$\overline{Y}(j+1) = T\overline{Y}(j) = T\left(T\overline{Y}(j-1)\right) = \cdots = T^j T\overline{Y}(0), \tag{3.4}$$

where $\overline{Y}(0)$ is the initial probability distribution. Since the line (i)–(vii) is an ergodic system, we have

$$\overline{Y} = \lim_{j\to\infty} \overline{Y}(j+1) = \lim_{j\to\infty} \overline{Y}(j), \tag{3.5}$$

that is,

$$\begin{aligned} \overline{Y}([0,\dots,0]) &= \lim_{j\to\infty} \overline{Y}([0,\dots,0],j), \\ \overline{Y}([0,\dots,1]) &= \lim_{j\to\infty} \overline{Y}([0,\dots,1],j), \\ &\vdots \\ \overline{Y}([N_1,\dots,N_{M-1}]) &= \lim_{j\to\infty} \overline{Y}([N_1,\dots,N_{M-1}],j). \end{aligned} \tag{3.6}$$

Let $\overline{X}_i(b_i)$, $0 \le b_i \le N_i$, $i = 1,\dots,M-1$, be the steady state probability of buffer B_i containing b_i parts. Then

$$\overline{X}_i(b_i) = \sum_{K \text{ such that } k_i = b_i} \overline{Y}(K), \quad 0 \le b_i \le N_i,\ i = 1,\dots,M-1. \tag{3.7}$$

Based on (3.7), the numerical solutions of the problem are obtained as fo1lows.

(a) Since the production rate is equal to the average steady-state number of parts produced by the line during a cycle, the last machine must be operational, p_M, and not starved, which is equal to the absence of emptiness of the last buffer, $[1-\overline{X}_{M-1}(0)]$. Therefore, the production rate can be expressed as

$$\overline{\mathrm{PR}} = p_M\left[1-\overline{X}_{M-1}(0)\right]. \tag{3.8}$$

(b) The average steady-state buffer occupancy can be calculated as

$$E[\overline{b}_i] = \sum_{j=0}^{N_1} j\overline{X}_i(j), \quad i = 1,\dots,M-1. \tag{3.9}$$

Unfortunately, there is no closed-form expression for the transition matrix T in a long production line. However, it is possible to find out for a computation procedure. The input data for the algorithm are given in the following.

(1) p_i, the isolation production rate of the machine m_i, $i = 1,\dots,M-1$. Let $P = [p_1,\dots,p_M]$.

(2) c_i, the isolation working rates of QC devices Q_i, $i = 1,\dots,M-1$. Let $C = [c_1,\dots,c_{M-1}]$.

(3) d_i, the bad part ratios, made defective by the machine m_i, $i = 1,\dots,M-1$. Let $D = [d_1,\dots,d_{M-1}]$.

(4) N_i, every capacity of buffers B_i, $i = 1,\dots,M-1$. Let $N = [N_1,\dots,N_{M-1}]$.

To obtain the transition matrix T, we need to know all its elements, $G(K \mid K')$, in (3.1). Let $\alpha = [\alpha_1,\dots,\alpha_M]$ be the state of consecutive machines m_i, $i = 1,\dots,M-1$, where $\alpha_i = 1$ or 0, represents m_i being up or down, respectively. Let $\beta = [\beta_1,\dots,\beta_M]$ be the state of consecutive QC devices Q_i, $i = 1,\dots,M-1$, where $\beta_i = 1$ or 0, represents Q_i being working or not, respectively. Let $\gamma = [\gamma_1,\dots,\gamma_M]$ be the state of consecutive defectives stage i, $i = 1,\dots,M-1$, $\gamma_i = 1$ or 0, represents m_i making defective or not, respectively. The quantity of $G(K \mid K')$ can be obtained according to the following rule.

Rule 3.1. The transition probability from the state K' to K is given as follows:

$$G(K \mid K') = \sum_{\alpha,\beta,\gamma} W([\alpha,\beta,\gamma]), \tag{3.10}$$

where

$$W([\alpha,\beta,r]) = \begin{cases} p_M^{\alpha_M}(1-p_M)^{(1-\alpha_M)} \prod_{i=1}^{M-1} Z_i, & \text{from } K' \text{ to } K \text{ under } [\alpha,\beta,r], \\ 0, & \text{otherwise}, \end{cases} \tag{3.11}$$

where

$$Z_i = p_i^{\alpha_i}(1-p_i)^{(1-\alpha_i)} c_i^{\beta_i}(1-c_i)^{(1-\beta_i)} d_i^{\gamma_i}(1-d_i)^{(1-\gamma_i)}. \tag{3.12}$$

A procedure is developed by using the above rule to calculate all elements of the matrix T.

Numerical procedure

Step 3.2. Establish a program function to check whether there is a transition from the state K' to K under the condition $[\alpha,\beta,\gamma]$. Then, the transition probability $G(K \mid K')$ can be calculated by (3.10) and (3.11).

Step 3.3. For each state from K' to K, using the function established in Step 3.2, calculate $G(K \mid K')$. Then continue the same process until the elements in matrix T are obtained, then stop.

A useful justification for the correctness of the computer program finding the elements in the matrix T is provided as

$$\sum_{K} G(K \mid K') = 1, \quad \forall K. \tag{3.13}$$

Step 3.4. Computing the steady-state probability distribution of consecutive buffers $B_1, \ldots, B_{M-1}, \overline{Y}$.

From (3.4), by using T and the initial value $\overline{Y}(0)$, let

$$\overline{Y}(0) = \frac{1}{\prod_{i=1}^{M-1}(N_i+1)} V, \tag{3.14}$$

where V is a vector of 1's $\prod_{i=1}^{M-1}(N_i+1)$ dimensions. Then,

$$\overline{Y} = \lim_{j\to\infty} T^j \overline{Y}(0). \tag{3.15}$$

The following equation can be used to justify the correctness of the resulting $\overline{Y}$:

$$\overline{Y}^T V = 1. \tag{3.16}$$

Step 3.5. From Step 3.4, (3.15), (3.7), (3.8), and (3.9), we have the PR and $E[b_i]$.

4. Analytical approach

This section is devoted to develop the performance evaluation of the production line with QC devices. The performance measures of interest in this paper are the production rate, $\overline{\mathrm{PR}}$, and the average steady-state buffer occupancy, $E[\overline{b}_i]$, $i = 1,\ldots,M-1$. However, these functions cannot be calculated in closed form for the long production line. Therefore, we derive their estimates, PR and $E[\overline{b}_i]$, $i = 1,\ldots,M-1$, below and then evaluate their accuracy. The accuracy analysis is based on the results obtained in Section 3.

To characterize the steady-state behavior of production line (i)–(vii), introduce the function

$$F(x,Y,N) = \begin{cases} \dfrac{y-x}{y-x\eta^N}, & x \neq y, \\[2ex] \dfrac{1-x}{1-x+N}, & x = y, \end{cases} \tag{4.1}$$

where

$$\eta = \frac{x(1-y)}{y(1-x)}. \tag{4.2}$$

Lemma 4.1. *Serial production line (i)–(vii) with $M = 2$ has production rate*

$$\begin{aligned} \overline{\mathrm{PR}} &= p_2[1 - F(p_1c_1(1-d_1), p_2, N_1)] \\ &= p_1c_1(1-d_1)[1 - F(p_2, p_1c_1(1-d_1), N_1)], \end{aligned} \tag{4.3}$$

which is a monotonically increasing function of p_1, c_1, p_2, and N_1, and decreasing of d_1.

Proof. See the appendix. □

Lemma 4.2. *Serial production line (i)–(vii) with $M = 2$ has average steady-state buffer occupancy*

$$E[\overline{b}_i] = \begin{cases} \left(\dfrac{F(x,y,N_1)}{1-y}\right)\left[\dfrac{\eta - (1+N_1 - N_1\eta)\eta^{N_1+1}}{(1-\eta)^2}\right], & x \neq y, \\ \dfrac{N_1(N_1+1)}{2(N_1+1-x)}, & x = y, \end{cases} \tag{4.4}$$

where $x = p_1c_1(1-d_1)$ and $y = p_2$. Functions $F(x,y,N_1)$ and η are defined in (4.1).

Proof. See the appendix. □

These two lemmas play a critical role for developing the analytic performance evaluation for a production line with $M > 2$.

Next, consider now a line with M machines, $(M > 2)$, $(M-1)$ QC devices, and $(M-1)$ buffers. Based on Lemma 4.1 and an aggregating technique consisting of two principal components, a forward and a backward aggregation, a recursive procedure is developed.

In the forward aggregation procedure, the first two machines, and the middle QC device and buffer, being repeatedly replaced by a single machine, such that reducing the length of the line, until the overall line has been reduced to a single machine. That is,

$$\begin{aligned} p_2^f &= p_2[1 - F(p_1c_1(1-d_1), p_2, N_1)], \\ p_3^f &= p_3\left[1 - F\left(p_2^f c_2(1-d_2), p_3, N_2\right)\right], \\ &\vdots \\ p_i^f &= p_i\left[1 - F\left(p_{i-1}^f c_{i-1}(1-d_1), p_i, N_{i-1}\right)\right], \\ &\vdots \\ p_M^f &= p_M\left[1 - F\left(p_{M-1}^f c_{M-1}(1-d_{M-1}), p_M, N_{M-1}\right)\right], \end{aligned} \tag{4.5}$$

where parameter p_i^f is the machine parameter of a single machine replacing the first i machines and $(i-1)$ QC devices and $(i-1)$ buffers.

It follows from above that the blockage of the aggregated machine results in the blockage of every machine involved in the aggregation which is not the same amount of blockage in the real system. Therefore, to eliminate this undesirable phenomenon, let us aggregate the line going from the back (backward aggregation) taking the results of the forward aggregation into account. In other words, in the backward aggregation procedure, the last two machines, and the middle QC device and buffer, being repeatedly replaced by a single machine until the overall line has again been reduced to a single machine. Here the

blockage probabilities are calculated from the forward aggregation. Specifically, we write

$$\begin{aligned}
p_{M-1}^b &= p_{M-1}c_{M-1}\Big[1-(1-d_{M-1})F\Big(p_M, p_{M-1}^f c_{M-1}(1-d_{M-1}), N_{M-1}\Big)\Big],\\
p_{M-2}^b &= p_{M-2}c_{M-2}\Big[1-(1-d_{M-2})F\Big(p_{M-1}^b, p_{M-2}^f c_{M-2}(1-d_{M-2}), N_{M-2}\Big)\Big],\\
&\vdots\\
p_i^b &= p_i c_i\Big[1-(1-d_i)F\Big(p_{i+1}^b, p_i^f c_i(1-d_i), N_i\Big)\Big],\\
&\vdots\\
p_1^b &= p_1 c_1\Big[1-(1-d_1)F\Big(p_2^b, p_1^f c_1(1-d_1), N_1\Big)\Big],
\end{aligned} \tag{4.6}$$

where parameter p_i^b is the machine parameter of the single machine replacing machines m_j, $j = i,\dots,M$, QC devices Q_j, and buffers B_j, $j = i,\dots,M-1$.

Now we iterate the process by constructing a new forward aggregation based on this backward aggregation, and so on. As a result, we obtain the following *recursive procedure*:

$$\begin{aligned}
p_i^f(s) &= p_i\Big[1-F\Big(p_{i-1}^f(s)c_{i-1}(1-d_{i-1}), p_i^b(s-1), N_{i-1}\Big)\Big], \quad 2 \le i \le M,\\
p_i^b(s) &= p_i c_i\Big[1-(1-d_i)F\Big(p_{i+1}^b(s), p_i^f(s)c_i(1-d_i), N_i\Big)\Big], \quad 1 \le i \le M-1,\\
p_i^f(s) &= p_1, \quad p_M^b(s) = p_M, \quad s = 1,2,3,\dots
\end{aligned} \tag{4.7}$$

with initial condition

$$p_i^b(0) = p_i, \quad i = 1,\dots,M, \tag{4.8}$$

where $F(x,y,N)$ is defined by (4.1).

The question of convergence of the resulting $p_i^f(s)$ and $p_i^b(s)$, $s = 1,2,3,\dots$, is answered in the following.

Lemma 4.3. *The recursive procedure (4.7) is convergent, so that the following limits exist:*

$$\lim_{s\to\infty} p_i^f(s) = p_i^f, \quad \lim_{s\to\infty} p_i^b(s) = p_i^b, \quad i = 1,\dots,M. \tag{4.9}$$

Proof. See the appendix. □

Parameters p_i^f and p_i^b can be interpreted as

$$\begin{aligned}
p_i^f &\approx \text{Prob}\{\text{machine } i \text{ produces a part} \mid \text{machine } i \text{ is not blocked}\},\\
p_i^b &\approx \text{Prob}\{\text{machine } i \text{ produces a part} \mid \text{machine } i \text{ is not starved}\}.
\end{aligned} \tag{4.10}$$

Therefore, since the last machine is never blocked, the production rate estimate for the line (i)–(vii) is defined as

$$\text{PR}(p_1,\dots,p_M,c_1,\dots,c_{M-1},d_1,\dots,d_{M-1},N_1,\dots,N_{M-1}) = p_M^f. \tag{4.11}$$

Table 4.1. Estimation error of PR.

Case	p_1	p_2	p_3	p_4	δ	$\overline{\mathrm{PR}}$	PR	ρ
I	0.99	0.99	0.99	0.99	0.0048	0.7978	0.7980	0.00025
II	0.7	0.95	0.7	0.95	0.0418	0.548	0.5604	0.0210
III	0.99	0.65	0.65	0.99	0.0098	0.5063	0.5035	0.0055

Consider now $(M-1)$ two machines, one QC device, and one buffer line L_i, $i = 1,\dots,$ $M-1$, where the first machine is defined by p_i^f, the second by p_{i+1}^b, and the buffer is of capacity N_i. From Lemma 4.2, the average steady-state buffer occupancy of buffer B_i, $i = 1,\dots,M-1$, can be calculated as

$$E[b_i] = \begin{cases} \dfrac{F(x,y,N_i)}{1-y}\left[\dfrac{\eta(1+N_i-N_i\eta)\eta^{N_i+1}}{(1-\eta)^2}\right], & x \neq y, \\[2ex] \dfrac{N_i(N_i+1)}{2(N_i+1-x)}, & x = y, \end{cases} \tag{4.12}$$

where $x = p_i c_i(1-d_i)$ and $y = p_{i+1}$, $i = 1,\dots,M-1$, and $F(x,y,N_i)$ and η are defined in (4.1).

To describe the relationships between the real production rate with its estimate (4.11), and between the real $E[\overline{b}_i]$ with estimate $E[b_i]$ (4.12), introduce function $\overline{X}_{i,\dots,j}(b_i,\dots,b_j)$ the joint steady-state probability that consecutive buffers $B_i,\dots,B_j$, $i \le i < j \le M-1$, contain $b_i,\dots,b_j$ parts, respectively. In general, these joint probabilities cannot equal to the product of their marginal distributions, that is, $\overline{X}_{i,\dots,j}(b_i,\dots,b_j) \neq \overline{X}_i(b_i)\overline{X}_{i+1,\dots,j}(b_{i+1},\dots,$ $b_j)$. It turns out, however, that for certain values of $b_i, b_{i+1},\dots,b_j$, related to blockages and starvations, they are indeed close. Specifically, from [10], define

$$\begin{aligned} \delta_{i,j}(A) &= \left|\overline{X}_{i,\dots,j}(0,A,N_{i+2},\dots,N_j) - \overline{X}_i(0)\overline{X}_{i+1,\dots,j}(A,N_{i+2},\dots,N_j)\right| \\ \delta^{i,j}(B) &= \left|\overline{X}_{i,\dots,j}(B,N_{i+1},\dots,N_j) - \overline{X}_i(B)\overline{X}_{i+1,\dots,j}(N_{i+1},\dots,N_j)\right|. \end{aligned} \tag{4.13}$$

Then, we have the following numerical fact.

Numerical fact. For serial production line defined by assumptions (i)–(vii),

$$\begin{gathered} \delta \ll 1, \\ \delta = \max_{i,j}\max_{A,B}\{\delta_{i,j}(A), \delta^{i,j}(B)\}. \end{gathered} \tag{4.14}$$

Numerical justification. The result is justified by extensive numerical experimentation, which is developed in Section 3. As it follows from the experimentation, the value of δ is always small. An illustration is given in Tables 4.1 and 4.2 for several lines with $c_i = 0.95$, $d_i = 0.05$, and $N_i = 3$, $i = 1,2,3$. Although we do not have an analytical proof that $\delta \ll 1$, on account of this evidence, we conclude that numerical fact holds.

Table 4.2. Estimation error of $E[b_i]$.

Case	$E[\overline{b}_1]$	$E[b_1]$	$E[\overline{b}_2]$	$E[b_2]$	$E[\overline{b}_3]$	$E[b_3]$	ϑ
I	1.5501	1.5488	1.2478	1.2449	0.8386	0.8394	0.000483
II	1.1534	1.0811	1.7505	1.7354	0.6155	0.6318	0.0121
III	2.8355	2.8267	1.6415	1.6412	0.5135	0.5138	0.0015

The relationship between the exact production rate, $\overline{\mathrm{PR}}$, and the calculated one, PR, and the relationship between the exact steady-state buffer occupancy, $E[\overline{b}_i]$, and the calculated one, $E[b_i]$, $i = 1,\dots,M-1$, are given by Theorems 4.4 and 4.5, respectively.

Theorem 4.4. *Under assumption (i)–(vii), production rate estimate (4.11) results in $O(\delta)$ accuracy, that is,*

$$\rho = \frac{|\overline{\mathrm{PR}} - \mathrm{PR}|}{\overline{\mathrm{PR}}} \sim O(\delta), \tag{4.15}$$

where δ is defined in (4.14).

Proof. See the appendix. □

Theorem 4.5. *Under assumption (i)–(vii), the average steady-state buffer occupancy estimate (4.12) results in $O(\delta)$ accuracy, that is,*

$$\vartheta = \max_i \left\{ \left(\frac{2}{N_i(N_i+1)} \right) |E[\overline{b}_i] - E[b_i]| \right\} \sim O(\delta), \quad i = 1,\dots,M-1, \tag{4.16}$$

where δ is defined in (4.14).

Proof. See the appendix. □

Remark 4.6. The meaning of Theorem 4.5 is that the average buffer occupancy for B_i equals $(1+2+3+\cdots+N_i)$, which is equal to $N_i(N_i+1)/2$. Therefore we have the normalized estimated error which is equal to the difference between real buffer occupancy and the estimated buffer occupancy divided by the average buffer occupancy.

Although these estimates are just numerical experimentation (since δ is not an asymptotic parameter), illustrated in Tables 4.1 and 4.2, they show that the proportionality constant in $O(\delta)$ is quite small, and the estimate (4.11) and (4.12) results in high accuracy.

The following theorem is a useful property of production system.

Theorem 4.7. *The estimated production rate defined by (4.11) is monotonically increasing with respect to p_i, $i = 1,\dots,M$, c_i, and N_i, $i = 1,\dots,M-1$, and decreasing with respect to d_i, $i = 1,\dots,M-1$.*

Proof. See the appendix. □

In addition, the monotonic properties of the line production rate can help us to determine which machines, *QC* devices, or buffers should be improved so that we will have the best possible performance improvement.

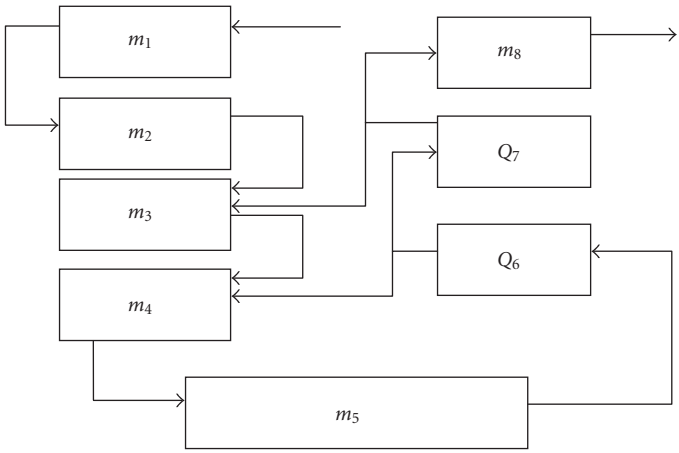

Figure 5.1. Structure of the system.

Table 5.1. Case study system parameters.

	m_1	m_2	m_3	m_4	m_5	Q_6	Q_7	m_8
F_1 (min)	198	198	98	98	98	198	198	198
R_i (min)	2	2	2	2	2	2	2	2
Rework ratio	0	0	0	0	0	0.0547	0.0724	0
Physical buffer size	5	5	10	10	10	10	30	

5. Case study

The performance analysis described above has been used in the continuous improvement project of a production line at a picture tube plant. The structure of the production line is shown in Figure 5.1. It consists of 8 operations, 6 processing machines ($m_1 - m_5$ and m_8), and 2 QC devices (Q_6 and Q_7), and a conveyor in-process material handling system. The material handling system is structured as a set of palletized conveyors connecting each pair of consecutive operations and providing buffering capacities. Parameters of the machines and buffers identified are given in Table 5.1.

F_i and R_i, listed in Table 5.1, refer to the mean time to failure and mean time to repair of machine m_i or QC device Q_i, respectively. Thus, the isolated machine production rate, or the isolated working rate, can be calculated as

$$\begin{aligned} p_i &= \frac{F_i}{F_i + R_i}, \quad i = 1,\dots,5,8, \\ c_i &= \frac{F_i}{F_i + R_i}, \quad i = 6,7. \end{aligned} \tag{5.1}$$

To make the system shown in Figure 5.2 fit the model defined in Section 2, consider the rework ratios as the bad part ratios. The capacity of the physical buffer was taken as

Table 5.2. The corresponding parameters of the case study.

Parameters	1	2	3	4	5	6	7	8
p_i	0.99	0.99	0.98	0.98	0.99	1	1	0.99
c_i	1	1	1	1	1	0.99	0.99	$*$
d_i	0	0	0	0	0	0.0547	0.0724	$*$
N_i	2	3	3	2	2	2	4	$*$

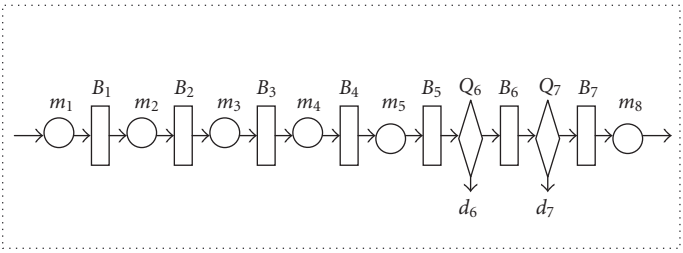

Figure 5.2. Implementation of the system model.

the difference of the actual number of pallets between two operations and the number of pallets necessary to sustain a continuous operation if no break down occurs. Then, by a transfer technique, we have the buffer capacity from the physical buffer size to the Bernoulli one in [11]. The transformation is based on the idea of choosing the buffer size in Bernoulli model to be the minimum value between the average downtime of upstream machine and that of the downstream one. Thus, the model of the system amenable to the performance evaluation techniques developed in Section 4 is identified. The corresponding Bernoulli parameters are listed in Table 5.2, and the simplified system is given in Figure 5.2. The reasons for neglecting the feedback effect of the rework parts are as follows. First, at present, we do not have a theory for analyzing such production lines, nor the current literature offers such a method. Second, some of the rework parts are discarded from the system, and some of them go to m_3 or m_4 to be reworked. The former term, in fact, can be viewed as the bad parts, and contribute to the bad part ratios. The latter term can cause the decrease in starvation probabilities of m_3 and m_4, and increase in blockage probabilities of m_2, m_3, and m_4. We assume the effects of starvations and blockages are neutralized mutually. Although this assumption leads to a loss in accuracy, we believe the loss would be bounded in small quantity. Thus, the simplified system, shown in Figure 5.2, is appropriate for the analysis of the actual production system.

The estimate of the average production rate of the production system, PR, has been calculated, using the recursive procedure (4.7), and the result has been compared with the actual average production rate $\overline{\mathrm{PR}}$. The results are as follows:

$$\begin{aligned} \overline{\mathrm{PR}} &= 262 \text{ parts/hour}, \\ \mathrm{PR} &= 255 \text{ parts/hour}. \end{aligned} \tag{5.2}$$

Table 5.3. The average buffer occupancy.

Buffer	1	2	3	4	5	6	7
$E[b_i]$	1.7336	2.5034	2.0693	1.03062	1.2571	1.0197	0.9112

Thus, based on the analytical results, the production rate estimated matches the actual system with an error 2.67%. Therefore, we accept the model identified as validated by the actual performance.

According to the technique of performance analysis results, two improvement measures are performed as follows.

(I) Since all machines of the system are highly reliable, it is difficult to obtain a significant improvement by improving machines' reliability. Therefore, we focus on the reconstruction of the material handling system. To obtain some ideas on how to reconstruct the material handling system, the average buffers occupancy of the system are calculated by using (4.12), and the calculated results are listed in Table 5.3. It follows from this result and Table 4.2, that buffer 7 has low buffer occupancy and large buffer capacity and, on the contrary, buffers 2 and 3 have high buffer occupancies and low buffer capacities. Usually, the factory space is limited, and therefore, the total capacities of all buffers are considered to be fixed. Under this constraint, we reallocate two units of the capacity of buffer 7 to buffers 2 and 3 (each one obtains one unit of capacity). The production rate of the above modified system is increasing 1 part/hour.

(II) From the discussion with the engineer on the factory, we have the following statement: the effort of reducing the bad part ratio of m_4 (d_6) by 50% is closed to the effort of reducing the bad part ratio of m_3 (d_7) by 50%. To know which reduction will gain a better improvement, the production rates of the systems using the technique of the performance analysis developed in this paper. The resulting production rate increase of reducing the bad part ratio of m_4 (d_6) by 50% is 7 parts/hour, and the production rate increase of reducing the bad part ratio of m_3 (d_7) by 50% is 10 parts/hour.

Based on the above results, the improvement effort should be directed to m_3 to obtain the best possible improvement. The plant management has accepted this recommendation.

6. Conclusion

The performance evaluation of production lines with Bernoulli machines and *QC* devices is obtained in a numerical form and a closed form within an acceptable error bound. These two results are justified each other, and are in a good agreement. Some important properties of the production line are also obtained in this work as follows. The estimated production rate is monotonically increasing in machines' production rate and *QC* devices working rates in isolation, and decreasing in bad part ratios. The results can be utilized for analysis and design of serial production lines with *QC* devices. Since the best test of a theoretical result is practice, it has been used in a project carried out at a picture tube plant. It is hoped that this approach will prove to be useful for other practitioner and theoreticians alike. The extension of the result to production lines with Markovian machines is the subject of the future work.

Appendix

To prove Lemma 4.1, we recall some results from [10]: function $F(x,y,N)$, $0 < x < 1$, $0 < y < 1$, $N \in Z^+$, defined in (4.1), has the following properties:

(a) monotonically decreasing in x,
(b) monotonically increasing in y,
(c) monotonically decreasing in N,
(d) takes value in $(0,1)$.

Proof of Lemma 4.1. Let $\overline{X}(j,s)$ denote the probability that the buffer B_1 contains j parts at time s. This is a closed irreducible Markov chain, which therefore converges to a unique equilibrium distribution. Let

$$\overline{X}(j) = \lim_{s\to\infty} \overline{X}(j,s), \quad 0 \le j \le N_1. \tag{A.1}$$

This equilibrium distribution must satisfy the following equilibrium equation of the Markov transition equation:

$$\begin{aligned}
\overline{X}(0) &= (1 - p_1c_1 + p_1c_1d_1)\overline{X}(0) + [(1 - p_1c_1 + p_1c_1d_1)p_2]\overline{X}(1) \\
\overline{X}(1) &= p_1c_1(1-d_1)\overline{X}(0) + [(1 - p_1c_1 + p_1c_1d_1)(1-p_2) + p_1c_1(1-d_1)p_2]\overline{X}(1) \\
&\quad + [(1 - p_1c_1 + p_1c_1d_1)p_2]\overline{X}(2) \\
\overline{X}(j) &= p_1c_1(1-d_1)\overline{X}(j-1) + [(1 - p_1c_1 + p_1c_1d_1)(1-p_2) + p_1c_1(1-d_1)p_2]\overline{X}(j) \\
&\quad + [(1 - p_1c_1 + p_1c_1d_1)p_2]\overline{X}(j+1), \quad 2 \le j \le N_1 - 1, \\
\overline{X}(N_1) &= [p_1c_1(1-d_1)(1-p_2)]\overline{X}(N_1-1) + [(1-p_2) + p_1c_1(1-d_1)p_2]\overline{X}(N_1).
\end{aligned} \tag{A.2}$$

Solving (A.2), we obtain

$$\overline{X}(j) = \overline{X}(0)\left(\frac{1}{1-p_2}\right)Z^j, \quad 1 \le j \le N_1, \tag{A.3}$$

where

$$Z = \frac{[p_1c_1(1-d_1)](1-p_2)}{[1 - p_1c_1(1-d_1)]p_2}. \tag{A.4}$$

Since $\sum_{j=0}^{N_1} \overline{X}(j) = 1$, we have

$$\overline{X}(0)\left[1 + \frac{Z}{1-p_2} + \frac{Z^2}{1-p_2} + \cdots + \frac{Z^{N_1}}{1-p_2}\right] = 1. \tag{A.5}$$

After some algebra, this simplifies to

$$\overline{X}(0) = \begin{cases} \dfrac{p_2 - p_1c_1(1-d_1)}{p_2 - p_1c_1(1-d_1)Z^{N_1}}, & p_1c_1(1-d_1) \neq p_2, \\ \dfrac{1-p_2}{1-p_2+N_1}, & p_1c_1(1-d_1) \neq p_2. \end{cases} \tag{A.6}$$

For the line to produce a part during a cycle, the second machine must be operational and not starved. Therefore, the production rate, $\overline{\mathrm{PR}}$, can be calculated as follows:

$$\overline{\mathrm{PR}} = p_2[1-\overline{X}(0)] = p_2[1-F(p_1c_1(1-d_1),p_2,N_1)], \tag{A.7}$$

where $F(x,y,N)$ is defined by (4.1).

From (A.3) and (A.6), after simplifying, we have

$$\overline{X}(N_1) = \begin{cases} \left(\dfrac{1}{1-p_2}\right)\dfrac{p_1c_1(1-d_1)-p_2}{p_1c_1(1-d_1)-p_2Z^{N_1}}, & p_1c_1(1-d_1) \neq p_2, \\ \dfrac{1-p_2}{1-p_2+N_1}, & p_1c_1(1-d_1) \neq p_2. \end{cases} \tag{A.8}$$

Since the first machine produces a part if it is operational and not blocked. However, the parts produced by the line added to the bad parts equal the parts produced by the first machine. Therefore,

$$\begin{aligned} \overline{\mathrm{PR}} &= p_1[1-(1-c_1)-c_1(1-d_1)(1-p_2)\overline{X}(N_1)] - p_1c_1d_1 \\ &= p_1c_1(1-d_1)[1-(1-p_2)\overline{X}(N_1)] \\ &= p_1c_1(1-d_1)[1-F(p_2,p_1c_1(1-d_1),N_1)]. \end{aligned} \tag{A.9}$$

The monotonicity of the production rate in p_1, c_1, p_2, and N_1 (or d_1) follows directly form (A.7), (A.9), and properties (a)–(d) of function $F(x,y,N)$. □

Proof of Lemma 4.2. By (A.3), the average steady-state buffer occupancy of buffer B_1 is

$$\begin{aligned} E[\overline{b}_1] &= \sum_{j=0}^{N_1} j\overline{X}(j) = \sum_{j=0}^{N_1} j\left(\frac{Z^j}{1-p_2}\right)\overline{X}(0) \\ &= \sum_{j=0}^{N_1} j\left(\frac{Z^j}{1-p_2}\right)F(p_1c_1(1-d_1),p_2,N_1). \end{aligned} \tag{A.10}$$

After some algebra and by (4.1), we can derive

$$E[\overline{b}_1] = \begin{cases} \dfrac{F(x,y,N_1)}{1-y}\left[\dfrac{Z-(1+N_1-N_1Z)Z^{N_1+1}}{(1-Z)^2}\right], & x \neq y, \\ \dfrac{N_1(N_1+1)}{2(N_1+1-x)}, & x = y, \end{cases} \tag{A.11}$$

where $x = p_1c_1(1-d_1)$, $y = p_2$. Functions $F(x,y,N)$ and Z are given in (4.1). □

The proof of Lemma 4.3, in addition to the above properties of function F, we also require the following fact.

Lemma A.1. *Consider $p_i^f(s)$ and $p_i^b(s)$, $i = 1,\ldots,M$, defined by procedure (4.7). Sequences $p_i^f(s)$ and $p_i^b(s)$, $s \in N$, are monotonically increasing and decreasing, respectively.*

Proof. By induction, for $s = 1$, due to the properties of function $F(x,y,N)$, we have

$$\begin{aligned} p_j^b(1) &= p_jc_j\left[1-(1-d_j)F\left(p_{j+1}^b(1), p_j^f(1)c_j(1-d_j), N_j\right)\right] \\ &< p_j = p_j^f(0), \quad 1 \le j \le M-1. \end{aligned} \tag{A.12}$$

Assume that for $s > 1$,

$$p_j^b(s) < p_j^b(s-1), \quad 1 \le j \le M-1. \tag{A.13}$$

Then from the recursive procedure (4.7), for $j = 2$, by (A.13),

$$\begin{aligned} p_2^f(s) &= p_2\left[1-F\left(p_1c_1(1-d_1), p_2^b(s-1), N_1\right)\right] \\ &< p_2\left[1-F\left(p_1c_1(1-d_1), p_2^b(s), N_1\right)\right] = p_2^f(s+1). \end{aligned} \tag{A.14}$$

For $j = 3,\ldots,M$,

$$\begin{aligned} p_j^f(s) &= p_j\left[1-F\left(p_{j-1}^f(s)c_{j-1}(1-d_{j-1}), p_j^b(s-1), N_{j-1}\right)\right] \\ &< p_j\left[1-F\left(p_{j-1}^f(s+1)c_{j-1}(1-d_{j-1}), p_j^b(s-1), N_{j-1}\right)\right] \\ &< p_j\left[1-F\left(p_{j-1}^f(s+1)c_{j-1}(1-d_{j-1}), p_j^b(s), N_{j-1}\right)\right] = p_j^f(s+1). \end{aligned} \tag{A.15}$$

Hence,

$$p_j^f(s) < p_j^f(s+1), \quad 2 \le j \le M. \tag{A.16}$$

Then for $j = M-1$, by (A.16),

$$\begin{aligned} p_{M-1}^{b}(s) &= p_{M-1}c_{M-1}\Big[1-(1-d_{M-1})F\Big(p_M, p_{M-1}^{f}(s)c_{M-1}(1-d_{M-1}), N_{M-1}\Big)\Big] \\ &> p_{M-1}c_{M-1}\Big[1-(1-d_{M-1})F\Big(p_M, p_{M-1}^{f}(s+1)c_{M-1}(1-d_{M-1}), N_{M-1}\Big)\Big] \\ &= p_{M-1}^{b}(s+1). \end{aligned} \tag{A.17}$$

For $j = M-2,\dots,1$,

$$\begin{aligned} p_j^{b}(s) &= p_j c_j\Big[1-(1-d_j)F\Big(p_{j+1}^{b}(s), p_j^{f}(s)c_j(1-d_j), N_j\Big)\Big] \\ &> p_j c_j\Big[1-(1-d_j)F\Big(p_{j+1}^{b}(s+1), p_j^{f}(s)c_j(1-d_j), N_j\Big)\Big] \\ &> p_j c_j\Big[1-(1-d_j)F\Big(p_{j+1}^{b}(s+1), p_j^{f}(s+1)c_j(1-d_j), N_j\Big)\Big] \\ &= p_j^{b}(s+1). \end{aligned} \tag{A.18}$$

Therefore, sequences $p_j^{f}(s)$ and $p_j^{b}(s)$, $s = 1,2,3,\dots$, are monotonically increasing and decreasing, respectively. □

Proof of Lemma 4.3. Since the sequences $p_j^{f}(s)$ and $p_j^{b}(s)$, $1 < j < M$, are monotonic (Lemma A.1) and bounded (result in (d)), so they are convergent. □

The proof of Theorem 4.4 requires the following lemmas.

Introduce the following conditional probabilities:

$$\begin{aligned} \overline{p}_i^{f} &= \text{Prob}\{m_i \text{ produces} \mid m_i \text{ is not blocked}\}, \\ \overline{p}_i^{b} &= \text{Prob}\{m_i \text{ produces} \mid m_i \text{ is not starved}\}. \end{aligned} \tag{A.19}$$

These probabilities play a crucial role in the proof of Theorem 4.4. Specifically, we show below (Lemma A.3) that if $\overline{p}_i^{f}$ and $\overline{p}_{i+1}^{b}$ are known, then the stationary probability distribution of buffer occupancy, $\overline{X}_i(\cdot)$, can be calculated with the error $O(\delta)$. Further, Lemma A.5 shows that $\overline{p}_i^{f}$ and $\overline{p}_{i+1}^{b}$ can be calculated from the steady state of recursive procedure (4.7), with the error $O(\delta)$. Therefore, since the production rate can be calculated from (3.8), $\overline{\text{PR}} = p_M[1-\overline{X}_{M-1}(0)]$, the claim of Theorem 4.4 will follow.

Lemma A.2. *Under numerical fact, the conditional probabilities $\overline{p}_i^{f}$, $\overline{p}_i^{b}$ take the following forms:*

$$\overline{p}_i^{f} = p_i[1-\overline{X}_{i-1}(0)] + O(\delta), \quad i = 2,\dots,M, \tag{α}$$

$$\begin{aligned} \overline{p}_i^{b} = p_i\Bigg\{c_i - c_i(1-d_i)\sum_{j=i+1}^{M}\Bigg[\prod_{r=i+1}^{j-1} p_r c_r(1-d_r)\Bigg](1-p_j c_j)\overline{X}_{i,\dots,j-1}(N_i,\dots,N_{j-1})\Bigg\} \\ + O(\delta), \quad i = 1,\dots,M-1,\ c_M = 1, \end{aligned} \tag{β}$$

where $\overline{X}_{i,\ldots,j}(b_i,\ldots,b_j)$ is the steady-state probability that consecutive buffers $B_i,\ldots,B_j$ in production line (i)–(vii) contain $b_i,\ldots,b_j$ parts, respectively.

Proof. The probability that machine m_i is blocked can be expressed as follows:

$$\begin{aligned}&\text{Prob}\{m_i \text{ is blocked}\}\\&\quad=\left\{(1-c_i)-c_i(1-d_i)\sum_{j=i+1}^{M}\left[\prod_{r=i+1}^{j-1}p_rc_r(1-d_r)\right](1-p_jc_j)\overline{X}_{i,\ldots,j-1}(N_i,\ldots,N_{j-1})\right\}.\end{aligned} \tag{A.20}$$

Since machine m_i is not starved when buffer B_{i-1} contains one or more parts, using the conditional probability formula and numerical fact, we write

$$\begin{aligned}&\text{Prob}\{m_i \text{ is blocked} \mid m_i \text{ is not starved}\}\\&\quad=\left\{(1-c_i)+c_i(1-d_i)\sum_{j=i+1}^{M}\left[\prod_{r=i+1}^{j-1}p_rc_r(1-d_r)\right]\right.\\&\qquad\left.\times(1-p_jc_j)\frac{\sum_{c=1}^{N_{i-1}}\overline{X}_{i-1,\ldots,j-1}(c,N_i,\ldots,N_{j-1})}{1-\overline{X}_{i-1}(0)}\right\},\end{aligned} \tag{A.21}$$

where

$$\begin{aligned}&\frac{\sum_{c=1}^{N_{i-1}}\overline{X}_{i-1,\ldots,j-1}(c,N_i,\ldots,N_{j-1})}{1-\overline{X}_{i-1}(0)}\\&\quad=\frac{\overline{X}_{i,\ldots,j-1}(N_i,\ldots,N_{j-1})-\overline{X}_{i-1}(0)\overline{X}_{i,\ldots,j-1}(N_i,\ldots,N_{j-1})}{1-\overline{X}_{i-1}(0)}+O(\delta)\\&\quad=\overline{X}_{i,\ldots,j-1}(N_i,\ldots,N_{j-1})+O(\delta).\end{aligned} \tag{A.22}$$

From above and (A.19), we obtain

$$\text{Prob}\{m_i \text{ is blocked} \mid m_i \text{ is not starved}\}=\text{Prob}\{m_i \text{ is blocked}\}+O(\delta). \tag{A.23}$$

Therefore,

$$\begin{aligned}\overline{p}_i^b&=\text{Prob}\{m_i \text{ produces} \mid m_i \text{ is not starved}\}\\&=\text{Prob}\{m_i \text{ is not blocked} \mid m_i \text{ is not starved}\}\\&=p_i(1-\text{Prob}\{m_i \text{ produces} \mid m_i \text{ is not starved}\})\\&=p_i(1-\text{Prob}\{m_i \text{ is blocked}\})+O(\delta).\end{aligned} \tag{A.24}$$

By multiplication rule of condition probability formula, the definition of $\overline{p}_i^f$, and (A.23),

we obtain

$$\begin{aligned}
\overline{p}_i^f &= \text{Prob}\{m_i \text{ produces} \mid m_i \text{ is not blocked}\} \\
&= \text{Prob}\{m_i \text{ is up, not blocked, and not starved} \mid m_i \text{ is not blocked}\} \\
&= \frac{\text{Prob}\{m_i \text{ is up, not blocked, and not starved}\}}{\text{Prob}\{m_i \text{ is not blocked}\}} \\
&= \text{Prob}\{m_i \text{ is not starved}\} \frac{\text{Prob}\{m_i \text{ is not blocked} \mid m_i \text{ is not starved}\}}{\text{Prob}\{m_i \text{ is not blocked}\}} \\
&\quad \cdot \text{Prob}\{m_i \text{ is up} \mid m_i \text{ is not blocked or starved}\} \\
&= p_i[1-\overline{X}_{i-1}(0)] \frac{\text{Prob}\{m_i \text{ is not blocked} \mid m_i \text{ is not starved}\}}{\text{Prob}\{m_i \text{ is not blocked}\}} \\
&= p_i[1-\overline{X}_{i-1}(0)] \frac{1-\text{Prob}\{m_i \text{ is not blocked} \mid m_i \text{ is not starved}\}}{1-\text{Prob}\{m_i \text{ is blocked}\}} \\
&= p_i[1-\overline{X}_{i-1}(0)] + O(\delta).
\end{aligned} \tag{A.25}$$

Consider now $(M-1)$ two machines, one QC device, and one-buffer lines L_i, $i = 1,\dots,$ $M-1$, where the first machine is defined by $\overline{p}_i^f$, the second by $\overline{p}_{i+1}^b$, and the buffer is of capacity N_i. Let $X_i(\cdot)$ be the equilibrium probability distribution of buffer occupancy of line L_i. Along with these $M-1$ lines, consider the line (i)–(vii) with M machines. Let $\overline{X}_i(\cdot)$, as before, be the equilibrium probability distribution of buffer occupancy of buffer B_i. Then, we have the following lemma. □

Lemma A.3. *Under numerical fact, the following is true:*

$$\left|\overline{X}_i(j) - X_i(j)\right| \approx O(\delta), \quad i = 1,\dots,M-1,\ j = 0,\dots,N_i, \tag{A.26}$$

where δ is defined by (4.13).

Proof. Consider a line (i)–(vii) with M machines. Let $K_i = [k_1,\dots,k_{i-1},k_{i+1},\dots,k_{M_1}]$, $1 \le i \le M-1$, $0 \le k_j \le N_j$, $j \ne i$, be an $(M-2)$-dimensional vector. Let $n_i(t+1) = (b_i, K_i)$, $1 \le i \le M-1$, denote the state that there are b_i parts in buffer B_i, and k_j parts in buffer B_j, for all $j \ne i$ at time slot t. Let $\alpha_i(1,t)$ (or $\alpha_i(0,t)$) denote that the state of machine m_i is up (or down), $\beta_i(1,t)$ (or $\beta_i(0,t)$) denote that the state of QC device Q_i is working (or not working), and $\gamma_i(1,t)$ (or $\gamma_i(0,t)$) denote that the state of a part at stage i is made defective (or nondefective) at time slot t. Specially, let

$$\zeta_i(0,t) = [\alpha_i(0,t)] \cup [\alpha_i(1,t),\beta_i(0,t)] \cup [\alpha_i(1,t),\beta_i(1,t),\gamma_i(0,t)] \tag{A.27}$$

be the state of buffer B_i that cannot input a part at the ending of the time slot t. Then by

Markov process, we have

$$\begin{aligned}
&\operatorname{Prob}\{n_i(t+1) = (0,K_i)\} \\
&\quad = \sum_{K_i'} \operatorname{Prob}\{\zeta_i(0,t),\ n_i(t) = (0,K_i'),\ n_i(t+1) = (0,K_i)\} \\
&\quad = \sum_{K_i'} \operatorname{Prob}\{\zeta_i(0,t),\ \alpha_{i+1}(t) = 1,\ n_i(t) = (1,K_i'),\ n_i(t+1) = (0,K_i)\} \\
&\quad = \sum_{K_i'} \lfloor \operatorname{Prob}\{n_i(t) = (0,K_i')\} \cdot \operatorname{Prob}\{\zeta_i(0,t) \mid n_i(t) = (0,K_i')\} \\
&\qquad\quad \cdot \operatorname{Prob}\{n_i(t+1) = (0,K_i') \mid \zeta_i(0,t),\ n_i(t) = (0,K_i')\}\rfloor \\
&\qquad + \sum_{K_i'} \lfloor \operatorname{Prob}\{n_i(t) = (1,K_i')\} \cdot \operatorname{Prob}\{\zeta_i(0,t),\ \alpha_{i+1}(t) \mid n_i(t) = (1,K_i')\} \\
&\qquad\quad \cdot \operatorname{Prob}\{n_i(t+1) = (0,K_i') \mid \zeta_i(0,t),\ \alpha_{i+1},\ n_i(t) = (1,K_i')\}\rfloor.
\end{aligned} \tag{A.28}$$

Let $n_i(b_i,K_i)$, $1 \le i \le M-1$, denote the steady state that there are b_i parts in buffer B_i and k_j parts in buffer B_j, for all $j \neq i$. Let $\overline{Y}_i(b_i,K_i)$, $1 \le i \le M-1$, denote the probability that there are b_i parts in buffer B_i and k_j parts in buffer B_j, for all $j \neq i$. Since the line (i)–(vii) can be described by an ergodic Markov chain with states in $\overline{Y}_i(b_i,K_i)$ the steady state, we write

$$\begin{aligned}
\overline{Y}_i(0,K_i) = &\sum_{K_i'} \overline{Y}(0,K_i') \cdot \operatorname{Prob}\{B_i \text{ does not input a part} \mid n_i = (0,K_i')\} \\
&\cdot \operatorname{Prob}\{n_i = (0,K_i) \mid B_i \text{ does not input a part},\ n_i = (0,K_i')\} \\
&+ \sum_{K_i'} \overline{Y}(1,K_i') \cdot \operatorname{Prob}\{B_i \text{ does not input a part},\ m_{i+1} \text{ produce} \mid n_i = (1,K_i')\} \\
&\cdot \operatorname{Prob}\{n_i = (0,K_i) \mid B_i \text{ does not input a part},\ m_{i+1} \text{ produce},\ n_i = (1,K_i')\},
\end{aligned} \tag{A.29}$$

where $\operatorname{Prob}\{B_i \text{ does not input a part} \mid n_i = (b_i,K_i)\}$ denotes the conditional probability that buffer B_i does not input a part during a cycle, given that buffer B_i contains b_i parts and buffer B_j contains k_j parts, for all $j \neq i$, and $\operatorname{Prob}\{n_i = (b_i,K_i) \mid B_i \text{ does not input a part},\ n_i = (b_i,K_i')\}$ denotes the conditional probability that there are b_i parts in buffer B_i and k_j parts in buffer B_j, for all $j \neq i$, given that buffer B_i does not input a part during a cycle and buffer B_i contains b_i parts and buffer B_j contains k_j parts, for all $j \neq i$. Then

$$\begin{aligned}
\overline{X}_i(0) &= \sum_{K_i'} \overline{Y}_i(0,K_i) \\
&= \sum_{K_i'} \overline{Y}(0,K_i') \cdot \operatorname{Prob}\{B_i \text{ does not input a part} \mid n_i = (0,K_i')\} \\
&\quad \cdot \sum_{K_i} \operatorname{Prob}\{n_i = (0,K_i) \mid B_i \text{ does not input a part},\ n_i = (0,K_i')\}
\end{aligned}$$

$$+\sum_{K_i'} Y(1,K_i') \cdot \text{Prob}\{B_i \text{ does not input a part, } m_{i+1} \text{ produce} \mid n_i = (1,K_i')\}$$
$$\cdot \sum_{K_i} \text{Prob}\{n_i = (0,K_i) \mid B_i \text{ does not input a part, } m_{i+1} \text{ produce, } n_i = (0,K_i')\}. \tag{A.30}$$

Since

$$\sum_{K_i} \text{Prob}\{n_i = (0,K_i) \mid B_i \text{ does not input a part, } n_i = (0,K_i')\} = 1,$$
$$\sum_{K_i} \text{Prob}\{n_i = (0,K_i) \mid B_i \text{ does not input a part, } m_{i+1} \text{ produce, } n_i = (0,K_i')\} = 1, \tag{A.31}$$

$$\overline{X}_i(0) = \sum_{K_i'} \overline{Y}_i(0,K_i') \cdot \text{Prob}\{B_i \text{ does not input a part } \mid n_i = (0,K_i')\}$$
$$+\sum_{K_i'} \overline{Y}_i(1,K_i') \cdot \text{Prob}\{B_i \text{ does not input a part, } m_{i+1} \text{ produce} \mid n_i = (1,K_i')\}. \tag{A.32}$$

Consider now the first term of the right-hand side of (A.32):

$$\sum_{K_i'} \overline{Y}_i(0,K_i') \cdot \text{Prob}\{B_i \text{ does not input a part} \mid n_i = (0,K_i')\}$$
$$= \sum_{K_i' \text{ such that } k_{i-1} \geq 1} \overline{Y}_i(0,K_i') \cdot \text{Prob}\{B_i \text{ does not input a part} \mid n_i = (0,K_i')\}$$
$$+ \sum_{K_i' \text{ such that } k_{i-1} = 0} \overline{Y}_i(0,K_i') \cdot \text{Prob}\{B_i \text{ does not input a part} \mid n_i = (0,K_i')\}. \tag{A.33}$$

When buffer B_{i-1} contains at least one part, machine m_i is not starved, and when buffer B_i contains zero parts, machine m_i is not blocked. Therefore, the probability in the first term on the right-hand side of (A.33) is equal to $[(1-p_i)+p_i(1-c_i)+p_ic_id_i]$. When buffer B_{i-1} contains zero parts, machine m_i is starved, and the probability in the second term on the right-hand side of (A.33) is equal to one. Consequently,

$$\begin{aligned}
&\sum_{K_i'} \overline{Y}_i(0,K_i') \text{Prob}\{B_i \text{ does not input a part} \mid n_i = (0,K_i')\} \\
&\quad = [(1-p_i)+p_i(1-c_i)+p_ic_id_i][\overline{X}_i(0)-\overline{X}_{i-1,i}(0,0)]+\overline{X}_{i-1,i}(0,0) \\
&\quad = [1-p_ic_i(1-d_i)][\overline{X}_i(0)-\overline{X}_{i-1,i}(0,0)]+\overline{X}_{i-1,i}(0,0) \\
&\quad = \overline{X}_i(0)[1-p_ic_i(1-d_i)]+\overline{X}_{i-1,i}(0,0)p_ic_id_i.
\end{aligned} \tag{A.34}$$

Using numerical fact, this can be rewritten as

$$\begin{aligned}
&\sum_{K_i'} \overline{Y}_i(0,K_i')\,\text{Prob}\,\{B_i \text{ does not input a part} \mid n_i = (0,K_i')\} \\
&\quad = \overline{X}_i(0)[1 - p_i c_i(1-d_i)] + \overline{X}_{i-1}(0)\overline{X}_i(0)p_i c_i d_i + O(\delta) \\
&\quad = \overline{X}_i(0)[1 - p_i c_i(1-d_i)(1-\overline{X}_{i-1}(0))] + O(\delta).
\end{aligned} \tag{A.35}$$

By Lemma A.2, we finally obtain

$$\begin{aligned}
&\sum_{K_i'} Y_i(0,K_i')\,\text{Prob}\,\{B_i \text{ does not input a part} \mid n_i = (0,K_i')\} \\
&\quad = \overline{X}_i(0)[1 - \overline{p}_i^f c_i(1-d_i) + O(\delta)].
\end{aligned} \tag{A.36}$$

Analysis of the second term on the right-hand side of (A.32) proceeds analogously and results in

$$\overline{X}_i(0) = \overline{X}_i(0)[1 - \overline{p}_i^f c_i(1-d_i) + O(\delta)] + \overline{X}_i(1)[1 - \overline{p}_i^f c_i(1-d_i)]\overline{p}_{i+1}^b. \tag{A.37}$$

Similar arguments can be used to obtain equations for $\overline{X}_i(j)$, $j = 1,\ldots,N_i$. As a result, we obtain the following set of equations:

$$\begin{aligned}
\overline{X}_i(0) &= \left[1 - \overline{p}_i^f c_i(1-d_i) + O(\delta)\right]\overline{X}_i(0) + \left[\left(1 - \overline{p}_i^f c_i(1-d_i)\right)\overline{p}_{i+1}^b\right]\overline{X}_i(1), \\
\overline{X}_i(1) &= \left[\overline{p}_i^f c_i(1-d_i)\right]\overline{X}_i(0) \\
&\quad + \left[\left(1 - \overline{p}_i^f c_i(1-d_i)\right)\left(1 - \overline{p}_{i+1}^b\right) + \overline{p}_i^f c_i(1-d_i)\overline{p}_{i+1}^b + O(\delta)\right]\overline{X}_i(1) \\
&\quad + \left[\left(1 - \overline{p}_i^f c_i(1-d_i)\right)\overline{p}_{i+1}^b\right]\overline{X}_i(2), \\
\overline{X}_i(j) &= \left[\overline{p}_i^f c_i(1-d_i)\left(1 - \overline{p}_{i+1}^b\right)\right]\overline{X}_i(j-1) \\
&\quad + \left[\left(1 - \overline{p}_i^f c_i(1-d_i)\right)\left(1 - \overline{p}_{i+1}^b\right) + \overline{p}_i^f c_i(1-d_i)\overline{p}_{i+1}^b + O(\delta)\right]\overline{X}_i(j) \\
&\quad + \left[\left(1 - \overline{p}_i^f (1-d_i)\right)\overline{p}_{i+1}^b\right]\overline{X}_i(j+2), \quad 2 \le j \le N_i - 1, \\
\overline{X}_i(N_i) &= \left[\overline{p}_i^f c_i(1-d_i)\left(1 - \overline{p}_{i+1}^b\right)\right]\overline{X}_i(N_i-1) \\
&\quad + \left[\left(1 - \overline{p}_{i+1}^b\right) + \overline{p}_i^f c_i(1-d_i)\overline{p}_{i+1}^b + O(\delta)\right]\overline{X}_i(N_i).
\end{aligned} \tag{A.38}$$

These equations can be written in matrix form as

$$\overline{X}_i = (A + \Delta A)X_i, \qquad \overline{X}_i = [\overline{X}_i(0),\ldots,\overline{X}_i(N_i)]^T, \tag{A.39}$$

where

$$A=\begin{bmatrix} 1-\overline{p}_i^f c_i(1-d_i) & (1-\overline{p}_i^f c_i(1-d_i))\overline{p}_{i+1}^b & 0 & \cdots \\ \overline{p}_i^f c_i(1-d_i) & (1-\overline{p}_i^f c_i(1-d_i))(1-\overline{p}_{i+1}^b)+\overline{p}_i^f c_i(1-d_i)\overline{p}_{i+1}^b & (1-\overline{p}_i^f c_i(1-d_i))\overline{p}_{i+1}^b & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \overline{p}_i^f c_i(1-d_i)(1-\overline{p}_{i+1}^b) & (1-\overline{p}_{i+1}^b)+\overline{p}_i^f c_i(1-d_i)\overline{p}_{i+1}^b \end{bmatrix} \tag{A.40}$$

and ΔA is a diagonal matrix with diagonal elements all of the order $O(\delta)$, and therefore $\|\Delta A\| \sim O(\delta)$. It follows from (A.2) of Lemma A.1, the equilibrium distribution of parts $X_i(\cdot)$ of line L_i is described by $X_i = AX_i$, where A is given in (A.40). Since A is the state transition matrix of an ergodic Markov chain, $\lambda = 1$ is an eigenvalue of multiplicity 1 of A. Therefore, using the perturbation theory, we obtain

$$\left|\overline{X}_i(j) - X_i(j)\right| \sim O(\delta), \quad 1 \le i \le M-1,\ 0 \le j \le N_i. \tag{A.41}$$

Our next goal is to show how p_i^f and p_i^b, the steady state values determined via recursive procedure (4.7), can be used to determine p_i^f and p_i^b. Before we do so, we will need a few preliminary results.

As it follows from Lemma 4.3, the steady state equation of recursive procedure (4.7), that is,

$$\begin{aligned} p_i^f &= p_i\left[1-F\left(p_{i-1}^f c_{i-1}(1-d_{i-1}), p_i^b, N_{i-1}\right)\right], \quad 2 \le i \le M, \\ p_i^b &= p_i c_i\left[1-(1-d_i)F\left(p_{i+1}^b, p_i^f c_i(1-d_i), N_i\right)\right], \quad 1 \le i \le M-1, \\ p_1^f &= p_1, \qquad p_M^b = p_M, \end{aligned} \tag{A.42}$$

has at least one solution $P_{\text{agg}} = [p_1^f, \ldots, p_M^f, p_1^b, \ldots, p_M^b]^T$. We prove below that this solution is in fact unique. To accomplish this we introduce $(M-1)$ two-machine one-buffer serial production lines, L_i', $i = 1, \ldots, M-1$, where the first machine has the isolation production rate p_i^f, the second p_{i+1}^b, and the buffer capacity is N_i. The following properties hold. □

Lemma A.4. *Let* PR_i *be the production rate of line* L_i', $i = 1, \ldots, M-1$. *Then* $\text{PR}_i = p_{i+1}^f p_{i+1}^b / p_{i+1}$, $i = 1, \ldots, M-1$. *Moreover,* $\text{PR}_i > \text{PR}_{i+1}$, $i = 1, \ldots, M-1$.

Proof. From Lemma A.1 and (A.42), for $1 \le i \le M-1$,

$$\begin{aligned} \text{PR}_i &= p_{i+1}^b\left[1-F\left(p_i^f c_i(1-d_i), p_{i+1}^b, N_i\right)\right] \\ &= \left(\frac{p_{i+1}^b}{p_{i+1}}\right) p_{i+1}\left[1-F\left(p_i^f c_i(1-d_i), p_{i+1}^b, N_i\right)\right] \\ &= \frac{p_{i+1}^b p_{i+1}^f}{p_{i+1}}, \quad i = 1, \ldots, M-1, \end{aligned} \tag{A.43}$$

$$\text{PR}_{M-1} = \frac{p_M^b p_M^f}{p_M} = p_M^f = \text{PR}. \tag{A.44}$$

This proves the first statement of the lemma. Moreover,

$$\begin{aligned}\mathrm{PR}_i &= p_i^f c_i(1-d_i)\left[1-F\left(p_{i+1}^b, p_i^f c_i(1-d_i), N_i\right)\right] \\ &= \frac{p_i^f}{p_i^b - p_i^f c_i d_i} = \frac{p_i^f p_i^b}{p_i} - p_i^f c_i d_i \\ &= \mathrm{PR}_{i-1} - p_i^f c_i d_i, \quad i = 2,\ldots,M.\end{aligned} \tag{A.45}$$

Therefore, $\mathrm{PR}_i \geq \mathrm{PR}_{i+1}$, $i = 1,\ldots,M-1$. The equality holds if and only if $d_i = 0$, $i = 1,\ldots,M-1$. □

Lemma A.5. *The equilibrium equation (A.42) of recursive procedure (4.7) has a unique solution.*

Proof. By contradiction, assume that, along with the solution $P_{\mathrm{agg}} = [p_1^f,\ldots,p_M^f,p_1^b,\ldots,p_M^b]^T$ to (A.42), there exists another solution denoted by $\widehat{P}_{\mathrm{agg}} = [\widehat{p}_1^f,\ldots,\widehat{p}_M^f,\widehat{p}_1^b,\ldots,\widehat{p}_M^b]^T$. Suppose that $\widehat{p}_M^f > p_M^f$. Then, by Lemma A.4, we have $\mathrm{PR}(\widehat{p}_{M-1}^f, p_M, N_{M-1}) > \mathrm{PR}(p_{M-1}^f, p_M, N_{M-1})$. By the monotonic property of PR in Lemma 4.1, $\widehat{p}_{M-1}^f > p_{M-1}^f$. Therefore, by properties of function $F(x,y,N)$ and (A.42),

$$\begin{aligned}\widehat{p}_{M-1}^b &= p_{M-1}c_{M-1}\left[1-(1-d_{M-1})F\left(p_M, \widehat{p}_{M-1}^f c_{M-1}(1-d_{M-1}), N_{M-1}\right)\right] \\ &< p_{M-1}c_{M-1}\left[1-(1-d_{M-1})F\left(p_M, p_{M-1}^f c_{M-1}(1-d_{M-1}), N_{M-1}\right)\right] \\ &= p_{M-1}^b.\end{aligned} \tag{A.46}$$

Now proceed inductively. Assume $\widehat{p}_j^f > p_j^f$, $\widehat{p}_j^b > p_j^b$, and $\mathrm{PR}(\widehat{p}_j^f, p_{j+1}, N_{j1}) > \mathrm{PR}(p_j^f, p_{j+1}, N_j)$. The case $(j-1 = M)$ has already been established. Since

$$\begin{aligned}\widehat{p}_j^f &= p_j\left[1-F\left(\widehat{p}_{j-1}^f c_{j-1}(1-d_{j-1}), \widehat{p}_j^b, N_{j-1}\right)\right] \\ &> p_j\left[1-F\left(p_{j-1}^f c_{j-1}(1-d_{j-1}), \widehat{p}_j^b, N_{j-1}\right)\right] = p_j^f.\end{aligned} \tag{A.47}$$

Thus, $F(\widehat{p}_{j-1}^f c_{j-1}(1-d_{j-1}), \widehat{p}_j^b, N_{j-1}) < F(p_{j-1}^f c_{j-1}(1-d_{j-1}), \widehat{p}_j^b, N_{j-1})$. Then by the properties of function $F(x,y,N)$, $\widehat{p}_{j-1}^f < p_{j-1}^f$. Therefore, by (A.42),

$$\begin{aligned}\widehat{p}_{j-1}^b &= p_{j-1}c_{j-1}\left[1-(1-d_{j-1})F\left(\widehat{p}_j^b, \widehat{p}_{j-1}^f c_{j-1}(1-d_{j-1}), N_{j-1}\right)\right] \\ &< p_{j-1}c_{j-1}\left[1-(1-d_{j-1})F\left(p_j^b, p_{j-1}^f c_{j-1}(1-d_{j-1}), N_{j-1}\right)\right] = p_{j-1}^b.\end{aligned} \tag{A.48}$$

Thus, the inductive hypothesis is established, and therefore $\widehat{p}_j^f > p_j^f$ and $\widehat{p}_j^b > p_j^b$, $1 \leq j \leq M-1$. In particular, $\widehat{p}_1^f > p_1^f$ which contradicts with (A.42) (i.e., $p_1^f = p_1$). We therefore conclude that $\widehat{p}_M^f \leq p_M^f$.

Assuming that $\hat{p}_M^f \le p_M^f$, proceeding analogously, yields $\hat{p}_M^f \ge p_M^f$. Therefore, $\hat{p}_M^f = p_M^f$.

The equality of the remaining components of $\hat{P}_{\text{agg}} = P_{\text{agg}}$ will be shown by induction. Note that $\hat{p}_M = p_M = p_M^b$, and that $\hat{p}_M^f = p_M^f$. Assume that $p_i^b = \hat{p}_i^b$ and $p_i^f = \hat{p}_i^f$. Let $f(x) = p_j\lfloor 1 - F(x, p_j^b, N_{j-1})\rfloor - p_j^f$. By properties of function $F(x,y,N)$ which is a monotonic function of x, $f(x)$ is also a monotonic function of x. Therefore, $f(x)$ can have at most one root. By the inductive hypothesis, $p_i^f = \hat{p}_i^f$ and $p_i^b = \hat{p}_i^b$, therefore both $\hat{p}_{j-1}^f$ and p_{j-1}^f must be roots of $f(x)$, which proves $p_{j-1}^f = \hat{p}_{j-1}^f$. It may now be calculated that

$$\begin{aligned}\hat{p}_{j-1}^b &= p_{j-1}c_{j-1}\Big\lfloor 1 - (1-d_{j-1})F\Big(\hat{p}_j^b, \hat{p}_{j-1}^f c_{j-1}(1-d_{j-1}), N_{j-1}\Big)\Big\rfloor \\ &= p_{j-1}c_{j-1}\Big[-(1-d_{j-1})F\Big(p_j^b, p_{j-1}^f c_{j-1}(1-d_{j-1}), N_{j-1}\Big)\Big] = p_{j-1}^b,\end{aligned} \tag{A.49}$$

which establishes the inductive hypothesis.

Lemma A.3 showed that if the conditional probabilities $\hat{p}_i^f$ and $\hat{p}_i^b$, $i = 1,\dots,M$, are known, then it is possible to determine, approximately, the steady-state buffer occupancy probability distributions $\overline{X}_i(\cdot)$, $i = 1,\dots,M-1$. Therefore, the following lemma shows that they are given, approximately, by recursive procedure (4.7). □

Lemma A.6. *The following relationships hold:*

$$|\overline{p}_i^f - p_i^f| \sim O(\delta), \quad |\overline{p}_i^b - p_i^b| \sim O(\delta), \quad i = 1,\dots,M, \tag{A.50}$$

where p_i^f and p_i^b are given by Lemma 4.3 and δ is defined in (4.13).

Proof. Let $X_i(\cdot)$ be the equilibrium probability distribution of buffer occupancy of line L_i, $i = 1,\dots,M-1$, as described earlier, and let $\overline{X}_i(\cdot)$ be the equilibrium probability distribution of buffer occupancy for buffer i of line (i)–(vii). Let the conditional probabilities $\overline{p}_i^f$ and $\overline{p}_i^b$, $i = 1,\dots,M$, be as defined in (A.19). Then, by Lemma A.2, $\overline{p}_i^f$ can be expressed in terms of $\overline{X}_{i-1}(0)$ as

$$\overline{p}_i^f = p_i(1 - \overline{X}_i(0)) + O(\delta), \quad i = 2,\dots,M. \tag{A.51}$$

By Lemma A.3, this can be approximated with the distribution of parts on line L_i by

$$\overline{p}_i^f = p_i(1 - X_i(0)) + O(\delta), \quad i = 2,\dots,M. \tag{A.52}$$

Using Lemma 4.1, this can be rewritten as

$$\overline{p}_i^f = p_i\Big\lfloor 1 - F\Big(\overline{p}_{i-1}^f c_{i-1}(1-d_{i-1}), \overline{p}_i^b, N_{i-1}\Big)\Big\rfloor + O(\delta), \quad i = 2,\dots,M. \tag{A.53}$$

Analogously, by Lemma A.2,

$$\begin{aligned}\overline{p}_i^b &= p_i\left\{c_i - c_i(1-d_i)\sum_{j=i+1}^{M}\left[\prod_{r=i+1}^{j-1} p_r c_r(1-d_r)\right](1-p_j c_j)\overline{X}_{i,\dots,j-1}(N_i,\dots,N_{j-1})\right\} \\ &\quad + O(\delta) \\ &= p_i\left\{c_i - c_i(1-d_i)(1-p_{i+1}c_{i+1})\overline{X}_i(N_i)\right. \\ &\qquad \left. - c_i(1-d_i)\sum_{j=i+2}^{M}\left[\prod_{r=i+2}^{j-1} p_r c_r(1-d_r)\right](1-p_j c_j)\overline{X}_{i+1,\dots,j-1}(N_{i+1},\dots,N_{j-1})\right\}.\end{aligned} \tag{A.54}$$

By numerical fact, this can be approximated by

$$\begin{aligned}\overline{p}_i^b &= p_i c_i[1-(1-d_i)(1-p_{i+1}c_{i+1})\overline{X}_i(N_i)] - p_i c_i(1-d_i)\overline{X}_i(N_i) \\ &\quad \cdot \sum_{j=i+2}^{M}\left[\prod_{r=i+2}^{j-1} p_r c_r(1-d_r)\right](1-p_j c_j)\overline{X}_{i+1,\dots,j-1}(N_{i+1},\dots,N_{j-1}) + O(\delta).\end{aligned} \tag{A.55}$$

Using Lemma A.3, this may be rewritten as

$$\begin{aligned}\overline{p}_i^b &= p_i c_i[1-(1-d_i)(1-p_{i+1}c_{i+1})X_i(N_i)] - p_i c_i(1-d_i)X_i(N_i) \\ &\quad \cdot \sum_{j=i+2}^{M}\left[\prod_{r=i+2}^{j-1} p_r c_r(1-d_r)\right](1-p_j c_j)X_{i+1,\dots,j-1}(N_{i+1},\dots,N_{j-1}) + O(\delta).\end{aligned} \tag{A.56}$$

Rearranging and using Lemma A.2, we obtain

$$\begin{aligned}\overline{p}_i^b &= p_i c_i[1-(1-d_i)(1-p_{i+1}c_{i+1})X_i(N_i)] - p_i c_i(1-d_i)X_i(N_i)(\overline{p}_{i+1}^b - p_{i+1}c_{i+1}) + O(\delta) \\ &= p_i c_i\{1 - X_i(N_i)[(1-d_i)(1-p_{i+1}c_{i+1}) - (1-d_i)(\overline{p}_{i+1}^b - p_{i+1}c_{i+1})]\} + O(\delta) \\ &= p_i c_i\{1 - X_i(N_i)[(1-d_i)(1-\overline{p}_{i+1}^b)]\} + O(\delta).\end{aligned} \tag{A.57}$$

Using Lemma A.5, this may be written as

$$\overline{p}_i^b = p_i c_i\left[1-(1-d_i)F\left(\overline{p}_{i+1}^b, \overline{p}_i^f c_i(1-d_i), N_i\right)\right] + O(\delta). \tag{A.58}$$

By Lemma A.5, the equilibrium equation (A.42) has a unique solution p_i^f, p_i^b, $i = 1,\dots,M$. Equations (A.53) and (A.58) show that the conditional probabilities $\overline{p}_i^f$, $\overline{p}_i^b$, $i = 1,\dots,M$, solve (A.42) with error $O(\delta)$. Under numerical fact, $\delta \ll 1$, and we therefore conclude that

$$\left|\overline{p}_i^f - p_i^f\right| \sim O(\delta), \quad \left|\overline{p}_i^b - p_i^b\right| \sim O(\delta), \quad i = 1,\dots,M. \tag{A.59}$$

The logic of the proof of Theorem 4.4 is as follows.

First, if the distribution of the last buffer occupancy, $\overline{X}_{M-1}(\cdot)$, is known, the production rate can be estimated immediately in (3.8) $\mathrm{PR} = p_M[1-\overline{X}_{M-1}(0)]$. Second, $\overline{X}_i(\cdot)$, $i = 1,\dots,M-1$, can be calculated, under numerical fact, with accuracy $O(\delta)$ if the conditional probabilities $\overline{p}_i^f = \mathrm{Prob}\{m_i$ produces a part $\mid m_i$ is not blocked$\}$ and $\overline{p}_i^b = \mathrm{Prob}\{m_i$ produces a part $\mid m_i$ is not starved$\}$, $i = 1,\dots,M$, are known (Lemma A.3). Third, these conditional probabilities are $O(\delta)$ close to p_i^f and p_i^b, $i = 1,\dots,M$, the limits of the sequences $p_i^f(s)$ and $p_i^b(s)$, $s \in N$, generated by the recursive procedure (4.7) (Lemma A.5). Finally, these limits do exist (Lemma 4.3). □

Proof of Theorem 4.4. Using Lemma A.3, the production rate may be calculated as

$$\overline{\mathrm{PR}} = [1-X_{M-1}(0)]p_M = [1-X_{M-1}(0)]p_M + O(\delta). \tag{A.60}$$

Using Lemma 4.1, this may be expressed as

$$\overline{\mathrm{PR}} = \left[1-F\left(p_{M-1}^f c_{M-1} d_{M-1}, p_M, N_{M-1}\right)\right]p_M + O(\delta). \tag{A.61}$$

By Lemma A.6, we obtain

$$\overline{\mathrm{PR}} = \left[1-F\left(p_{M-1}^f c_{M-1} d_{M-1}, p_M, N_{M-1}\right)\right]p_M + O(\delta). \tag{A.62}$$

By Lemma A.5, we may finally conclude that

$$\overline{\mathrm{PR}} = p_M^f + O(\delta) = \mathrm{PR} + O(\delta). \tag{A.63}$$

□

Proof of Theorem 4.5. Using the results in Lemma 4.2, consider now $(M-1)$ two machines, one QC device, and one buffer line L_i, $i = 1,\dots,M-1$, where the first machine is defined by p_i^f, the second by p_{i+1}^b, and the buffer is of capacity N_i. From Lemma 4.2, the average steady-state buffer occupancy of buffer B_i, $i = 1,\dots,M-1$, can be calculated as

$$E[\overline{b}_i] = \sum_{j=0}^{N_i} j\overline{X}_i(j). \tag{A.64}$$

By Lemma A.3

$$E[\overline{b}_i] = \sum_{j=0}^{N_i} j\overline{X}_i(j) = \sum_{j=0}^{N_i} j\left(\overline{X}_i(j) + O(\delta)\right) = E[b_i] + \sum_{j=0}^{N_i} j\left(O(\delta)\right). \tag{A.65}$$

Therefore,

$$\left(\frac{2}{N_i(N_i+1)}\right)E[\overline{b}_i] - E[b_i] \mid \sim O(\delta). \tag{A.66}$$

□

Proof of Theorem 4.7. Consider two serial production lines (i)–(vii), the first of which is described by parameters p_i, $i = 1,\dots,M$, c_i, d_i, and N_i, $i = 1,\dots,M-1$, and the second by parameters $\widehat{p}_i \geq p_i$, $i = 1,\dots,M$, $\widehat{c}_i \geq c_i$, $\widehat{d}_i \leq d_i$, and $\widehat{N}_i \geq N_i$, $i = 1,\dots,M-1$. Let p_i^f, p_i^b, $\widehat{p}_i^f$, $\widehat{p}_i^b$ denote the steady state of the recursive procedure (4.7) for the first and second lines, respectively. We prove Theorem 4.7 by contradiction. □

Assume

$$\widehat{p}_M^f < p_M^f. \tag{A.67}$$

Then, by (A.44), $\widehat{P}R_{M-1} < \mathrm{PR}_{M-1}$. Since

$$\mathrm{PR}\left(\widehat{p}_{M-1}^f c_{M-1}(1-d_{M-1}), p_M, N_{M-1}\right) < \mathrm{PR}\left(p_{M-1}^f c_{M-1}(1-d_{M-1}), p_M, N_{M-1}\right) \tag{A.68}$$

by the monotonic property of $\overline{\mathrm{PR}}$ in Lemma 4.1, then we have

$$\widehat{p}_{M-1}^f c_{M-1}(1-d_{M-1}) < p_{M-1}^f c_{M-1}(1-d_{M-1}). \tag{A.69}$$

That implies $\widehat{p}_{M-1}^f < p_{M-1}^f$. Therefore, by properties of function $F(x,y,N)$ and (A.42),

$$\begin{aligned}\widehat{p}_{M-1}^b &= p_{M-1}c_{M-1}\left[1-(1-d_{M-1})F\left(p_M, \widehat{p}_{M-1}^f c_{M-1}(1-d_{M-1}), N_{M-1}\right)\right] \\ &> p_{M-1}c_{M-1}\left[1-(1-d_{M-1})F\left(p_M, p_{M-1}^f c_{M-1}(1-d_{M-1}), N_{M-1}\right)\right] = p_{M-1}^b.\end{aligned} \tag{A.70}$$

Now proceed inductively. Assume $\widehat{p}_j^f < p_j^f$, $\widehat{p}_j^b > p_j^b$, and $\widehat{P}R_j > \mathrm{PR}_j$. The base case ($j = M-1$) has already been established. Since

$$\begin{aligned}\widehat{p}_j^f &= p_j\left[1-F\left(\widehat{p}_{j-1}^f c_{j-1}(1-d_{j-1}), \widehat{p}_j^b, N_{j-1}\right)\right] \\ &> p_j\left[1-F\left(p_{j-1}^f c_{j-1}(1-d_{j-1}), \widehat{p}_j^b, N_{j-1}\right)\right] = p_j^f.\end{aligned} \tag{A.71}$$

Thus, $F(\widehat{p}_{j-1}^f c_{j-1}(1-d_{j-1}), \widehat{p}_j^b, N_{j-1}) > F(p_{j-1}^f c_{j-1}(1-d_{j-1}), \widehat{p}_j^b, N_{j-1})$. Then by the properties of function $F(x,y,N)$, $\widehat{p}_{j-1}^f < p_{j-1}^f$. Therefore,

$$\begin{aligned}\widehat{p}_{j-1}^b &= p_{j-1}c_{j-1}\left[1-(1-d_{j-1})F\left(\widehat{p}_j^b, \widehat{p}_{j-1}^f c_{j-1}(1-d_{j-1}), N_{j-1}\right)\right] \\ &> p_{j-1}c_{j-1}\left[1-(1-d_{j-1})F\left(p_j^b, p_{j-1}^f c_{j-1}(1-d_{j-1}), N_{j-1}\right)\right] = p_{j-1}^b.\end{aligned} \tag{A.72}$$

Then by the monotonic property of $\overline{\mathrm{PR}}$ in Lemma 4.1,

$$\mathrm{PR}\left(\widehat{p}_{j-1}^f c_{j-1}(1-d_{j-1}), p_j, N_{j-1}\right) < \mathrm{PR}\left(p_{j-1}^f c_{j-1}(1-d_{j-1}), p_j, N_{j-1}\right). \tag{A.73}$$

The inductive hypothesis is therefore established, and $\hat{p}_i^b > p_i^b$, $\hat{p}_i^f < p_i^f$, $i = 1,\dots,M-1$. In particular, $\hat{p}_1^f < p_1^f = p_1$, which contradicts (A.67). Therefore, $\hat{p}_M^f \geq p_M^f$ and estimate production rate in PR, (4.7), is monotonically increasing with respect to p_i, $i = 1,\dots,M$, c_i, and N_i, $i = 1,\dots,M-1$, and decreasing with respect to d_i, $i = 1,\dots,M-1$.

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