Research Article
A Self-Adaptive Method for Solving a System of Nonlinear Variational Inequalities

Chaofeng Shi

Recommended by A. Isidori

The system of nonlinear variational inequalities (SNVI) is a useful generalization of variational inequalities. Verma (2001) suggested and analyzed an iterative method for solving SNVI. In this paper, we present a new self-adaptive method, whose computation cost is less than that of Verma's method. The convergence of the new method is proved under the same assumptions as Verma's method. Some preliminary computational results are given to illustrate the efficiency of the proposed method.

1. Introduction

A large number of problems arising in various branches of pure and applied sciences can be studied in the unified framework of variational inequalities. In recent years, classical variational inequality and complementarity problems have been extended and generalized to study a wide range of problems arising in mechanics, physics, optimization, and applied sciences, see [1–13] and the references therein. A useful and important generalization of variational inequalities is the system of nonlinear variational inequalities.

Many authors suggested and analyzed various iterative methods for solving different types of variational inequalities in Hilbert spaces or Banach spaces based on the auxiliary principle and resolvent operators technique. Usually, they discuss the convergence for those methods, but most of them have not discussed the efficiency for the proposed method, especially through the numerical result.

On the other hand, much attention has been given to develop efficient and implemental projection method and its variant forms for solving variational inequalities and related optimization problems in $\mathbb{R}^n$, see, for example, Glowinski et al. [2], Harker and Pang [3], He [4], He and Liao [5], Shi et al. [8], Y. J. Wang et al. [12, 13], and the references therein.

However, this type of numerical methods can be only applied for solving the classical variational inequality.

Verma [11] investigated the approximation solvability of a new system of nonlinear variational inequalities involving strongly monotone mappings. In 2005, Bnouhachem [1] presented a new self-adaptive method for solving general mixed variational inequalities. In this paper, inspired and motivated by the results of Verma [11] and Bnouhachem [1], the author proposed a new self-adaptive iterative method for solving SNVI. The author also proved the convergence of the proposed method under the same assumptions as those by Verma [11]. The numerical examples are given to illustrate the efficiency of the proposed method.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $T : K \to H$ be any mapping and K a closed convex subsets of H. The author considered a system of nonlinear variational inequalities (SNVI): determine elements $x^*, y^* \in K$ such that

$$\begin{aligned}
\langle \rho T(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K, \text{ for } \rho > 0, \\
\langle \gamma T(x^*) + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in K, \text{ for } \gamma > 0.
\end{aligned} \tag{2.1}$$

The SNVI (2.1) is first introduced and studied by Verma [11] in 2001. For the applications, formulation, and numerical methods of SNVI (2.1), we refer the reader to Verma [11].

For $y^* = x^*$ and $\rho = \gamma = 1$, the SNVI (2.1) reduces to the following standard nonlinear variational inequality (NVI) problem: find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K. \tag{2.2}$$

Let K be a closed convex cone of H. The SNVI (2.1) is equivalent to a system of nonlinear complementarities (SNC): find the elements $x^*, y^* \in K$ such that $T(x^*), T(y^*) \in K^*$,

$$\begin{aligned}
\langle \rho T(y^*) + x^* - y^*, x^* \rangle = 0, \quad \text{for } \rho > 0, \\
\langle \gamma T(x^*) + y^* - x^*, y^* \rangle = 0, \quad \text{for } \gamma > 0,
\end{aligned} \tag{2.3}$$

where K^* is a polar cone to K defined by

$$K^* = \{ f \in H : \langle f, x \rangle \geq 0, \ \forall x \in K \}. \tag{2.4}$$

For $y^* = x^*$ and $\rho = \gamma = 1$, the SNC (2.3) reduces to the nonlinear complementarity problem: find an element $x^* \in K$ such that $T(x^*) \in K^*$ and

$$\langle T(x^*), x^* \rangle = 0. \tag{2.5}$$

The projection of a point $x \in H$ onto the closed convex set K, denoted by $P_K[x]$, is defined as the unique solution of the problem

$$\min_{y \in K} \|x - y\|. \tag{2.6}$$

For any closed convex set $K \subset H$, a basic property of the projection operator $P_K[\cdot]$ is

$$\langle x - P_K[x], y - P_K[x] \rangle \leq 0, \quad \forall x \in H, \ y \in K. \tag{2.7}$$

From the above inequality and the Cauchy-Schwartz inequality, it follows that the projection operator $P_K[\cdot]$ is nonexpansive, that is,

$$\|P_K[x] - P_K[y]\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n. \tag{2.8}$$

LEMMA 2.1 [11]. *Elements $x^*, y^* \in K$ form a solution set of the SNVI (2.1) if and only if*

$$x^* = P_K[y^* - \rho T(y^*)], \quad \text{for } \rho > 0, \tag{2.9}$$

where $y^* = P_K[x^* - \gamma T(x^*)]$, for $\gamma > 0$.

In [11], Verma used Lemma 2.1 to suggest and analyze the following algorithm for solving SNVI.

ALGORITHM 2.2. *For an arbitrarily chosen initial point $x \in K$, compute sequences $\{x^k\}$ and $\{y^k\}$ by an iterative procedure (for $k \geq 0$)*

$$\langle \rho T(y^k) + x^{k+1} - y^k, x - x^{k+1} \rangle \geq 0, \quad \forall x \in K, \text{ for } \rho > 0, \tag{2.10}$$

where

$$\langle \gamma T(x^k) + y^k - x^k, x - y^k \rangle \geq 0, \quad \forall x \in K, \text{ for } \gamma > 0. \tag{2.11}$$

For $\gamma = 0$ and $y^k = x^k$, Algorithm 2.2 reduces to the following algorithm.

ALGORITHM 2.3. *Compute a sequence $\{x^k\}$ by the following iteration for an initial point $x^0 \in K$:*

$$\langle \rho T(x^k) + x^{k+1} - x^k, x - x^{k+1} \rangle \geq 0, \tag{2.12}$$

for all $x \in K$ and for $\rho > 0$.

3. Convergence of projection methods

Unlike Algorithms 2.2-2.3, we construct the following algorithm.

ALGORITHM 3.1
Step 1. Given $\varepsilon > 0$, $\gamma \in [1,2)$, $\mu \in (0,1)$, $\rho > 0$, $\delta \in (0,1)$, $\delta_0 \in (0,1)$, and $u^\circ \in H$, set $k = 0$.

Step 2. Set $\rho_k = \rho$, if $\|r(x^k,\rho)\| < \varepsilon$, then stop; otherwise, find the smallest nonnegative integer m_k, such that $\rho_k = \rho\mu^{m_k}$ satisfying

$$\|\rho_k(T(x^k) - T(w^k))\| \leq \delta\|r(x^k,\rho_k)\|,\tag{3.1}$$

where $w^k = P_K[x^k - \rho_k T(x^k)]$.

Step 3. Compute

$$d(x^k,\rho_k) := r(x^k,\rho_k) - \rho_k T(x^k) + \rho_k T(P_K[x^k - \rho_k T(x^k)]),\tag{3.2}$$

where $r(x,\rho) := x - P_K[x - \rho Tx]$.

Step 4. Get the next iterate:

$$\begin{aligned} y^k &= P_K[x^k - \gamma d(x^k,\rho_k) - \gamma Tx^k]; \\ x^{k+1} &= P_K[y^k - \rho Ty^k]. \end{aligned}\tag{3.3}$$

Step 5. If $\|\rho_k(T(x^k) - T(w^k))\| \leq \delta_0\|r(x^k,\rho_k)\|$, then set $\rho = \rho_k/\mu$, else set $\rho = \rho_k$. Set $k := k+1$, and go to Step 2.

Remark 3.2. Note that Algorithm 3.1 is obviously a modification of the standard procedure. In Algorithm 3.1, the searching direction is taken as $-\gamma d(x^k,\rho_k) - \gamma Tx^k$, which is closely related to the projection residue, and differs from the standard procedure. In addition, the self-adaptive strategy of step-size choice is used. The numerical results show that these modifications can introduce computational efficiency substantially.

THEOREM 3.1. *Let H be a real Hilbert space and $T : K \rightarrow H$ an r-strongly monotone and s-Lipschitz continuous mapping from a nonempty closed convex subset K of H into H. Let $x^*, y^* \in K$ form a solution set for the SNVI (2.1) and let the sequences $\{x^k\}$ and $\{y^k\}$ be generated by Algorithm 3.1. If $0 < \theta = \sqrt{1 - 2\rho r + \rho^2 s^2}(1 + \gamma s) < 1$, then the sequence $\{x^k\}$ converges to x^* and the sequence $\{y^k\}$ converges to y^*, for $0 < \rho < 2r/s^2$.*

Proof. Since (x^*, y^*) is a solution of SNVI (2.1), it follows from Lemma 2.1 that

$$x^* = P_K[y^* - \rho T(y^*)], \qquad y^* = P_K[x^* - \gamma T(x^*)].\tag{3.4}$$

Applying Algorithm 3.1, we know

$$\|x^{k+1} - x^*\| = \|P_K[y^k - \rho Ty^k] - P_K[y^* - \rho Ty^*]\| \leq \|y^k - y^* - \rho Ty^k + \rho Ty^*\|.\tag{3.5}$$

Since T is r-strongly monotone and s-Lipschitz continuous, we know

$$
\begin{aligned}
||y^k - y^* &- \rho(T(y^k) - T(y^*))||^2 \\
&= ||y^k - y^*||^2 - 2\rho\langle T(y^k) - T(y^*), y^k - y^*\rangle + \rho^2||T(y^k) - T(y^*)||^2 \\
&\leq ||y^k - y^*||^2 - 2\rho r||y^k - y^*||^2 + \rho^2||T(y^k) - T(y^*)||^2 \\
&\leq ||y^k - y^*||^2 - 2\rho r||y^k - y^*||^2 + \rho^2 s^2||y^k - y^2||^2 \\
&= (1 - 2\rho r + \rho^2 s^2)||y^k - y^*||^2.
\end{aligned}
\tag{3.6}
$$

It follows that

$$
||x^{k+1} - x^*|| \leq \sqrt{1 - 2\rho r + \rho^2 s^2}||y^k - y^*||.
\tag{3.7}
$$

Next, we consider

$$
\begin{aligned}
||y^k - y^*|| &= ||P_K[x^k - \gamma d(x^k, \rho_k) - \gamma T(x^k)] - P_K[x^* - \gamma T(x^*)]|| \\
&\leq ||x^k - \gamma d(x^k, \rho_k) - \gamma T(x^k) - x^* + \gamma T(x^*)|| \\
&\leq ||x^k - x^* - \gamma d(x^k, \rho_k)|| + \gamma||T(x^k) - T(x^*)||,
\end{aligned}
\tag{3.8}
$$

where we use the property of the operator P_K.

Now, we consider

$$
\begin{aligned}
||x^k - x^* &- \gamma d(x^k, \rho_k)||^2 \\
&\leq ||x^k - x^*||^2 - 2\gamma\langle x^k - x^*, d(x^k, \rho_k)\rangle + \gamma^2||d(x^k, \rho_k)||^2 \leq ||x^k - x^*||^2,
\end{aligned}
\tag{3.9}
$$

where we use the definition of $d(x^k, \rho_k)$.

It follows that

$$
||y^k - y^*|| \leq (1 + \gamma s)||x^k - x^*||.
\tag{3.10}
$$

From (3.7) and (3.10), we know

$$
||x^{k+1} - x^*|| \leq \sqrt{1 - 2\rho r + \rho^2 s^2}\,(1 + \gamma s)||x^k - x^*||.
\tag{3.11}
$$

Since $0 < \theta < 1$, from (3.11), we know $x^k \to x^*$. Thus from (3.10), we know $y^k \to y^*$. □

4. Preliminary computational results

In this section, we presented some numerical results for the proposed method.

Table 4.1

n	Algorithm 3.1		Algorithm 2.2	
	Iteration no.	CPU(s)	Iteration no.	CUP(s)
10	656	0.2820	24546	6.45
50	656	0.312	21397	6.56
100	656	0.359	21397	8.09
200	656	0.61	21397	13.20

Table 4.2

n	Algorithm 3.1		Algorithm 2.2	
	Iteration no.	CPU(s)	Iteration no.	CUP(s)
10	155	0.047	4801	1.563
50	150	0.063	4795	2.016
100	100	0.109	4423	2.20
200	200	0.141	3899	3.047

Example 4.1. Let $T(x) = Dx + C$, where

$$D = \begin{pmatrix} 4 & -2 & & & & \\ 1 & 4 & -2 & & & \\ \cdots & \cdots & \cdots & & & \\ & & & \cdots & & \\ & & & & 4 & -2 \\ & & & & 1 & 4 \end{pmatrix}, \tag{4.1}$$

$c = (-1, -1, \ldots, -1)^T$, let $K = [l, u]$, where $l = (0, 0, \ldots, 0)^T$, $u = (1, 1, \ldots, 1)^T$.

The calculations are started with a vector $x^0 = (0, 0, \ldots, 0)^T$, and stopped whenever $\|r(x_n, \gamma)\|_\infty < 10^{-5}$. Table 4.1 gives the numerical results of Algorithms 3.1 and 2.2.

Table 4.1 show that Algorithm 3.1 is very effective for the problem tested. In addition, for our method, it seems that the computational time and the iteration numbers are not very sensitive to the problem size.

Example 4.2. Let $T(x) = N(x) + Dx + C$, where $N(x)$ and $Dx + c$ are the nonlinear and linear parts of $F(x)$. The element of $N(x)$ is $N_j(x) = a_j * \arctan x_j$, where a_j are randomly chosen in $(0, 1)$, D and c are the same as Example 4.1.

In all tests, the calculations are started with a vector $x^0 = (0, 0, \ldots, 0)^T$, and stopped whenever $\|r(x_n, \gamma)\|_\infty < 10^{-4}$. All codes are written in Matlab and run on a desk computer. The iteration numbers and the computational time from Algorithms 2.2 and 3.1 with different dimensions are given in Table 4.2.

Table 4.2 shows that Algorithm 3.1 is also very effective for the problem tested. In addition, for our method, it seems that the computational time and iteration numbers are not very sensitive to the problem size.

Acknowledgments

References

[1] A. Bnouhachem, "A self-adaptive method for solving general mixed variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 1, pp. 136–150, 2005.

[2] R. Glowinski, J.-L. Lions, and R. Trémolières, *Numerical Analysis of Variational Inequalities*, vol. 8 of *Studies in Mathematics and Its Applications*, North-Holland, Amsterdam, The Netherlands, 1981.

[3] P. T. Harker and J.-S. Pang, "Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications," *Mathematical Programming*, vol. 48, no. 2, pp. 161–220, 1990.

[4] B. S. He, "Inexact implicit methods for monotone general variational inequalities," *Mathematical Programming*, vol. 86, no. 1, pp. 199–217, 1999.

[5] B. S. He and L. Z. Liao, "Improvements of some projection methods for monotone nonlinear variational inequalities," *Journal of Optimization Theory and Applications*, vol. 112, no. 1, pp. 111–128, 2002.

[6] A. N. Iusem and B. F. Svaiter, "A variant of Korpelevich's method for variational inequalities with a new search strategy," *Optimization*, vol. 42, no. 4, pp. 309–321, 1997.

[7] J.-L. Lions and G. Stampacchia, "Variational inequalities," *Communications on Pure and Applied Mathematics*, vol. 20, pp. 493–519, 1967.

[8] C. Shi, S. Y. Liu, J. L. Lian, and B. D. Fang, "A modified prediction-correction method for a general monotone variational inequality," *Mathematica Numerica Sinica*, vol. 27, no. 2, pp. 113–120, 2005.

[9] R. U. Verma, "A class of quasivariational inequalities involving cocoercive mappings," *Advances in Nonlinear Variational Inequalities*, vol. 2, no. 2, pp. 1–12, 1999.

[10] R. U. Verma, "An extension of a class of nonlinear quasivariational inequality problems based on a projection method," *Mathematical Sciences Research Hot-Line*, vol. 3, no. 5, pp. 1–10, 1999.

[11] R. U. Verma, "Projection methods, algorithms, and a new system of nonlinear variational inequalities," *Computers & Mathematics with Applications*, vol. 41, no. 7-8, pp. 1025–1031, 2001.

[12] Y. J. Wang, N. H. Xiu, and C. Y. Wang, "Unified framework of extragradient-type methods for pseudomonotone variational inequalities," *Journal of Optimization Theory and Applications*, vol. 111, no. 3, pp. 641–656, 2001.

[13] Y. J. Wang, N. H. Xiu, and C. Y. Wang, "A new version of extragradient method for variational inequality problems," *Computers & Mathematics with Applications*, vol. 42, no. 6-7, pp. 969–979, 2001.

Chaofeng Shi: School of Mathematics, Sichuan University, Chengdou, Sichuan 610064, China; Department of Mathematics, Xianyang Normal College, Xianyang, Shaanxi 712000, China
Email address: shichf@163.com