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Research Article

Stabilization of Linear Sampled-Data Systems by a Time-Delay Feedback Control

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We consider one-dimensional, time-invariant sampled-data linear systems with constant feedback gain, an arbitrary fixed time delay, which is a multiple of the sampling period and a zero-order hold for reconstructing the sampled signal of the system in the feedback control. We obtain sufficient conditions on the coefficients of the characteristic polynomial associated with the system. We get these conditions by finding both lower and upper bounds on the coefficients. These conditions let us give both an estimation of the maximum value of the sampling period and an interval on the controller gain that guarantees the stabilization of the system.

1. Introduction

The sampled-data systems are particular cases of a general type of systems called networked control systems, that have an important value in applications (see Hespanha et al. [1], Hespanha et al. [2], Hikichi et al. [3], Meng et al. [4], Naghshtabrizi and Hespanha [5] Ögren et al. [6], Seiler and Sengupta [7] and Shirmohammadi and Woo [8]). The networked control systems can be studied either from the approach of control theory or communication theory (see Hespanha et al. [1]). Among the reported papers in control theory that have researched about networked control systems it is worth to mention the works by Zhang, Tipsuwan, and Hespanha (see Zhang et al. [9], Tipsuwan and Chow [10], and Hespanha et al. [1]).

When in networked control systems it is satisfied that the plant outputs and the control inputs are delivered at the same time, then we obtain a sampled-data system. In this paper, we focus our attention on sampled-data systems. These systems have widely been studied due to their importance in engineering applications (see Åström and Wittenmark [11], Chen and Francis [12], Franklin et al. [13], and Kolmanovskii and Myshkis [14]).

A sampled-data linear system with fixed time delay in the feedback is a continuous plant such that the feedback control of the closed loop system is discrete and has a delay r, namely,

$$\dot{x} = Ax(t) + Bu_{k-r}(t), \tag{1.1}$$

$$u_{k-r}(t) = Kx\left(\left[\frac{t}{h}\right]h - r\right), \quad h = t_{k+1} - t_k, \tag{1.2}$$

where $[\alpha]$ denotes the integer part of α, A is an $n \times n$ matrix, $B \in R^n$, $r \in R$, and h is the interval between the successive sample instants t_k and t_{k+1}. If h is a constant, it is called the sampling period and $t_k = kh$. Recommendable references about time-delay systems are the books by Hale and Verduyn Lunel [15], and Kolmanovskii and Myshkis [14]. On the other hand, the theory about n-dimensional sampled-data control systems can be studied in the books by Åström and Wittenmark [11] or Chen and Francis [12]. In relation with the study of sampled-data systems and the problem of proving the existence of a stabilizing control, it is worth to mention the work by Fridman et al. [16], which is based on solving a linear matrix inequality. The application of this approach has been very successful in subsequent works (see Fridman et al. [17], and Mirkin [18]). Another idea is to propose a control depending on a parameter ϵ and then prove that the control stabilizes the system when ϵ is small enough. This idea was developed by Yong and Arapostathis [19]. Since the existence has been proved for these last authors, now we focus on estimating an interval for ϵ. In order to reduce the difficulty of the problem, we will restrict our study to the one-dimensional sampled-data systems. These systems have attracted the attention of several researchers as they can model interesting phenomena in engineering (see, e.g., Busenberg and Cooke [20] and Cooke and Wiener [21]). We will consider the one-dimensional case of (1.1), that is, we will study the differential equation

$$\dot{x} = ax(t) + bu_{k-r}(t), \tag{1.3}$$

where a and b are given constants. Our problem is to find the values of the (gain) parameter K and of the period h so that the discrete control (zero-order hold) with delay r

$$u_{k-r} = Kx\left(\left[\frac{t}{h}\right]h - r\right) \tag{1.4}$$

makes the system (1.3) an asymptotically stable one. The time delay is considered an integer multiple of the sampling period h in the sense that $r = Nh$, where N is a natural number.

For $t \in [kh, (k+1)h)$, $k \in Z^+$, the function $x([t/h]h - Nh)$ is constant and the solution of the differential equation (1.3) is

$$x(t) = e^{a(t-kh)}x(kh) + \int_0^{t-kh} e^{a\tau}\, d\tau bKx\big((k-N)h\big). \tag{1.5}$$

Therefore by continuity

$$x\big((k+1)h\big) = e^{ah}x(kh) + \int_0^h e^{a\tau}bKx\big((k-N)h\big)d\tau. \tag{1.6}$$

We now define

$$A_d = e^{ah}, \qquad B_d = b\int_0^h e^{a\tau}\,d\tau, \qquad \varepsilon(k) = x(kh). \tag{1.7}$$

From (1.6) we obtain the following difference equation:

$$\varepsilon(k+1) = A_d\varepsilon(k) + B_d K\varepsilon(k-N). \tag{1.8}$$

By making the change of variable $J = k - N$, the difference equation (1.8) becomes a homogeneous difference equation of order $N+1$, namely,

$$\varepsilon(J+N+1) - A_d\varepsilon(J+N) - B_d K\varepsilon(J) = 0. \tag{1.9}$$

This homogeneous difference equation of order $N+1$ can be rewritten as the following system of $N+1$ difference equations of order one. Indeed let

$$\begin{aligned} \varepsilon(J) &= x_1(J) \\ \varepsilon(J+1) &= x_1(J+1) = x_2(J) \\ &\vdots \\ \varepsilon(J+N) &= x_N(J+1) = x_{N+1}(J) \\ \varepsilon(J+N+1) &= x_{N+1}(J+1). \end{aligned} \tag{1.10}$$

Using (1.9), we obtain the following system of difference equations:

$$\begin{aligned} x_1(J+1) &= x_2(J) \\ x_2(J+1) &= x_3(J) \\ &\vdots \\ x_N(J+1) &= x_{N+1}(J) \\ x_{N+1}(J+1) &= A_d x_{N+1}(J) + B_d K x_1(J), \end{aligned} \tag{1.11}$$

which in matrix form becomes

$$X(J+1) = AX(J), \tag{1.12}$$

where

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ B_d K & 0 & \cdots & 0 & A_d \end{bmatrix}, \qquad X(J) = \begin{bmatrix} x_1(J) \\ x_2(J) \\ x_3(J) \\ \vdots \\ x_N(J) \\ x_{N+1}(J) \end{bmatrix}. \tag{1.13}$$

To give stability conditions of the system of difference equations, we first obtain the characteristic polynomial of the matrix A:

$$P(\lambda) = \lambda^{N+1} - A_d\lambda^N - B_d K. \tag{1.14}$$

Thus the problem of stabilizing system (1.3) is equivalent to giving conditions on the coefficients of the characteristic polynomial (1.14) so that this polynomial is Schur stable. The problem of characterizing the stability region of (1.12) [or equivalently (1.14)] is considered an interesting problem [1] although it is known that it is very difficult [9]. Our objective in this paper is to find information about the stability region, which is explained below.

System (1.1) has been studied, and necessary and sufficient conditions on $[A, B]$ for the r-stabilization of the system have been obtained (see Yong and Arapostathis [19]), but they are not easily verifiable. For the one-dimensional case (1.3), their result is the following. Suppose $-(N+1)/N < -A_d < -1$. Then the polynomial (1.14) is Schur stable if $-B_d K = -(-A_d) - 1 + \epsilon$ for a sufficiently small ϵ. However in a design problem we need to say how to find such an ϵ, or to obtain an estimation of the maximum sampling interval for which the stability is guaranteed, that is very important (see Hespanha et al. [1]).

In this paper, we find a $\tilde{\tilde{\epsilon}}_0$ such that the polynomial

$$P(\lambda) = \lambda^{N+1} - A_d\lambda^N - \left(-A_d\right) - 1 + \epsilon \tag{1.15}$$

is Schur stable if $0 < \epsilon < \tilde{\tilde{\epsilon}}_0$. That is, we get an estimation of the largest $\epsilon_{\max}$ with the property that the polynomial (1.15) is Schur stable for $0 < \epsilon < \epsilon_{\max}$.

Some general results about the stability for retarded differential equations with piecewise constant delays were obtained by Cooke and Wiener [21]. Problems (1.1) and (1.3) for continuous-time systems were studied by Yong [22, 23] with an analogous approach.

2. Main result

Consider a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + a_0$ such that $-n/(n-1) < a_{n-1}/a_n < -1$. Our objective is to give values of the coefficient a_0 such that $P(z)$ is Schur stable. The result is the following. Choose $a_0 = -a_{n-1} + a_n(\epsilon - 1)$, then $P(z)$ is Schur stable if ϵ satisfies the inequality $0 < \epsilon < 3n/(2n-1) + (3(n-1)/(2n-1))(a_{n-1}/a_n)$.

We begin by establishing the result when the degree of $P(z)$ is two (in fact, we have here necessary and sufficient conditions).

Theorem 2.1. *Let $P(z) = a_2 z^2 + a_1 z + a_0$ a polynomial such that $-2 < a_1/a_2 < -1$, where $a_0 = -a_1 + a_2(\epsilon - 1)$. Then $P(z)$ is Schur stable if and only if*

$$0 < \epsilon < 2 + \frac{a_1}{a_2}. \tag{2.1}$$

Proof. $P(z)$ is Schur stable if and only if its coefficients satisfy [24] the following:

$$\begin{aligned} \left|a_2\right| &> \left|a_2(\epsilon - 1) - a_1\right|, \\ \left|a_1\right| &< \left|a_2 + a_2(\epsilon - 1) - a_1\right|; \end{aligned} \tag{2.2}$$

or equivalently

$$\begin{aligned} 0 &> a_2^2\epsilon^2 - \left(2a_2^2 + 2a_2a_1\right)\epsilon + 2a_2a_1 + a_1^2, \\ 0 &< \epsilon\left(a_2^2\epsilon - 2a_2a_1\right). \end{aligned} \tag{2.3}$$

To prove this last part, we define

$$g(\epsilon) = a_2^2\epsilon^2 - (2a_2^2 + 2a_2a_1)\epsilon + 2a_2a_1 + a_1^2. \tag{2.4}$$

Then

$$g(\epsilon) = 0 \Longleftrightarrow \left(\epsilon = 2 + \frac{a_1}{a_2} \text{ or } \epsilon = \frac{a_1}{a_2}\right). \tag{2.5}$$

Since the coefficient of ϵ^2 is positive, $g(\epsilon) < 0$ if and only if

$$\frac{a_1}{a_2} < \epsilon < 2 + \frac{a_1}{a_2}. \tag{2.6}$$

We have, it holds that.

On the other hand, $\epsilon(a_2^2\epsilon - 2a_2a_1) > 0$ if and only if ($\epsilon > 0$ and $\epsilon > 2(a_1/a_2)$). Now since $a_1/a_2 < -1$, it holds that $2(a_1/a_2) < -2$. Therefore, $\epsilon(a_2^2\epsilon - 2a_2a_1) > 0$ if and only if $\epsilon > 0$, so that $[g(\epsilon) < 0$ and $\epsilon(a_2^2\epsilon - 2a_2a_1) > 0]$ if and only if $0 < \epsilon < 2 + a_1/a_2$. □

The arbitrary degree proof depends on the following lemma and several technical propositions that can be checked in the appendix.

Lemma 2.2. *Fix an arbitrary integer* $n > 2$. *Given* $P(z) = a_{n+1}z^{n+1} + a_nz^n + a_0$ *with* $-(n+1)/n < a_n/a_{n+1} < -1$ *and* $a_0 = -a_n + a_{n+1}(\epsilon - 1)$, *define* $Q(z) = a_0z^{n+1} + a_nz + a_{n+1}$ *and*

$$R(z) = \frac{1}{z}\left[P(z) - \frac{a_0}{a_{n+1}}Q(z)\right] = \frac{1}{a_{n+1}}\left[A_nz^n + A_{n-1}z^{n-1} + A_0\right], \tag{2.7}$$

where $A_n = a_{n+1}^2 - a_0^2$, $A_{n-1} = a_{n+1}a_n$ *and* $A_0 = -a_0a_n$. *If* ϵ *satisfies* $0 < \epsilon < 3(n+1)/(2n+1) + (3n/(2n+1))(a_n/a_{n+1})$, *then* $(|a_{n+1}| > |a_0|$ *and* $|A_n| > |A_0|)$.

Proof. We have that $|a_{n+1}| > |a_0|$ if and only if (Proposition A.1)

$$0 < \epsilon < 2 + \frac{a_n}{a_{n+1}}. \tag{2.8}$$

Hence to prove the lemma, it is sufficient to show that

$$\frac{3(n+1)}{2n+1} + \frac{3n}{2n+1}\frac{a_n}{a_{n+1}} < 2 + \frac{a_n}{a_{n+1}}. \tag{2.9}$$

A straightforward calculation shows that inequality (2.9) holds if and only if $((n-1)/(2n+1))(a_n/a_{n+1}) < (n-1)/(2n+1)$, which is true because $a_n/a_{n+1} < -1$.

We now show that $|A_n| > |A_0|$. It can be seen that $A_n = a_{n+1}^2 - a_0^2$ and $A_0 = -a_0a_n$, from where

$$\begin{aligned}
|A_n| > |A_0| &\Longleftrightarrow |a_{n+1}^2 - a_0^2| > |a_0a_n| \\
&\Longleftrightarrow (a_{n+1}^2 - a_0^2)^2 > (a_0a_n)^2 \\
&\Longleftrightarrow \left[(a_{n+1}^2 - a_0^2) - a_0a_n\right]\left[a_{n+1}^2 - a_0^2 + a_0a_n\right] > 0 \\
&\Longleftrightarrow \left\{\left[(a_{n+1}^2 - a_0^2) - a_0a_n > 0,\ a_{n+1}^2 - a_0^2 + a_0a_n > 0\right]\right. \\
&\qquad \left.\text{or } \left[(a_{n+1}^2 - a_0^2) - a_0a_n < 0,\ a_{n+1}^2 - a_0^2 + a_0a_n < 0\right]\right\}.
\end{aligned} \tag{2.10}$$

We will split the analysis into the following two cases:

$$\left[\left(a_{n+1}^2 - a_0^2\right) - a_0 a_n > 0,\ a_{n+1}^2 - a_0^2 + a_0 a_n > 0\right] \tag{2.11}$$

or

$$\left[\left(a_{n+1}^2 - a_0^2\right) - a_0 a_n < 0,\ a_{n+1}^2 - a_0^2 + a_0 a_n < 0\right]. \tag{2.12}$$

We analyze (2.11). By Proposition A.2, the first inequality in (2.11) is satisfied if and only if

$$1 + \frac{1}{2}\frac{a_n}{a_{n+1}} - \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} < \epsilon < 1 + \frac{1}{2}\frac{a_n}{a_{n+1}} + \sqrt[2]{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{2.13}$$

Since $1 + (1/2)(a_n/a_{n+1}) > 0$, it follows that

$$1 + \frac{1}{2}\frac{a_n}{a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} > 0. \tag{2.14}$$

By straightforward calculations,

$$1 + \frac{1}{2}\frac{a_n}{a_{n+1}} - \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} < 0, \tag{2.15}$$

and since $\epsilon > 0$, it must satisfy

$$0 < \epsilon < 1 + \frac{1}{2}\frac{a_n}{a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{2.16}$$

For the second inequality in (2.11), we use Proposition A.3. So $a_{n+1}^2 - a_0^2 + a_0 a_n > 0$ if and only if

$$1 + \frac{3a_n}{2a_{n+1}} - \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} < \epsilon < 1 + \frac{3a_n}{2a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{2.17}$$

Since $-(n+1)/n < a_n/a_{n+1} < -1$, we have the following two inequalities:

$$\begin{gathered} 1 + \frac{3a_n}{2a_{n+1}} - \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} < 0, \\ 1 + \frac{3a_n}{2a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} > 0. \end{gathered} \tag{2.18}$$

Now, since we are interested in $\epsilon > 0$, it must satisfy

$$0 < \epsilon < 1 + \frac{3a_n}{2a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{2.19}$$

By straightforward calculations, it follows that

$$1 + \frac{3a_n}{2a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} < 1 + \frac{1}{2}\frac{a_n}{a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{2.20}$$

Therefore both inequalities in (2.11) are satisfied if and only if

$$0 < \epsilon < 1 + \frac{3a_n}{2a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{2.21}$$

Note that not depending on (2.12), we get that $|A_n| > |A_0|$ if (2.21) is satisfied, so we can omit the analysis of (2.12).

Now by hypothesis $\epsilon < 3(n+1)/(2n+1) + (3n/(2n+1))(a_n/a_{n+1})$ and by Proposition A.4, it holds that

$$\frac{3(n+1)}{2n+1} + \frac{3n}{2n+1}\frac{a_n}{a_{n+1}} < 1 + \frac{3a_n}{2a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{2.22}$$

It follows that

$$\epsilon < 1 + \frac{3a_n}{2a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}, \tag{2.23}$$

and consequently $|A_n| > |A_0|$. □

We now prove the main result for an arbitrary degree.

Theorem 2.3 (fix an arbitrary integer $n \geq 2$). *Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + a_0$ be a polynomial such that $-n/(n-1) < a_{n-1}/a_n < -1$, where $a_0 = -a_{n-1} + a_n(\epsilon - 1)$. If ϵ satisfies $0 < \epsilon < 3n/(2n-1) + (3(n-1)/(2n-1))(a_{n-1}/a_n)$, then we have that $|a_n| > |a_0|$ and P is a polynomial Schur stable.*

Proof. We make induction over n. The case $n = 2$ is part of Theorem 2.1. Now suppose that the theorem holds for $n \geq 2$, and let $P(z) = a_{n+1}z^{n+1} + a_n z^n + a_0$ be a polynomial of degree $n+1$ such that

$$-\frac{n+1}{n} < \frac{a_n}{a_{n+1}} < -1, \qquad a_0 = -a_n + a_{n+1}(\epsilon - 1). \tag{2.24}$$

If we define the polynomials Q and R as in lemma, then replacing P and Q in the polynomial R, we obtain

$$R(z) = \frac{(a_{n+1}^2 - a_0^2)z^n + a_{n+1}a_n z^{n-1} - a_0 a_n}{a_{n+1}}. \tag{2.25}$$

If $|a_{n+1}| > |a_0|$, then $a_{n+1}R(z) = (a_{n+1}^2 - a_0^2)z^n + a_{n+1}a_n z^{n-1} - a_0 a_n$ is a Schur stable polynomial if and only if P is Schur stable [25]. The inequality $|a_{n+1}| > |a_0|$ was proved in the lemma.

If we define $A_n = a_{n+1}^2 - a_0^2$, $A_{n-1} = a_{n+1}a_n$ and $A_0 = -a_0 a_n$ and since the inequality $|A_n| > |A_0|$ is satisfied (which was proved in the lemma), then by induction hypothesis the polynomial $a_{n+1}R(z) = A_n z^n + A_{n-1}z^{n-1} + A_0$ is Schur stable if $A_0 = -A_{n-1} + A_n(\tilde{\epsilon} - 1)$ and $\tilde{\epsilon}$ satisfies $0 < \tilde{\epsilon} < 3n/(2n-1) + 3(n-1)/(2n-1)(A_{n-1}/A_n)$. From the equality $A_0 = -A_{n-1} + A_n(\tilde{\epsilon} - 1)$, it follows that

$$A_0 = -A_{n-1} - A_n + A_n\tilde{\epsilon}. \tag{2.26}$$

By (2.24), $-a_0 a_n = a_n^2 + a_n a_{n+1} - a_n a_{n+1}\epsilon$ or equivalently

$$-a_0 a_n = -a_n a_{n+1} - \left(a_{n+1}^2 - a_0^2\right) + \left[\frac{\left(a_{n+1}^2 - a_0^2\right) + a_n^2 + 2a_n a_{n+1} - a_n a_{n+1}\epsilon}{a_{n+1}^2 - a_0^2}\right]\left(a_{n+1}^2 - a_0^2\right). \tag{2.27}$$

That is

$$A_0 = -A_{n-1} - A_n + \left[\frac{\left(a_{n+1}^2 - a_0^2\right) + a_n^2 + 2a_n a_{n+1} - a_n a_{n+1}\epsilon}{a_{n+1}^2 - a_0^2}\right]A_n. \tag{2.28}$$

Comparing this with (2.26), we see that

$$\tilde{\epsilon} = \frac{a_{n+1}^2 - a_0^2 + a_n^2 + 2a_n a_{n+1} - a_n a_{n+1}\epsilon}{a_{n+1}^2 - a_0^2}. \tag{2.29}$$

Moreover by induction hypothesis $\tilde{\epsilon}$ must satisfy the condition

$$0 < \tilde{\epsilon} < \frac{3n}{2n-1} + \frac{3(n-1)}{2n-1}\left(\frac{A_{n-1}}{A_n}\right). \tag{2.30}$$

Substituting $\tilde{\epsilon}$, A_{n-1} and A_n into (2.30), we obtain

$$0 < \frac{a_{n+1}^2 - a_0^2 + a_n^2 + 2a_n a_{n+1} - a_n a_{n+1}\epsilon}{a_{n+1}^2 - a_0^2} < \frac{3n}{2n-1} + \frac{3(n-1)}{2n-1}\frac{a_{n+1}a_n}{a_{n+1}^2 - a_0^2}. \tag{2.31}$$

The first inequality in (2.31) is equivalent to

$$0 < a_{n+1}^2 - a_0^2 + a_n^2 + 2a_n a_{n+1} - a_n a_{n+1}\epsilon. \tag{2.32}$$

And by Proposition A.5 this holds if and only if

$$0 < \epsilon < 2 + \frac{a_n}{a_{n+1}}. \tag{2.33}$$

Now we will analyze the second inequality in (2.31) which is equivalent to

$$a_{n+1}^2 - a_0^2 + a_n^2 + 2a_n a_{n+1} - a_n a_{n+1}\epsilon < \frac{3n\left(a_{n+1}^2 - a_0^2\right)}{2n-1} + \frac{3(n-1)}{2n-1}a_{n+1}a_n. \tag{2.34}$$

By Proposition A.6, inequality (2.34) is obtained if and only if

$$1 + \frac{4n+1}{2(n+1)}c - \sqrt{H_n(c)} < \epsilon < 1 + \frac{4n+1}{2(n+1)}c + \sqrt{H_n(c)}, \tag{2.35}$$

where $c = a_n/a_{n+1}$ and $H_n(c) = 1 + ((n-2)/(n+1))c + ((n-1/2)/(n+1))^2 c^2$.

By Proposition A.7,

$$1 + \frac{4n+1}{2(n+1)}c - \sqrt{H_n(c)} < 0. \tag{2.36}$$

So that (2.34) is satisfied if and only if

$$0 < \epsilon < 1 + \frac{4n+1}{2(n+1)}c + \sqrt{H_n(c)}. \tag{2.37}$$

Moreover by Proposition A.8,

$$1 + \frac{4n+1}{2(n+1)}c + \sqrt{H_n(c)} \leq 2 + c \quad \forall n \geq 1. \tag{2.38}$$

Thus (2.32) and (2.34) are satisfied if and only if

$$0 < \epsilon < 1 + \frac{4n+1}{2(n+1)}c + \sqrt{H_n(c)}. \tag{2.39}$$

We now analyze the right-hand side of (2.39). Let

$$F(c) = 1 + \frac{4n+1}{2(n+1)}c + \sqrt{H_n(c)}. \tag{2.40}$$

By Proposition A.9, it holds that $F(c)$ is increasing and convex, $F(-(n+1)/n) = 0$ and $F'(-(n+1)/n) = 3n/(2n+1)$.

We now get the equation of the tangent line of the function F at the point $c = -(n+1)/n$. To do this, we use fact hat $F(-(n+1)/n) = 0$ and $F'(-(n+1)/n) = 3n/(2n+1)$. So that the equation of the tangent line passing through the point $(-(n+1)/n, 0)$ is $y = 3(n+1)/(2n+1) + (3n/(2n+1))c$. Therefore if $0 < \epsilon < 3(n+1)/(2n+1) + (3n/(2n+1))c$, then

$$0 < \epsilon < 1 + \frac{4n+1}{2(n+1)}c + \sqrt{H_n(c)} \tag{2.41}$$

from which Theorem 2.3 follows. □

Remark 2.4. Note that the inequality $-(N+1)/N < -A_d < -1$ implies that the number a in (1.3) must be positive since $A_d = e^{ah}$ with $h > 0$ and then: $-e^{ah} < -1$ is satisfied if and only if $a > 0$.

The next corollary is a consequence of our results.

Corollary 2.5. *Suppose that the system* (1.3) *has a proportional control* (1.4) *with delay* $r = Nh$ *and suppose that* $a, b > 0$*. If the sampling period and the gain of the controller satisfy*

$$h < \frac{\ln\left[3(N+1)/3N\right]}{a},$$
$$-\frac{a}{b}\left[1 - \frac{3N}{2N+1} + \frac{3}{(2N+1)(e^{ah}-1)}\right] < K < -\frac{a}{b}, \tag{2.42}$$

then the sampled-data system is stabilizable.

3. Example

We consider the sampled-data system

$$\dot{x} = x(t) + u_{k-r}(t),$$
$$u_{k-r}(t) = Kx\left(\left[\frac{t}{h}\right]h - 4h\right), \tag{3.1}$$

where the values of the parameters are $a = 1$, $b = 1$, $N = 4$, and $r = 4h$. The difference equation (1.8) is $\varepsilon(k+1) = e^h\varepsilon(k) + (e^h - 1)K\varepsilon(k-4)$ and the characteristic polynomial (1.12) associated with the system is $P(\lambda) = \lambda^5 - e^h\lambda^4 + e^h - 1 + \epsilon$ which is Schur stable for $0 < \epsilon < (5 - 4e^h)/3$ by Theorem 2.1. Furthermore by Corollary 2.5 the maximum sampling period is $h < \ln(5/4)$ and the interval for the gain of the controller is

$$\frac{e^h - 2}{3(e^h - 1)} < K < -1. \tag{3.2}$$

Now for $h = 0.22$, the interval of the gain that guaranties the stabilization of the system is

$$-1.02 < K < -1. \tag{3.3}$$

For $K = -1.01$ the sampled-data system is stable as the characteristic polynomial has roots with modulus less than one: $\lambda_1 = -0.605301$, $\lambda_2 = -0.0721534 - 0.637776i$, $\lambda_3 = -0.0721534 + 0.637776i$, $\lambda_4 = 0.997839 - 0.031305i$, and $\lambda_5 = 0.997839 + 0.031305i$.

Appendix

In what follows we prove several inequalities.

Proposition A.1. *If* $-(n+1)/n < a_n/a_{n+1} < -1$ *and* $a_0 = -a_n + a_{n+1}(\epsilon - 1)$*, then*

$$|a_{n+1}| > |a_0| \Longleftrightarrow 0 < \epsilon < 2 + \frac{a_n}{a_{n+1}}. \tag{A.1}$$

Proof. Replacing the value of a_0, we see that

$$\begin{aligned} |a_{n+1}| > |a_0| &\Longleftrightarrow a_{n+1}^2 > \left[-a_n + a_{n+1}(\epsilon-1)\right]^2, \\ &\Longleftrightarrow a_{n+1}^2\epsilon^2 - 2a_{n+1}(a_n + a_{n+1})\epsilon + a_n(2a_{n+1} + a_n) < 0. \end{aligned} \tag{A.2}$$

Let $h(\epsilon) = a_{n+1}^2\epsilon^2 - 2a_{n+1}(a_n + a_{n+1})\epsilon + a_n(2a_{n+1} + a_n)$. The roots of the equation $h(\epsilon) = 0$ are $\epsilon_1 = 2 + a_n/a_{n+1}$ and $\epsilon_2 = a_n/a_{n+1}$. Since the coefficient of ϵ^2 is positive, then $h(\epsilon) < 0$ if and only if $a_n/a_{n+1} < \epsilon < 2 + a_n/a_{n+1}$. But since $a_n/a_{n+1} < 0$, we obtain $0 < \epsilon < 2 + a_n/a_{n+1}$. $\square$

Proposition A.2. *If* $a_0 = -a_n + a_{n+1}(\epsilon - 1)$, *then*

$$(a_{n+1}^2 - a_0^2) - a_0 a_n > 0 \Longleftrightarrow 1 + \frac{1}{2}\frac{a_n}{a_{n+1}} - \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} < \epsilon < 1 + \frac{1}{2}\frac{a_n}{a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{A.3}$$

Proof. Substituting a_0 into the first inequality, we see that

$$a_{n+1}^2 - \left[-a_n + a_{n+1}(\epsilon-1)\right]^2 - \left[-a_n + a_{n+1}(\epsilon-1)\right]a_n > 0 \tag{A.4}$$

if and only if

$$-a_{n+1}^2\epsilon^2 + (2a_{n+1}^2 + a_n a_{n+1})\epsilon - a_n a_{n+1} > 0. \tag{A.5}$$

Let $g(\epsilon) = -a_{n+1}^2\epsilon^2 + (2a_{n+1}^2 + a_n a_{n+1})\epsilon - a_n a_{n+1}$, then

$$g(\epsilon) = 0 \Longleftrightarrow \epsilon = 1 + \frac{1}{2}\frac{a_n}{a_{n+1}} \pm \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{A.6}$$

Since the coefficient of ϵ^2 is negative,

$$g(\epsilon) > 0 \Longleftrightarrow 1 + \frac{1}{2}\frac{a_n}{a_{n+1}} - \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} < \epsilon < 1 + \frac{1}{2}\frac{a_n}{a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{A.7}$$

$\square$

Proposition A.3. *If* $a_0 = -a_n + a_{n+1}(\epsilon - 1)$, *then* $a_{n+1}^2 - a_0^2 + a_0 a_n > 0$ *if and only if*

$$1 + \frac{3a_n}{2a_{n+1}} - \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} < \epsilon < 1 + \frac{3a_n}{2a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{A.8}$$

Proof. Replacing a_0 into the inequality $a_{n+1}^2 - a_0^2 + a_0 a_n > 0$, we get

$$a_{n+1}^2 - \left[-a_n + a_{n+1}(\epsilon-1)\right]^2 + \left[-a_n + a_{n+1}(\epsilon-1)\right]a_n > 0. \tag{A.9}$$

That is,

$$-a_{n+1}^2\epsilon^2 + (2a_{n+1}^2 + 3a_n a_{n+1})\epsilon - (2a_n^2 + 3a_n a_{n+1}) > 0. \tag{A.10}$$

Let

$$h(\epsilon) = -a_{n+1}^2\epsilon^2 + \left(2a_{n+1}^2 + 3a_n a_{n+1}\right)\epsilon - \left(2a_n^2 + 3a_n a_{n+1}\right). \tag{A.11}$$

Then

$$h(\epsilon) = 0 \Longleftrightarrow \epsilon = 1 + \frac{3a_n}{2a_{n+1}} \pm \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{A.12}$$

Since the coefficient of ϵ^2 is negative, $h(\epsilon) > 0 \Leftrightarrow$

$$1 + \frac{3a_n}{2a_{n+1}} - \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2} < \epsilon < 1 + \frac{3a_n}{2a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{A.13}$$

□

Proposition A.4. *If $n \geq 2$ and $-(n+1)/n < a_n/a_{n+1} < -1$, then*

$$\frac{3(n+1)}{2n+1} + \frac{3n}{2n+1}\frac{a_n}{a_{n+1}} < 1 + \frac{3a_n}{2a_{n+1}} + \sqrt{1 + \frac{1}{4}\left(\frac{a_n}{a_{n+1}}\right)^2}. \tag{A.14}$$

Proof. If we let $c = a_n/a_{n+1}$, then the previous inequality becomes $3(n+1)/(2n+1) + (3n/(2n+1))c < 1 + (3/2)c + \sqrt{1 + (1/4)c^2}$, which is satisfied if and only if

$$\left[(2n+1)^2 - 9\right]c^2 + 12(n+2)c - 4\left[(n+2)^2 - (2n+1)^2\right] > 0. \tag{A.15}$$

But this is true because the discriminant $-4n^4 - 4n^3 + 15n^2 + 16n + 4$ of this quadratic function is negative for $n \geq 3$ and the coefficient of c^2 is positive.

If $n = 2$, the assumption for a_2/a_3 becomes $-3/2 < a_2/a_3 < -1$ and since $c = a_2/a_3$ we get that $-3/2 < c$, that is, $0 < 2c + 3$ then $(2c+3)^2 > 0$. On the other hand, the left-hand side of inequality (A.15) becomes $16c^2 + 48c + 36 = 4(2c+3)^2$ which is positive and the proposition follows. □

Proposition A.5. *If $-(n+1)/n < a_n/a_{n+1} < -1$ and $a_0 = -a_n + a_{n+1}(\epsilon - 1)$, then $0 < a_{n+1}^2 - a_0^2 + a_n^2 + 2a_n a_{n+1} - a_n a_{n+1}\epsilon$ in and only if $0 < \epsilon < 2 + a_n/a_{n+1}$.*

Proof. If $a_0 = -a_n + a_{n+1}(\epsilon - 1)$ is replaced in the first inequality, we obtain

$$\begin{aligned} &0 < a_{n+1}^2 - \left[-a_n + a_{n+1}(\epsilon - 1)\right]^2 + a_n^2 + 2a_n a_{n+1} - a_n a_{n+1}\epsilon \\ &\Longleftrightarrow 0 < \epsilon\left[-a_{n+1}^2\epsilon + 2a_{n+1}^2 + a_n a_{n+1}\right] \\ &\Longleftrightarrow 0 < \epsilon < 2 + \frac{a_n}{a_{n+1}}. \end{aligned} \tag{A.16}$$

By hypothesis $-(n+1)/n < a_n/a_{n+1} < -1$ or equivalently $(n-1)/n < 2 + a_n/a_{n+1} < 1$. Since $(n-1)/n > 0$ for all $n \geq 2$, $2 + a_n/a_{n+1} > 0$. Therefore the first inequality is satisfied if and only if $0 < \epsilon < 2 + a_n/a_{n+1}$. □

Proposition A.6. *If* $a_0 = -a_n + a_{n+1}(\epsilon - 1)$*, then*

$$\begin{aligned} a_{n+1}^2 - a_0^2 + a_n^2 + 2a_n a_{n+1} - a_n a_{n+1}\epsilon &< \frac{3n\left(a_{n+1}^2 - a_0^2\right)}{2n-1} + \frac{3(n-1)}{2n-1}a_{n+1}a_n \\ \Longleftrightarrow 1 + \frac{4n+1}{2(n+1)}c - \sqrt{H_n(c)} < \epsilon &< 1 + \frac{4n+1}{2(n+1)}c + \sqrt{H_n(c)}, \end{aligned} \tag{A.17}$$

where $c = a_n/a_{n+1}$ *and* $H_n(c) = 1 + ((n-2)/(n+1))c + ((n-1/2)/(n+1))^2c^2$.

Proof. The inequality

$$a_{n+1}^2 - a_0^2 + a_n^2 + 2a_n a_{n+1} - a_n a_{n+1}\epsilon < \frac{3n\left(a_{n+1}^2 - a_0^2\right)}{2n-1} + \frac{3(n-1)}{2n-1}a_{n+1}a_n \tag{A.18}$$

is equivalent to

$$-a_n a_{n+1}\epsilon < \frac{n+1}{2n-1}\left(a_{n+1}^2 - a_0^2\right) + \frac{-(n+1)}{2n-1}a_{n+1}a_n - a_n^2. \tag{A.19}$$

Replacing a_0 into the last inequality, we see that this is equivalent to say that $0 < f(\epsilon)$, where

$$f(\epsilon) = -(n+1)a_{n+1}^2\epsilon^2 + \left[2(n+1)a_{n+1}^2 + (4n+1)a_n a_{n+1}\right]\epsilon - 3na_n^2 - 3(n+1)a_n a_{n+1}. \tag{A.20}$$

Since

$$f(\epsilon) = 0 \Longleftrightarrow \epsilon = 1 + \frac{4n+1}{2(n+1)}c \pm \sqrt{H_n(c)}, \tag{A.21}$$

and the coefficient of ϵ^2 is negative, it holds that

$$f(\epsilon) > 0 \Longleftrightarrow 1 + \frac{4n+1}{2(n+1)}c - \sqrt{H_n(c)} < \epsilon < 1 + \frac{4n+1}{2(n+1)}c + \sqrt{H_n(c)}. \tag{A.22}$$

□

Proposition A.7. *Fix an arbitrary* $n \in N$*. If* $-(n+1)/n < c < -1$*, then*

$$1 + \frac{4n+1}{2(n+1)}c - \sqrt{H_n(c)} < 0 \quad \forall n \geq 1. \tag{A.23}$$

Proof. We have that $-(n+1)/n < c \leq -1$ if and only if

$$-\frac{(4n+1)}{2n} + 1 < \frac{4n+1}{2(n+1)}c + 1 \leq -\frac{4n+1}{2(n+1)} + 1 = \frac{-2n+1}{2(n+1)} < 0, \tag{A.24}$$

for all $n \geq 1$, and the result follows. □

Proposition A.8. *If* $-(n+1)/n < c < -1$*, it holds that*

$$1 + \frac{4n+1}{2(n+1)}c + \sqrt{H_n(c)} \leq 2 + c \quad \forall n \geq 1. \tag{A.25}$$

Proof. We have that

$$1+\frac{4n+1}{2(n+1)}c+\sqrt{H_n(c)}\leq 2+c \tag{A.26}$$

if and only if

$$\sqrt{H_n(c)}\leq 1+\left[1-\frac{4n+1}{2(n+1)}\right]c. \tag{A.27}$$

From the definition of $H_n(c)$ this is true if and only if

$$1+\frac{(n-2)}{(n+1)}c+\left(\frac{n-1/2}{n+1}\right)^2c^2\leq 1+\left(\frac{-2n+1}{n+1}\right)c+\frac{(-2n+1)^2}{4(n+1)^2}c^2; \tag{A.28}$$

if and only if $((3n-3)/(n+1))c\leq 0$. Since $c<0$ and $3n-3\geq 0\ \forall n\geq 1$. Then the inequality is satisfied. □

Proposition A.9. *Let*

$$F(c)=1+\frac{4n+1}{2(n+1)}c+\sqrt{1+\frac{(n-2)}{(n+1)}c+\left(\frac{n-1/2}{n+1}\right)^2c^2}, \tag{A.29}$$

for $c>-(n+1)/n$. *Then*

(a) $F'(c)>0$;

(b) $F''(c)>0$ *(F is convex);*

(c) $F(-(n+1)/n)=0$, *and* $F'(-(n+1)/n)=3n/(2n+1)$.

Proof. It is elementary. □

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