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Research Article

Vanishing Waves on Closed Intervals and Propagating Short-Range Phenomena

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This study presents mathematical aspects of wave equation considered on closed space intervals. It is shown that a solution of this equation can be represented by a certain superposition of traveling waves with null values for the amplitude and for the time derivatives of the resulting wave in the endpoints of this interval. Supplementary aspects connected with the possible existence of initial conditions for a secondorder differential system describing the amplitude of these localized oscillations are also studied, and requirements necessary for establishing a certain propagation direction for the wave (rejecting the possibility of reverse radiation) are also presented. Then it is shown that these aspects can be extended to a set of adjacent closed space intervals, by considering that a certain traveling wave propagating from an endpoint to the other can be defined on each space interval and a specific mathematical law (which can be approximated by a differential equation) describes the amplitude of these localized traveling waves as related to the space coordinates corresponding to the middle point of the interval. Using specific differential equations, it is shown that the existence of such propagating law for the amplitude of localized oscillations can generate periodical patterns and can explain fracture phenomena inside materials as well.

1. Introduction

Test-functions(whichdifferto zero only on a limited interval and have continuous derivatives of any order on the whole real axis) are widely used in the mathematical theory of distributions and in Fourier analysis of wavelets. Yet such test-functions, similar to the Dirac functions, cannot be generated by a differential equation. The existence of such an equation of evolution, beginning to act at an initial moment of time, would imply the necessity for a derivative of certain order to make a jump at this initial moment of time from the zero value to a nonzero value. But this aspect is in contradiction with the property of test-functions to have continuous derivatives of any order on the whole real axis, represented in this case by the time axis. So it results that an ideal test-function cannot be generated by a differential equation (see also [1]); the analysis has to be restricted at possibilities of generating practical test-functions (functions similar to test-functions, but having a finite number of continuous derivatives on the whole real axis) useful for wavelets analysis. Due to the exact form of the derivatives of test-functions, we cannot apply derivative free algorithms [2] or algorithms which can change in time [3]. Starting from the exact mathematical expressions of a certain test-function and of its derivatives, we must use specific differential equations for generating such practical test-functions.

Thisaspect is connected with causal aspects of generating apparently acausal pulses as solutions of the wave equation, presented in [4]. Such test-functions, considered at the macroscopic scale (that does not mean Dirac-functions), can represent solutions for certain equations in mathematical physics (an example being the wave-equation). The main consequence of this aspect consists in the possibility of certain pulses to appear as solutions of the wave-equation under initial null conditions for the function and for all its derivatives and without any free-term (a source-term) to exist. In order to prove the possibility of appearing acausal pulses as solutions of the wave-equation (not determined by the initial conditions or by some external forces) we begin by writing the wave-equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{1.1}$$

for a free string defined on the length interval $(0, l)$ (an open set), where ϕ represents the amplitude of the string oscillations and v represents the velocity of the waves inside the string medium. At the initial moment of time (the zero moment) the amplitude ϕ together with all its derivatives of first and second orders is equal to zero. From the mathematical theory of the wave-equation, we know that any solutionof this equation must be a superposition of a direct wave and of a reverse wave. For the beginning, we will restrict our analysis at direct waves by considering a supposed extension of the string on the whole Ox axis, ϕ being defined by the function

$$\phi(\tau) = \begin{cases} \exp\left(\dfrac{1}{(x - vt + 1)^2 - 1}\right) & \text{for } |x - vt + 1| < 1, \\ 0 & \text{for } |x - vt + 1| \geq 1, \end{cases} \tag{1.2}$$

where $t \geq 0$. This function for the extended string satisfies the wave-equation (being a function of $x - vt$, a direct wave). It is a continuous function, having continuous partial derivatives of any order for $x \in (-\infty, \infty)$ and for $t \geq 0$. For $x \in (0, l)$ (the real string) the amplitude ϕ and all its derivatives are equal to zero at the zero moment of time, as required by the initial null conditions for the real string (nonzero values appearing only for $x \in (-2, 0)$ for $t = 0$, while on this interval $|x - vt + 1| = |x + 1| < 1$). We can notice that for $t = 0$ the amplitude ϕ and its partial derivatives differ to zero only on a finite space interval, this being a property of the functions defined on a compact set (test-functions). But the argument of the exponential function is $x - vt$; this implies that the positive amplitude existing on the length interval $(-2, 0)$ at the zero moment of time will move along the Ox axis in the direction $x = +\infty$. So at some time moments t_k after the zero moment, a nonzero amplitude ϕ will appear inside the string, propagating from one edge to the other. It can be noticed that the pulse passes through the real string and at a certain time moment t_{fin} (when the pulse existing at the zero moment of time on the length interval $(-2, 0)$ has moved into the length interval $(l, l + 2)$) its action upon the real string ceases. We must point the fact that the limit points $x = 0$ and $x = l$ are not considered to belong to the string; but this is in accordance with the rigorous definition of derivatives (for this limit points cannot be defined as derivatives related to any direction around them).

This point of space (the limit of the open space interval considered) is very important for our analysis, while we will extend the study to closed space intervals. Considering small space intervals around the points of space where the sources of the generated field are situated (e.g., the case of electrical charges generating the electromagnetic field), it will be shown that causal aspects require the logical existence of a certain causal chain for transmitting interaction from one point of space to another, which can be represented by mathematical functions which vanish (its amplitude and all its derivatives) in certain points of space. From this point of space, an informational connection for transmitting the wave further could be considered (instead of a transmission based on certain derivatives of the wave). Thus a kind of granular approach for propagation along a certain axis can be considered suitable for application in quantum theory. As an important consequence, some directions of propagation for the generated wave will appear and the possibility of reverse radiation will be rejected. Moreover, specific applications for other propagating phenomena involving the generation of some spatial periodical patterns or an increasing amplitude of oscillations along a certain spatial axis can be also analyzed by this mathematical model.

2. Utility of test-functions in mathematical physics for half-closed space intervals

If we extend our analysis to half-closed intervals by adding one endpoint of the space interval to the previously studied open intervals (e.g., by adding the point $x = 0$ to the open interval $(0, l)$), we should take into account the fact that a complete mathematical analysis usually implies the use of a certain function $f(t)$ defined at the limit of the working space interval (the point of space $x = 0$, in the previous example). Some other supplementary functions can be met in mathematical physics.

The use of such supplementary functions defined on the limit of the half-closed interval could appear as a possible explanation for the problem of generating acausal pulses as solutions of the wave equation on bounded open intervals. The acausal pulse presented in the previous paragraph (similar to wavelets) traveling along the Ox axis requires a certain nonzero function of time $f_0(t)$ for the amplitude of the pulse for the limit of the interval $x = 0$. It could be argued that the complete mathematical problem of generating acausal pulses for null initial conditions on this interval and for null function $f_0(t)$ corresponding to function ϕ (the pulse amplitude) at this endpoint of the interval ($x = 0$, resp.) would reject the possibility of appearing the acausal pulse presented in the previous paragraph. The acausal pulse ϕ previously presented implies nonzero values for f_0 at certain time moments, which represents a contradiction with the requirement for this function f_0 to present null values at any time moment. By an intuitive approach, null external sources would imply null values for function f_0 and (as a consequence) null values for the pulse amplitude ϕ.

Yet it can be easily shown that the problem of generating acausal pulses on half-closed intervals cannot be rejected by using supplementary requirements for certain functions $f(t)$ defined at one limit of such bounded space intervals. Let us simply suppose that instead of function

$$\phi(\tau) = \begin{cases} \exp\left(\dfrac{1}{(x - vt + 1)^2 - 1}\right) & \text{for } |x - vt + 1| < 1, \\ 0 & \text{for } |x - vt + 1| \geq 1 \end{cases} \tag{2.1}$$

presented in the previous paragraph we must take into consideration two functions ϕ_0 and ϕ_l defined as

$$\phi_0(\tau) = \begin{cases} \exp\left(\dfrac{1}{(x - vt + m)^2 - 1}\right) & \text{for } |x - vt + m| < 1, \\ 0 & \text{for } |x - vt + m| \geq 1, \end{cases}$$
$$\phi_l(\tau) = \begin{cases} -\exp\left(\dfrac{1}{(x + vt - m)^2 - 1}\right) & \text{for } |x - vt + m| < 1, \\ 0 & \text{for } |x + vt - m| \geq 1, \end{cases} \tag{2.2}$$

with m selected as $m > 0$, $m - 1 > l$ (so as both functions ϕ_0 and ϕ_l to have nonzero values outside the real string and asymmetrical as related to the point of space $x = 0$. While function ϕ_0 corresponds to a direct wave (its argument being $(x - vt)$) and ϕ_l corresponds to a reverse wave (its argument being $(x + vt)$) it results that both functions ϕ_0 and ϕ_l arrive at the same space origin $x = 0$, the sum of these two external pulses being null all the time (functions ϕ_0 and ϕ_l being asymmetrical, $\phi_0 = -\phi_l$) at any moment of time. So by requiring that $\phi(t) = 0$ for $x = 0$ (the left limit of a half-closed interval $[0, l)$) we cannot reject the mathematical possibility of the appearance of an acausal pulse on a half-closed interval.

A possible mathematical explanation for this aspect consists in the fact that we have used a reverse wave (an acausal pulse) propagating from $x = \infty$ toward $x = -\infty$, which is first received at the right limit $x = l$ of the half-closed interval $[0, l)$ before arriving at the point of space $x = 0$. It can be argued that in case of a closed space interval $[0, l]$, we should consider the complete mathematical problem, consisting of two functions $f_0(t)$, $f_l(t)$ corresponding to both limits of the working space intervals (the points of space $x = 0$ and $x = l$). But in fact the wave equation corresponds to a physical model valid in the three-dimensional space, under the form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{2.3}$$

and the one-dimensional model previously used is just an approximation. Moreover, the source of the field is considered at a microscopic scale (e.g., quantum particles like electrons for the case of the electromagnetic field) and the emitted field for such elementary particles presents a spherical symmetry. Transforming the previous equation in polar coordinates and supposing that the function ϕ depends only on r (the distance from the source of the field to the point of space where this emitted field is received), it results that

$$\frac{\partial^2 U}{\partial r^2} - \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} = 0, \tag{2.4}$$

where

$$U = r\varphi. \tag{2.5}$$

An analysis of the field emitted from the point of space $r = 0$ (the source) toward a point of space $r = r_0$ (where the field is received) should be performed on the space interval $(0, r]$ (a half-closed interval); the point of space $r = 0$ cannot be included in the working interval as long as the solution $\phi(r)$ for the field is obtained by dividing the solution $U(r)$ of the previous equation (in spherical coordinates) through r (the denominator of the solution ϕ being zero, some supplementary aspects connected to the limit of functions should be added, but still without considering a function of time as condition for the space origin). This can be put in correspondence with the previously presented case of an acausal pulse defined on $[0, l)$ if we consider that (as a rule) (a) the endpoint where the function $\phi(t)$ is not defined represents the source of the field (a round bracket being added, while it cannot be considered as part of the working interval) and (b) the endpoint where the function φ vanishes represents a point of space where the propagating phenomenon is recreated (by reflection or by interaction with different particles, for the case of optical waves), a square bracket being added. The endpoint represented by square bracket (where the wave vanishes) can be considered as a source for the field propagating in a next space interval after an interaction, and so on.

Thus an asymmetry in the required methods for analyzing phenomena appears. Moreover, for the appearance of a certain direction for the transmission of interaction (from one space interval to another), it results that the possibility of retroradiation (a reverse wave generated by points of space where a direct wave has arrived) should be rejected (a memory of previous phenomena is determining the direction of propagation).

3. Applications for closed space intervals: applications in quantum physics

The pulse presented in the previous paragraph is in fact a traveling wave propagating from $x = \infty$ toward $x = 0$ and back which vanishes at the point of space $x = 0$ due to a kind of reflection. Yet we can extend our analysis by considering a subsequent reflection of this pulse at the limit point $x = l$ and so on. Thus a resulting traveling wave can be considered inside the *closed* space interval $[0, l]$ with null values at the endpoints $x = 0$, $x = l$ at any time moment *after* the first reflection.

At first sight, this localized oscillation is not useful for our mathematical analysis of acausal pulses. It does not correspond to initial null conditions on the closed bounded space interval $[0, l]$ and to null time functions defined at the endpoints $x = 0$, $x = l$ (while the traveling wave should already exist inside this interval when null conditions for the endpoints at any subsequent time moment are added). Yet we must take into consideration the fact that in quantum physics the operators corresponding to creation and annihilation of particles are obtained (in a heuristic manner) starting from an analysis of electromagnetic field performed on bounded space intervals and extended to unbounded intervals by simply replacing the space limits for a set of such intervals with infinite values [5]. However, the previously mentioned analysis on bounded intervals makes use of stationary waves which cannot be taken into consideration when a space limit equals $\pm\infty$ (no reflection can appear). This logical contradiction can be avoided if any extended space interval is considered as a sum of adjacent small space intervals with specific localized oscillations defined on each of them.

Supposing that a localized oscillation is generated on a certain limited space interval by an external force or by a received wave-train, we can consider that subsequent oscillations are generated on adjacent space intervals (as in the case of spherical waves) due to a kind of informational connection existing on the boundaries of these intervals. A mathematical connection described by wave-equation cannot be taken into consideration any more, and thus the previous model of causal chain corresponding to a sequence: *changes in the value of partial derivatives as related to space coordinates imply changes in the partial derivatives of the amplitude as related to time, which further imply changes in the value of the function,* should be replaced by a step-by-step transmission of interaction starting from an initial half-closed interval (e.g., its open limit corresponding to the source of the field) to adjacent space intervals. This corresponds to a granular aspect of space suitable for applications in quantum physics, where the generation and annihilation of quantum particles should be considered on limited space-time intervals (asymmetrical pulses could be also used [6]). A specific physical quantity (corresponding to the amplitude of localized oscillations) is transmitted from one space interval to another, according to a certain mathematical law.

4. Dynamical spatial generation of structural patterns

We will continue the study by presenting properties of spatial linear systems described by a certain physical quantity generated by a differential equation. This quantity can be represented by internal electric or magnetic field inside the material or by similar physical quantities, and corresponds to the amplitude of localized oscillations previously mentioned. A specific mathematical law which can be approximated by a differential equation generates this quantity considering as input the spatial alternating variations of a certain internal parameter. As a consequence, specific spatial linear variations of the corresponding physical quantity appear. In case of very short-range variations of this internal parameter, systems described by a differential equation able to generate a practical test-function [1] exhibit an output which appears to an external observer under the form of two distinct envelopes. These can be considered as two distinct structural patterns located in the same material along a certain linear axis. This aspect differs from the oscillations of unstable type second-order systems studied using difference equations [7] or advanced differential equations [8], and they differ also from the previous studies of the same author [9] where the frequency response of such systems to alternating inputs was studied (in conjunction with the ergodic hypothesis). For our purpose, we will use the function

$$\varphi(x) = \begin{cases} \exp\left(\dfrac{1}{x^2-1}\right) & \text{if } x \in (-1,1), \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

which is a test-function on $[-1,1]$. For a small value of the numerator of the exponent, a rectangular shape of the output is obtained. An example is the case of the function

$$\varphi(x) = \begin{cases} \exp\left(\dfrac{0.1}{x^2-1}\right) & \text{if } x \in (-1,1), \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Using the expression of $\varphi(x)$ and of its derivatives of first and second orders, a differential equation which admits as solution the function φ corresponding to a certain physical quantity can be obtained. However, a test-function cannot be the solution of a differential equation. Such an equation of evolution implies a jump at the initial space point for a derivative of certain order, and test-function must possess continuous derivatives of any order on the whole real axis. So it results that a differential equation which admits a test-function φ as solution can generate only a practical test-function f similar to φ, but having a finite number of continuous derivatives on the real Ox axis. In order to do this, we must add initial conditions for the function f (generated by the differential equation) and for some of its derivatives $f^{(1)}$, and/or $f^{(2)}$ and so on equal to the values of the test-function φ and of some of its derivatives $\varphi^{(1)}$, and/or $\varphi^{(2)}$ and so on at an initial space point x_{in} very close to the beginning of the working spatial interval. This can be written under the form

$$f_{x_{\text{in}}} = \varphi_{x_{\text{in}}}, \quad f^{(1)}_{x_{\text{in}}} = \varphi^{(1)}_{x_{\text{in}}}, \quad \text{and/or} \quad f^{(2)}_{x_{\text{in}}} = \varphi^{(2)}_{x_{\text{in}}}, \quad \text{and so on.} \tag{4.3}$$

If we want to generate spatial practical test-functions f which are symmetrical as related to the middle of the working spatial interval, we can choose as space origin for the Ox axis the middle of this interval, and so it results that the function f should be invariant under the transformation

$$x \longrightarrow -x. \tag{4.4}$$

Functions invariant under this transformation can be written in the form $f(x^2)$ (similar to aspects presented in [1]) and so the form of a general second-order differential equation generating this kind of functions should be

$$a_2(x^2)\frac{d^2 f}{d(x^2)^2} + a_1(x^2)\frac{df}{dx^2} + a_0(x^2)f = 0. \tag{4.5}$$

However, for studying the generation of structural patterns on such a working interval, we must add a free term corresponding to the cause for the variations of the external observable physical quantity. Thus, a model for generating a practical test-function using as input the internal parameter $u = u(x)$, $x \in [-1, 1]$, is

$$a_2(x^2)\frac{d^2 f}{d(x^2)^2} + a_1(x^2)\frac{df}{dx^2} + a_0(x^2)f = u \tag{4.6}$$

subject to

$$\lim_{x\to\pm 1} f^k(x) = 0 \quad \text{for } k = 0, 1, \dots, n, \tag{4.7}$$

which are the boundary conditions of a practical test-function. For u represented by alternating functions, we should notice periodical variations of the external observable physical quantity f.

According to the previous considerations for the form of a differential equation invariant at the transformation

$$x \longrightarrow -x, \tag{4.8}$$

a first-order system can be written under the form

$$\frac{df}{d(x^2)} = f + u \tag{4.9}$$

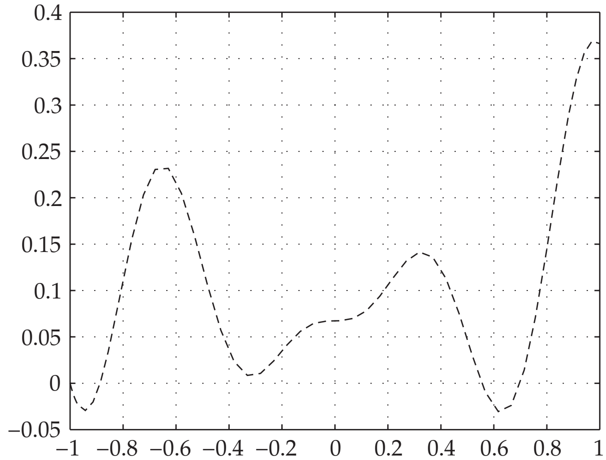

Figure 1: f versus distance for first-order system, input $u = \sin(10x)$.

which converts to

$$\frac{df}{dx} = 2xf + 2xu \tag{4.10}$$

representing a first-order dynamical system. For a periodical input (corresponding to the internal parameter) $u = \sin 10x$, numerical simulations performed using Runge-Kutta functions in MATLAB present an output of an irregular shape (Figure 1) not suitable for joining together the outputs for a set of adjoining linear intervals (the value of f at the end of the interval differs in a significant manner to the value of f at the beginning of the interval). A better form for the physical quantity f is obtained for variations of the internal parameter described by the equation $u = \cos 10x$. In this case, the output is symmetrical as related to the middle of the interval (as can be noticed in Figure 2) and the results obtained on each interval can be joined together on the whole linear spatial axis, without any discontinuities to appear. The resulting output would be represented by a sum of two great spatial oscillations (one at the end of an interval and another one at the beginning of the next interval) and two small spatial oscillations (around the middle of the next interval).

Similar results are obtained for an undamped dynamical system first order, represented by

$$\frac{df}{d(x^2)} = u \tag{4.11}$$

which is equivalent to

$$\frac{df}{dx} = 2xu. \tag{4.12}$$

When the internal parameter presents very short-range variations, some new structural patterns can be noticed. Considering an alternating input of the form $u = \sin(100x)$, it results in an observable physical quantity f represented in Figure 3; for an alternating cosine input represented by $u = \cos(100x)$, it results in the output f represented in Figure 4.

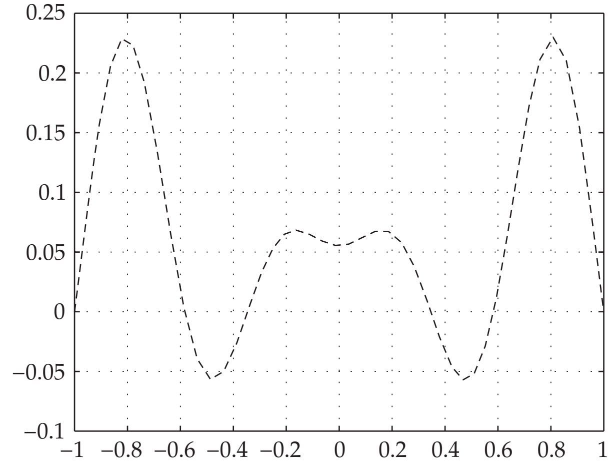

Figure 2: f versus distance for first-order system, input $u = \cos(10x)$.

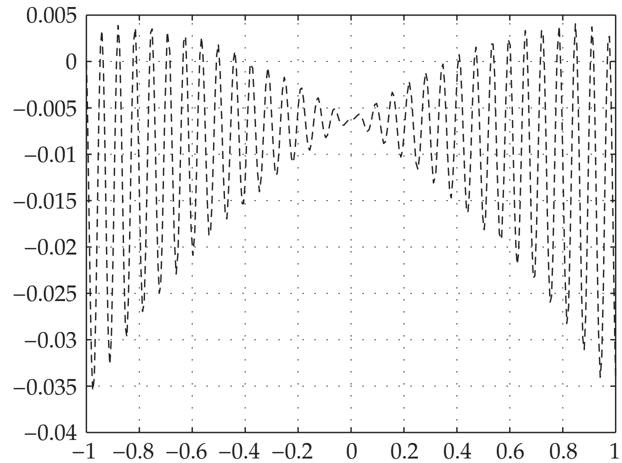

Figure 3: f versus distance for first-order system, input $u = \sin(100x)$.

Studying these two graphics, we can notice the presence of two distinct envelopes. Their shape depends on the phase of the input alternating component (the internal parameter), as related to the space origin. At first sight, an external observer could notice two distinct functions f inside the same material, along the Ox axis. These can be considered as two distinct structural patterns located in the same material, generated by a short-range alternating internal parameter u through a certain differential equation (invariant at the transformation $x \to -x$).

5. Aspects connected with short-range breaking phenomena

For simulating the generation of specific deformations inside a material medium under the action of external forces, it can be considered that some short wavelength vibrations appear in the area where the force acts. Usually the corresponding deformation is simulated inside the

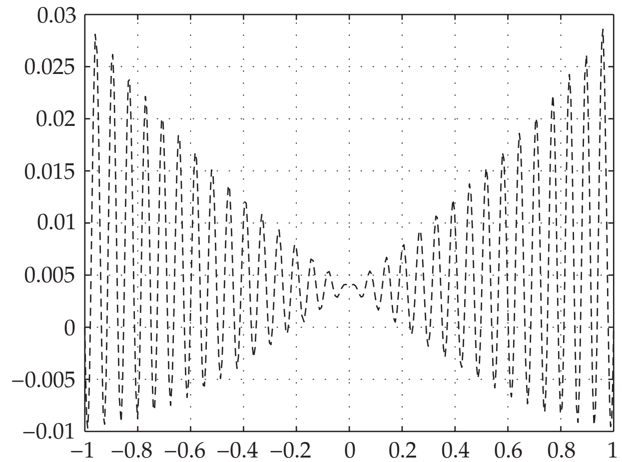

Figure 4: f versus distance for first-order system, input $u = \cos(100x)$.

material medium, using linear differential equations or equations with partial derivatives (similar to the wave equation or to the equation of diffusion). Yet such linear equations cannot explain the distance between the space area where the external force acts and the space area where fracture phenomena appear. Using differential equations of higher order, some slow variations of deformation along a certain direction could be obtained. Due to the fact that the mathematical model should explain the sharp deformations at a certain distance of the point of space where the force acts (leading to fracture phenomena), some different types of differential equations must be studied. For this reason, our study has taken into consideration some dynamical equations able to generate practical test-functions (similar to wavelets) [1] and delayed pulses (when a free term which corresponds to an external pulse is added) [10] for justifying fracture phenomena appearing in a certain material medium. It is considered that an external force (described by a short wavelength sine function multiplied by a Gaussian function) acts upon the material medium in a certain area. As a consequence, some localized vibrations (corresponding to localized oscillations on closed space intervals presented in the previous paragraphs) appear. These localized oscillations are transmitted from one space interval to another according to a certain mathematical law which puts into correspondence the amplitude of these local vibrations to spatial coordinates.

Using a specific differential equation (able to generate symmetrical functions for a null free term) for describing the generation of the corresponding deformation along an axis inside the material medium, it results that a significant deformation appears at a certain distance. This significant deformation justifies the fracture phenomena, while the inner structure of the material cannot allow significant sharp deformations without breaking. The main problem is represented by the search of an adequate free term $u(x)$ able to justify fracture phenomena. We start by using a constant free term, using an equation as

$$f^{(2)} = \frac{0.6x^4 - 0.36x^2 - 0.2}{\left(x^2 - 1\right)^4} f + u(x), \tag{5.1}$$

where $u(x)$ represents the external force (supposed to be constant in a first approximation on the working space interval $(-1, 1)$). The deformation $f(x)$ is supposed to be first time generated by the external force at the limit $x = -1$ of the working interval and then (according

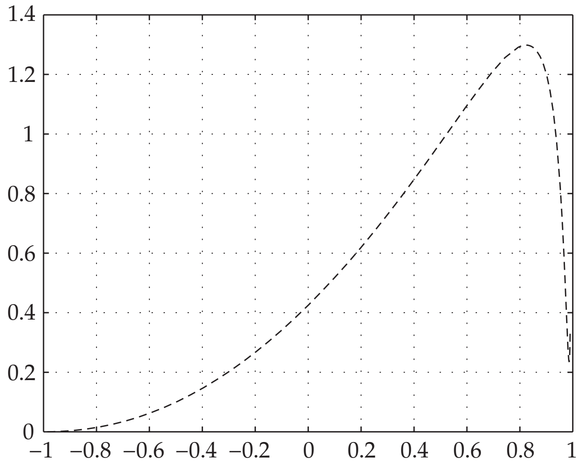

Figure 5: Deformation generated by an external constant force.

to the differential equation) it generates the corresponding deformation along the whole working interval, with the external constant force u acting in a continuous manner upon the material. The deformation generated by such a constant force u should be symmetrical as related to origin 0 (the previous differential equations being valid on the space interval $(-1, 1)$ with initial null conditions for $f(x)$ at the initial point of space $x = -1$). The property of symmetry previously mentioned is justified by invariance properties of this type of differential equations [1]. However, even for $u(x) = 1$ (the most simple external force acting upon the material which is symmetrical as related to space origin 0) numerical simulations in MATLAB present an asymmetry of the output signal, justified by numerical errors (see Figure 5). But numerical simulations present also a slow varying deformation along the axis, with no spatial oscillations; thus the fracture phenomenon cannot be explained.

A similar shape of the output can be noticed for an input represented by a Gaussian external force, acting around the point of space $x = -0.9$ and having a width ten times smaller than the working period—similar to the use of a Gaussian modulated signal for generating delayed pulses [10]. In such a case the differential equation generating the deformation along the working interval is represented by

$$f^{(2)} = \frac{0.6x^4 - 0.36x^2 - 0.2}{\left(x^2 - 1\right)^4} f + \exp - \frac{(x + 0.9)^2}{(0.01)^2} \tag{5.2}$$

and the corresponding output is represented in Figure 6.

So we must extend our search for adequate mathematical models, and we will try a free term $u(x)$ represented by

$$u(x) = \exp\left(-\frac{(x + 0.9)^2}{(0.01)^2}\right) \sin 10^4 x. \tag{5.3}$$

This mathematical expression describes an external force represented by a Gaussian multiplied by a sine function with short wavelength, being considered that the applied force is transformed by the surface of the material into a set of alternating internal efforts with very short wavelength (similar to a localized vibration).

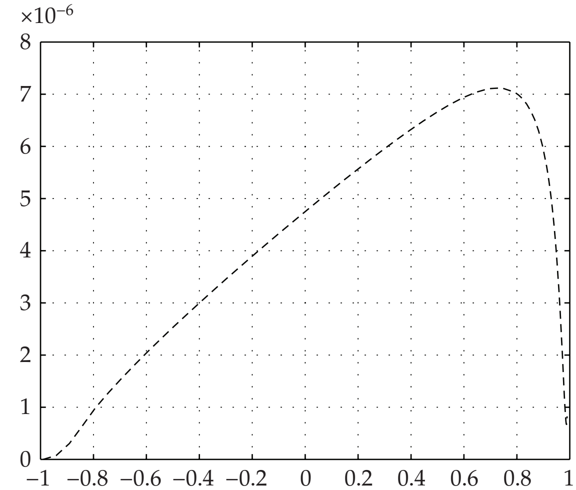

Figure 6: Deformation generated by an external Gaussian (localized) force.

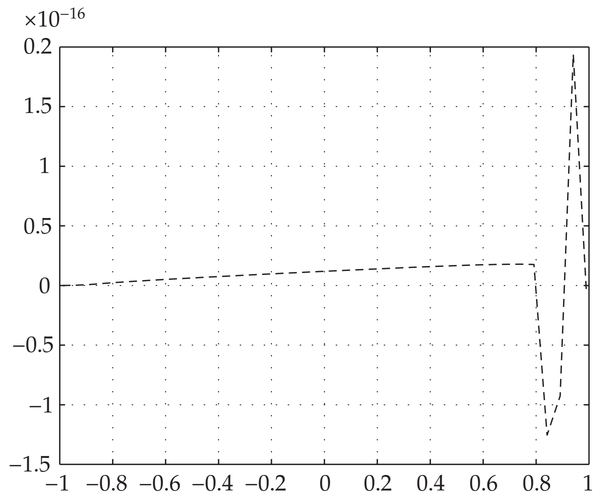

Figure 7: Deformation generated by a modulated Gaussian internal effort.

The corresponding output is represented in Figure 7. It can be noticed that we have finally obtained a sharp deformation appearing at a certain distance between the point of space where the external (modulated Gaussian) force acts and the point of space where the sharp deformation appears. Moreover, the sharp deformation appears as an alternating function localized on a very short spatial interval. It is quite obvious that such a deformation cannot be allowed by the inner structure of the material, leading to fracture phenomena. This simulation explains also the fact that the fracture point is usually situated at a certain distance from the point where the external force is applied (as can be noticed studying the deformation presented in Figure 7 generated by the internal efforts $u(x)$ presented in Figure 8)

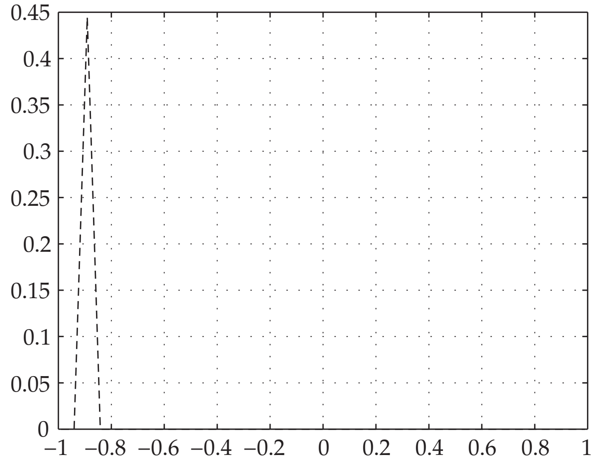

Figure 8: Modulated Gaussian internal effort by a sine function.

For the case when the Gaussian input is modulated by a cosine function, which means that

$$u(t) = \exp\left(-\frac{(\tau + 0.9)^2}{(0.01)^2}\right)\cos 10^4\tau, \tag{5.4}$$

we obtain an output represented by a slowly varying function, without alternate deformation. So a cosine modulation of a Gaussian input is not suitable for simulating fracture phenomena appearing at a certain distance from the point where the external force acts.

We must point the fact that such localized alternating deformations generated by systems working on a limited interval and situated at a certain distance from the point where the external force acts differ to wavelets resulting from PDE equations (see [11]) and to propagating wavelets through dispersive media [12], while the shape of the resulting deformation is not symmetrical as related to *Ox* axis (its mean value differs to zero). However, a multiscale analysis of such pulses should be performed for explaining the complex fracture phenomena in an extended area and for justifying why a certain direction for generating deformation has to be chosen.

6. Conclusions

This study has shown that some solutions of the wave equation for half-closed space interval are considered around the point of space where the sources of the generated field are situated (e.g., the case of electrical charges generating the electromagnetic field). These solutions can be mathematically represented by vanishing waves corresponding to a superposition of traveling test-functions. Then some properties of spatial linear systems described by a certain physical quantity (generated by a differential equation) are studied. This quantity can be represented by internal electric or magnetic field inside the material or by similar physical quantities, and corresponds to the amplitude of localized oscillations previously mentioned. A specific mathematical law which can be approximated by a differential equation generates this quantity considering as input the spatial alternating variations of this internal parameter. As a consequence, specific spatial linear variations of the corresponding physical quantity appear. Finally, a specific differential equation (able to generate symmetrical functions for a null free term) is used for describing the generation of the corresponding deformation along an axis inside the material medium. Numerical simulations have shown that a significant deformation appears at a certain distance. This deformation justifies the fracture phenomena, while the inner structure of the material cannot allow significant sharp deformations without breaking.

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