Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2007, Article ID 86852, 10 pages
doi:10.1155/2007/86852

Research Article

Chaos Synchronization Criteria and Costs of Sinusoidally Coupled Horizontal Platform Systems

Jianping Cai, Xiaofeng Wu, and Shuhui Chen

Received 24 September 2006; Revised 12 December 2006; Accepted 11 February 2007

Recommended by José Manoel Balthazar

Some algebraic sufficient criteria for synchronizing two horizontal platform systems coupled by sinusoidal state error feedback control are derived by the Lyapunov stability theorem for linear time-varying system and Sylvester's criterion. The state variables are restricted in a subregion in order to obtain easily verified criteria. The validity of these algebraic criteria is illustrated with some numerical examples. A new concept, synchronization cost, is introduced based on a measure of the magnitude of the feedback control. The minimal synchronization cost as well as optimal coupling strength is calculated numerically. The results are meaningful in engineering application.

1. Introduction

Horizontal platform devices are widely used in offshore engineering and earthquake engineering. Mechanical model for a horizontal platform system with an accelerometer is depicted in Figure 1.1. The platform can freely rotate about the horizontal axis, which penetrates its mass center. When the platform deviates from horizon, the accelerometer will give an output signal to the torque generator, which generates a torque to inverse the rotation of the platform about rotational axis. The equation governing this system is

$$A\ddot{y} + D\dot{y} + rg\sin y - \frac{3g}{R}(B - C)\cos y \sin y = F\cos\omega t, \tag{1.1}$$

where y denotes the rotation of the platform relative to the earth, A, B, and C are respectively the inertia moment of the platform for axis 1, 2, and 3, D is the damping coefficient,

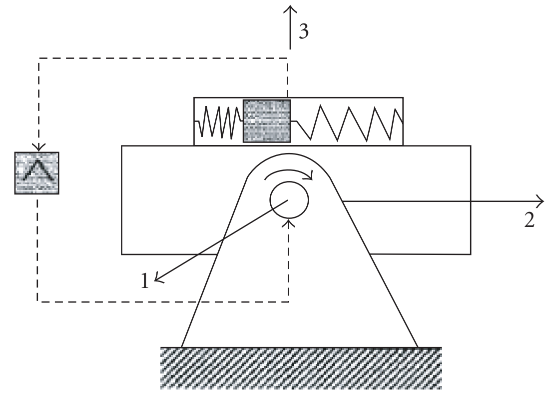

Figure 1.1. Mechanical model for a horizontal platform system with an accelerometer.

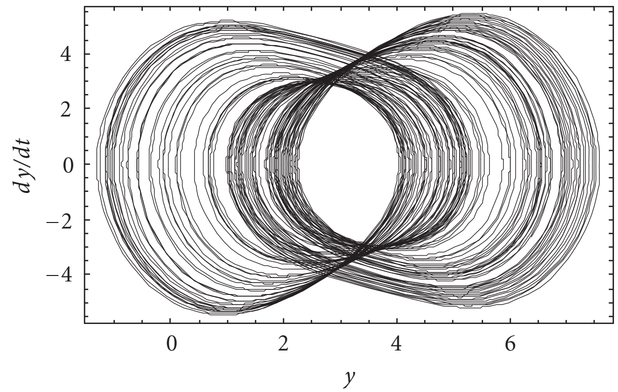

Figure 1.2. Double-scroll attractor of the horizontal platform system.

r is the proportional constant of the accelerometer, g is the acceleration constant of gravity, R is the radius of the earth, and $F\cos\omega t$ is harmonic torque. More details about this model can be found in [1, 2]. Such horizontal platform systems can reduce the swing of moving devices and keep the system close to horizontal position. They are used in modelling offshore platforms and earthquake-proof devices. As shown in Figure 1.2, the horizontal platform system has a double-scroll attractor when its parameter values are $A = 0.3$, $B = 0.5$, $C = 0.2$, $D = 0.4$, $r = 0.1155963$, $R = 6378000$, $g = 9.8$, $F = 3.4$, and $\omega = 1.8$. It was numerically verified in [1] that two identical horizontal platform systems coupled by a linear, sinusoidal, or exponential state error feedback control can achieve chaos synchronization. Analytic criteria for chaos synchronization have the advantage over numerical ones because they can reveal the relationship between the criteria and system parameters, and then they are convenient for design and analysis of the coupling controller [3–11]. Algebraic sufficient criteria for synchronizing the driving-response horizontal platform systems via linear state error feedback control were obtained in [12].

In this paper, some sufficient criteria for synchronizing the horizontal platform systems coupled by sinusoidal state error feedback control are further derived by the Lyapunov stability theory and the Sylvester's criterion. In order to obtain easily verified algebraic criteria, the state variables are restricted in a subregion, which is different from [12]. Furthermore, a new concept of synchronization cost is introduced based on a measure of the magnitude of the feedback control. The minimal synchronization cost, as well as optimal coupling strength is calculated numerically. Minimal cost means the lowest energy input, which is meaningful in engineering application.

2. Algebraic sufficient synchronization criteria

Let $x_1 = y, x_2 = \dot{y}$, and $x = (x_1, x_2)^T$, and rewrite the governing equation in form of vector

$$\dot{x} = Mx + f(x) + m(t) \tag{2.1}$$

with

$$M = \begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}, \qquad f(x) = \begin{pmatrix} 0 \\ -b\sin x_1 + c\cos x_1 \sin x_1 \end{pmatrix}, \qquad m(t) = \begin{pmatrix} 0 \\ h\cos\omega t \end{pmatrix},$$
$$a = \frac{D}{A} > 0, \qquad b = \frac{rg}{A} > 0, \qquad c = \frac{3g}{RA}(B - C), \qquad h = \frac{F}{A} > 0. \tag{2.2}$$

A driving-response synchronization scheme for two identical platform systems coupled by a sinusoidal state error feedback controller is constructed as follows:

$$\text{driving system: } \dot{x} = Mx + f(x) + m(t), \tag{2.3}$$
$$\text{response system: } \dot{y} = My + f(y) + m(t) + u(t), \tag{2.4}$$
$$\text{controller: } u(t) = \left(k_1 \sin(x_1 - y_1), k_2 \sin(x_2 - y_2)\right)^T, \tag{2.5}$$

where $y = (y_1, y_2)^T$, T means transpose, and k_1 and k_2 are constant coupling coefficients. Defining an error variable $e = x - y$, or $(e_1, e_2) = (x_1 - y_1, x_2 - y_2)$, we can obtain an error dynamical system

$$\dot{e} = M(x - y) - u(t) + f(x) - f(y) = (M - K(t) + N(t))e \tag{2.6}$$

with

$$K(t) = \begin{pmatrix} k_1 s_1(t) & 0 \\ 0 & k_2 s_2(t) \end{pmatrix}, \qquad s_1(t) = \frac{\sin(x_1 - y_1)}{x_1 - y_1}, \qquad s_2(t) = \frac{\sin(x_2 - y_2)}{x_2 - y_2},$$
$$N(t) = \begin{pmatrix} 0 & 0 \\ q(t) & 0 \end{pmatrix}, \qquad q(t) = \frac{-b(\sin x_1 - \sin y_1) + c(\sin x_1 \cos x_1 - \sin y_1 \cos y_1)}{x_1 - y_1}. \tag{2.7}$$

Our object is to select suitable coupling coefficients k_1 and k_2 such that $x(t)$ and $y(t)$ satisfy

$$\lim_{t\to+\infty} ||x(t)-y(t)|| = \lim_{t\to+\infty} ||e(t)|| = 0, \tag{2.8}$$

where $\|x(t)-y(t)\| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$ denotes the Euclidean norm of vector. By the theory of stability, chaos synchronization of systems (2.3) and (2.4) in the sense of (2.8) is equivalent to asymptotic stability of the error system (2.6) at the origin $e=0$.

Taking a quadratic Lyapunov function $V(e)=e^TPe$ with P a symmetric positive definite constant matrix, then the derivative of $V(e)$ with respect to time along the trajectory of system (2.6) is

$$\dot{V}(e) = \dot{e}^TPe + e^TP\dot{e} = e^T\left[P(M-K(t)+N(t)) + (M-K(t)+N(t))^TP\right]e. \tag{2.9}$$

By the Lyapunov stability theorem for linear time-varying system (see [13, Theorem 4.1]), a sufficient condition that the error system (2.6) is asymptotically stable at the origin is that the following matrix

$$Q(t) = P(M-K(t)+N(t)) + (M-K(t)+N(t))^TP \tag{2.10}$$

is negative definite, denoting it by

$$Q(t) < 0. \tag{2.11}$$

For simplicity, we choose $P=\operatorname{diag}\{p_1,p_2\}$ with $p_1>0$ and $p_2>0$, then

$$Q(t) = \begin{pmatrix} -2p_1k_1s_1(t) & p_1+p_2q(t) \\ p_1+p_2q(t) & -2p_2(k_2s_2(t)+a) \end{pmatrix}. \tag{2.12}$$

By the Sylvester's criterion, $Q(t)<0$ is equivalent to the following inequalities:

$$p_1k_1s_1(t) > 0, \qquad 4p_1p_2k_1s_1(t)(k_2s_2(t)+a) > (p_1+p_2q(t))^2. \tag{2.13}$$

Note that $s_1(t)>0$ and $s_2(t)>0$ if (x_1,x_2) and (y_1,y_2) are limited in the region $G=\{|x_1-y_1|<\pi, |x_2-y_2|<\pi\}$. So we conclude that under condition (2.13) the error system (2.6) is locally asymptotically stable at the origin in the region G. In order to get an easily verified algebraic condition, we further restrict the variables in the subregion $G_0=\{|x_1-y_1|\le 3\pi/4, |x_2-y_2|\le 3\pi/4\}$, then we have $2\sqrt{2}/3\pi \le s_1(t)\le 1$ and $2\sqrt{2}/3\pi \le s_2(t)\le 1$. Now, a simple algebraic sufficient criterion for synchronizing the systems (2.3) and (2.4) can be obtained from (2.13) as

$$k_1 > 0, \qquad k_2 > \frac{9\pi^2(p_1+p_2(b+|c|))^2}{32p_1p_2k_1} - a, \tag{2.14}$$

in which the inequality $|q(t)|<b+|c|$ has been used as in [12].

The synchronization criterion obtained here only renders a sufficient but not necessary condition. It is natural to expect that a sharp criterion can provide more choices of the

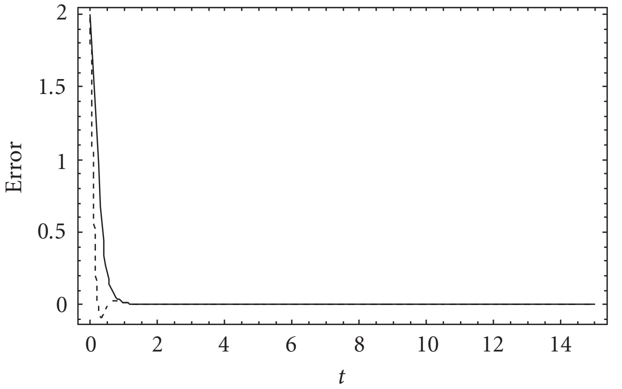

Figure 2.1. Error between the driving-response horizontal platform systems (2.3)–(2.5) with the coupling coefficients $k_1 = 5.6$ and $k_2 = 6.2$, solid curve for $x_1 - y_1$ and dashed curve for $x_2 - y_2$, initial conditions $(x_1(0), x_2(0)) = (1,1)$ and $(y_1(0), y_2(0)) = (-1,-1)$.

coupling coefficients. To this end, we can minimize the lower bound of k_2 in inequality (2.14) by choosing $p = \operatorname{diag}\{(b+|c|)p_2, p_2\}$ and obtain a sharper criterion

$$k_1 > 0, \qquad k_2 > \frac{9\pi^2(b+|c|)}{8k_1} - a. \tag{2.15}$$

Similarly, if the controller is chosen as $u(t) = (k_1 \sin(x_1 - y_1), 0)^T$, the sufficient criteria associated with inequalities (2.14) and (2.15) become, respectively,

$$k_1 > \frac{3\pi\left(p_1 + p_2(b+|c|)\right)^2}{8\sqrt{2}p_1 p_2 a}, \tag{2.16}$$

$$k_1 > \frac{3\pi(b+|c|)}{2\sqrt{2}a}. \tag{2.17}$$

The theoretical sufficient criteria are illustrated with the following examples. If we choose $p_2 = 1$ and $p_1 = (b+|c|)p_2 = 3.776615$, it is easy to verify that the coupling coefficients $k_1 = 5.6$ and $k_2 = 6.2$ satisfy inequalities (2.15). For this choice, the two coupled horizontal platform systems (2.3) and (2.4) can be asymptotically synchronized. The parameter values are chosen such that the system is in a state of chaos: $A = 0.3$, $B = 0.5$, $C = 0.2$, $D = 0.4$, $r = 0.1155963$, $R = 6378000$, $g = 9.8$, $F = 3.4$, and $\omega = 1.8$. The result is shown in Figure 2.1 with initial values $(x_1(0), x_2(0)) = (1,1)$ and $(y_1(0), y_2(0)) = (-1,-1)$, which are chosen arbitrarily in the region G_0. In this paper, software *Mathematica* is applied to implement relative calculations and plots.

For the controller $u(t) = (k_1 \sin(x_1 - y_1), 0)^T$, inequality (2.17) should be $k_1 > 9.43706$. Chaos synchronization for $k_1 = 9.5$ is illustrated in Figure 2.2, where p_1, p_2, and other parameter values are the same as above.

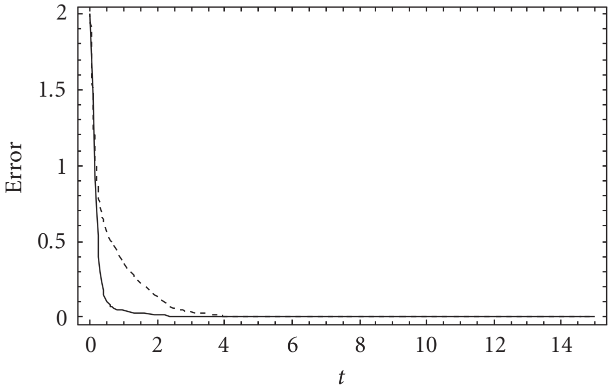

Figure 2.2. Error between the driving-response horizontal platform systems (2.3)–(2.5) with the coupling coefficients $k_1 = 9.5$ and $k_2 = 0$, solid curve for $x_1 - y_1$ and dashed curve for $x_2 - y_2$, initial conditions $(x_1(0), x_2(0)) = (1, 1)$ and $(y_1(0), y_2(0)) = (-1, -1)$.

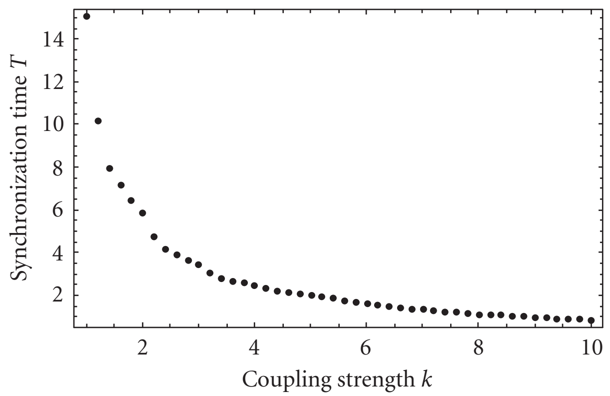

Figure 3.1. Synchronization time of systems (2.3) and (2.4) with sinusoidal controller $u(t) = (k\sin(x_1 - y_1), k\sin(x_2 - y_2))^T$, synchronization error measure $d < 0.001$, $L = 1000$, initial conditions $(x_1(0), x_2(0)) = (1,1)$ and $(y_1(0), y_2(0)) = (-1,-1)$.

3. Synchronization time and cost

Firstly, we numerically investigate the behavior of synchronization time T_{syn} as a function of coupling strength k_1 and/or k_2. The synchronization time is defined as the initial time when the error measure $d = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < \varepsilon$ is satisfied and maintains in a long enough time interval $[T_{\text{syn}}, T_{\text{syn}} + L]$, where ε is the precision of the synchronization, and L is a sufficiently large positive constant. As shown in Figures 3.1 and 3.2, the synchronization time T_{syn} gradually decreases with the increase of coupling strength, and approaches an asymptotic minimal value. This is a very interesting phenomenon,

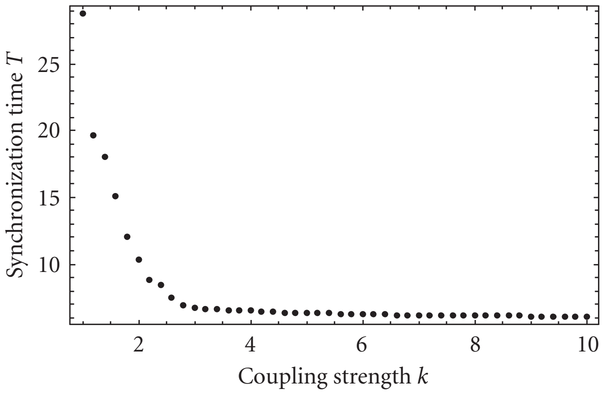

Figure 3.2. Synchronization time of systems (2.3) and (2.4) with sinusoidal controller $u(t) = (k\sin(x_1 - y_1), 0)^T$, synchronization error measure $d < 0.001$, $L = 1000$, initial conditions $(x_1(0), x_2(0)) = (1, 1)$ and $(y_1(0), y_2(0)) = (-1, -1)$.

since one might think that the synchronization could be led as fast as desired if coupling strength is large enough. Figures 3.1 and 3.2 confirm that very large values of coupling strength are not necessary to ensure the synchronization with approximately the minimum T_{syn}. Such phenomenon also occurred in synchronization scheme of single-well Duffing oscillators [14]. Generally, synchronizing two chaotic systems is not cost-free. In order to evaluate what price must be paid to achieve synchronization, a new concept of synchronization cost for scheme (2.3)–(2.5) is introduced as follows:

$$\int_0^{\infty} k_1 \,|\sin(x_1 - y_1)|\, dt + \int_0^{\infty} k_2 \,|\sin(x_2 - y_2)|\, dt. \tag{3.1}$$

The meaning of this definition refers to the cost to achieve a certain degree of synchronization in the sense of (2.8). Note that the magnitude of $|x_i - y_i|$ is very small once synchronization is nearly achieved. So a good approximation of cost should be

$$\int_0^{T_{\text{syn}}} k_1 \,|\sin(x_1 - y_1)|\, dt + \int_0^{T_{\text{syn}}} k_2 \,|\sin(x_2 - y_2)|\, dt, \tag{3.2}$$

which will be adopted in the following simulations. Another definition of synchronization cost adopted in [15] for linear control is

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} k_i \,|x_i - y_i|\, dt, \quad i = 1, 2, \tag{3.3}$$

which refers to the cost per unit time required to keep the synchronization going. The meaning is different from ours.

From the viewpoint of preventing from a useless increase of coupling strength, that is, from an unavailing waste of input energy, the calculation of minimal synchronization cost, as well as optimal coupling strength, is of great practical interest. Synchronization

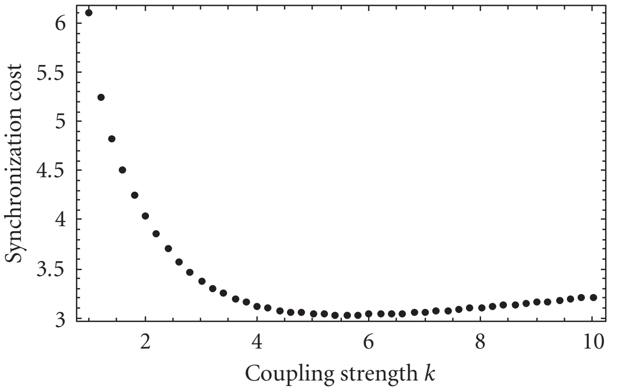

Figure 3.3. Synchronization cost of systems (2.3) and (2.4) with sinusoidal controller $u(t) = (k\sin(x_1 - y_1), k\sin(x_2 - y_2))^T$, initial conditions $(x_1(0), x_2(0)) = (1,1)$ and $(y_1(0), y_2(0)) = (-1,-1)$.

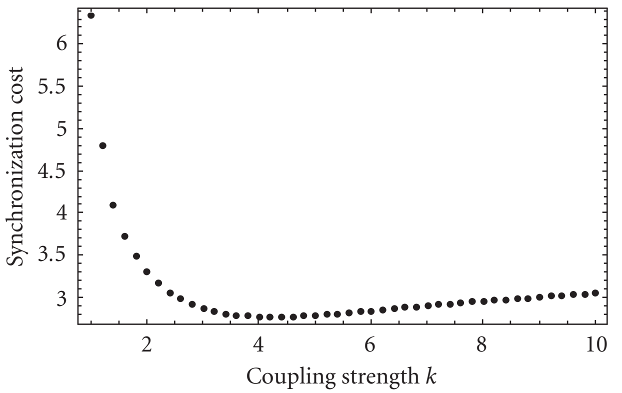

Figure 3.4. Synchronization cost of systems (2.3) and (2.4) with sinusoidal controller $u(t) = (k\sin(x_1 - y_1), 0)^T$, initial conditions $(x_1(0), x_2(0)) = (1,1)$ and $(y_1(0), y_2(0)) = (-1,-1)$.

cost versus coupling strength is simulated in Figures 3.3 and 3.4 with different controllers. From these figures we can see that the synchronization cost decreases rapidly at first, then reaches a minimal value and increases slowly with the increase of coupling strength at last. The explanation of this phenomenon is in agreement with the simulations of synchronization time shown in Figures 3.1 and 3.2. The critical coupling strength with the minimal synchronization cost can be chosen as the optimal coupled strength in the sense of consumed energy. The optimal coupling strength and minimal synchronization cost are 5.6 and 3.03922 in Figure 3.3, 4.2 and 2.77078 in Figure 3.4, respectively. Although double-variable-coupled configuration (x- and y-coupled) can lead to fast synchronization, its minimal synchronization cost is larger than that of single-variable-coupled configuration (x-coupled).

4. Conclusions

Some algebraic sufficient criteria for synchronizing driving-response horizontal platform systems coupled by sinusoidal state error feedback control are derived and their validity is illustrated with some numerical examples. Numerical simulations show that the synchronization time approaches an asymptotic minimal value with the increase of coupling strength. The concept of synchronization cost is introduced and the minimal synchronization cost as well as optimal coupling strength is calculated numerically. The minimal synchronization cost refers to the lowest-energy input, which is of great practical interest.

Acknowledgments

The authors are grateful to the anonymous referees for constructive comments. This research is supported by the National Natural Science Foundation of China under Grants no. 10672193, no. 60674049, and no. 10571184, and the Foundation of Advanced Research Center of Zhongshan University under Grant no. 06M13.

References

[1] Z.-M. Ge, T.-C. Yu, and Y.-S. Chen, "Chaos synchronization of a horizontal platform system," *Journal of Sound and Vibration*, vol. 268, no. 4, pp. 731–749, 2003.

[2] C.-L. Huang, "Nonlinear dynamics of the horizontal platform," M.S. thesis, National Chiao Tung University, Hsinchu, Taiwan, 1996.

[3] J. A. K. Suykens, P. F. Curran, and L. O. Chua, "Master-slave synchronization using dynamic output feedback," *International Journal of Bifurcation and Chaos*, vol. 7, no. 3, pp. 671–679, 1997.

[4] J. Lü, T. Zhou, and S. Zhang, "Chaos synchronization between linearly coupled chaotic systems," *Chaos, Solitons and Fractals*, vol. 14, no. 4, pp. 529–541, 2002.

[5] G.-P. Jiang, W. K.-S. Tang, and G. Chen, "A simple global synchronization criterion for coupled chaotic systems," *Chaos, Solitons and Fractals*, vol. 15, no. 5, pp. 925–935, 2003.

[6] E. M. Elabbasy, H. N. Agiza, and M. M. El-Dessoky, "Global synchronization criterion and adaptive synchronization for new chaotic system," *Chaos, Solitons and Fractals*, vol. 23, no. 4, pp. 1299–1309, 2005.

[7] J. Sun and Y. Zhang, "Some simple global synchronization criterions for coupled time-varying chaotic systems," *Chaos, Solitons and Fractals*, vol. 19, no. 1, pp. 93–98, 2004.

[8] Y. Lei, W. Xu, J. Shen, and T. Fang, "Global synchronization of two parametrically excited systems using active control," *Chaos, Solitons and Fractals*, vol. 28, no. 2, pp. 428–436, 2006.

[9] J. H. Park, "Stability criterion for synchronization of linearly coupled unified chaotic systems," *Chaos, Solitons and Fractals*, vol. 23, no. 4, pp. 1319–1325, 2005.

[10] J.-G. Wang and Y. Zhao, "Chaotic synchronization of the master slave chaotic systems with different structures based on BANG-BANG control principle," *Chinese Physics Letters*, vol. 22, no. 10, pp. 2508–2510, 2005.

[11] J.-H. Shen, S. Chen, and J. Cai, "Chaos synchronization criterion and its optimizations for a nonlinear transducer system via linear state error feedback control," *Chinese Physics Letters*, vol. 23, no. 6, pp. 1406–1409, 2006.

[12] X. Wu, J. Cai, and M. Wang, "Master-slave chaos synchronization criteria for the horizontal platform systems via linear state error feedback control," *Journal of Sound and Vibration*, vol. 295, no. 1-2, pp. 378–387, 2006.

[13] J. Slotine and W. P. Li, *Applied Nonlinear Control*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1991.

[14] Y. C. Kouomou and P. Woafo, "Stability and optimization of chaos synchronization through feedback coupling with delay," *Physics Letters, Section A*, vol. 298, no. 1, pp. 18–28, 2002.

[15] C. Sarasola, F. J. Torrealdea, A. D'Anjou, and M. Graña, "Cost of synchronizing different chaotic systems," *Mathematics and Computers in Simulation*, vol. 58, no. 4–6, pp. 309–327, 2002.

Jianping Cai: Department of Applied Mechanics and Engineering, Zhongshan University, Guangzhou 510275, China
Email address: stscip@mail.sysu.edu.cn

Xiaofeng Wu: Center for Control and Optimization, South China University of Technology, Guangzhou 510640, China
Email address: wuxiaof@21cn.com

Shuhui Chen: Department of Applied Mechanics and Engineering, Zhongshan University, Guangzhou 510275, China
Email address: stscsh@mail.sysu.edu.cn