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Research Article

On the Essential Instabilities Caused by Fractional-Order Transfer Functions

Farshad Merrikh-Bayat and Masoud Karimi-Ghartemani

Department of Electrical Engineering, Sharif University of Technology, Tehran, Iran

Correspondence should be addressed to Farshad Merrikh-Bayat, f.bayat@gmail.com

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The exact stability condition for certain class of fractional-order (multivalued) transfer functions is presented. Unlike the conventional case that the stability is directly studied by investigating the poles of the transfer function, in the systems under consideration, the branch points must also come into account as another kind of singularities. It is shown that a multivalued transfer function can behave unstably because of the numerator term while it has no unstable poles. So, in this case, not only the characteristic equation but the numerator term is of significant importance. In this manner, a family of unstable fractional-order transfer functions is introduced which exhibit essential instabilities, that is, those which cannot be removed by feedback. Two illustrative examples are presented; the transfer function of which has no unstable poles but the instability occurred because of the unstable branch points of the numerator term. The effect of unstable branch points is studied and simulations are presented.

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1. Introduction

The stability problem of linear systems has been the subject of many studies. In the field of classical linear time-invariant (LTI) systems, the well-known Routh-Hurwitz criterion is widely used for testing the stability of a given rational transfer function. In the literature, the system stability assessment of a given delayed system is usually performed with a graphical method, for example, Nyquist criterion [1], Mikhailov criterion [2], and the root-locus technique [3]. A common observation is that the open-loop stability can be examined by investigating the denominator of the transfer function for closed right half-plane (RHP) roots.

Although most LTI systems can be represented by rational transfer functions (possibly with delay) but there are also some important exceptions. For example,

$$H(s) = \frac{\tanh\sqrt{s}}{\sqrt{s}} \tag{1.1}$$

appears in a boundary controlled and observed diffusion process in a bounded domain [4]. The transfer function

$$H(s) = \frac{\cosh(\sqrt{s}x_0)}{\sqrt{s}\sinh\sqrt{s}}, \quad 0 < x_0 < 1, \tag{1.2}$$

corresponds to the heat equation with Neumann boundary control [4]. As a general observation, systems governed by the heat equation commonly lead to multivalued transfer functions; the domain of definition for which is a Riemann surface with two Riemann sheets where the origin is a branch point of order one. As another example,

$$H(s) = \frac{1}{\sqrt{s^2+1}} \tag{1.3}$$

is the transfer function of the causal Bessel function of the first kind and of order zero $J_0(t)$ [5]. The transfer function

$$H(s) = \frac{2\Phi(s)}{s+\Phi(s)}e^{s-\Phi(s)}, \quad \varepsilon > 0, \tag{1.4}$$

where

$$\Phi(s) = \sqrt{s^2 + \varepsilon s^{3/2} + 1}, \tag{1.5}$$

is involved in the description of a 1D wave equation in a flared duct of finite length, with viscothermal losses at the boundary [6]. The transfer function

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs^{1/2} + k} \tag{1.6}$$

corresponds to the fractionally damped model where m, c, and k represent the mass, damping, and stiffness, respectively, $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is the externally applied force, and $x(t) = \mathcal{L}^{-1}\{X(s)\}$ is the displacement [7]. Many other real-world examples of the similar type can be found in [4, 8]. All transfer functions that contain noninteger powers of the Laplace variable, s, are commonly entitled *fractional-order systems*.

Systems such as those described by (1.1)–(1.6) are of multivalued nature and this makes the related studies a challenging task. For example, the stability testing of such systems is not yet fully addressed. In the field of fractional-order systems, the most well-known analytic stability test is the one available for the particular case of fractional-order systems commonly known as *fractional differential systems of commensurate order* [5, 8, 9]. The test resembles the stability condition for classical systems and it concludes that a system described by the multivalued transfer function

$$H(s) = \frac{b_0 s^{m\alpha} + b_1 s^{(m-1)\alpha} + \cdots + b_m}{s^{n\alpha} + a_1 s^{(n-1)\alpha} + \cdots + a_n}, \tag{1.7}$$

where $m, n \in \mathbb{N},\ \alpha \in (0,1)$, and $a_k, b_l \in \mathbb{R},\ (k = 1, \ldots, n;\ l = 0, \ldots, m)$, is stable if and only if the roots of the equation

$$w^n + a_1 w^{n-1} + \cdots + a_n = 0 \tag{1.8}$$

lie in the sector defined by

$$|\arg(w)| > \frac{\pi}{2}\alpha, \tag{1.9}$$

where $w = s^\alpha$. The proof of the above fact is based on studying the asymptotic behavior of the inverse Laplace transform of the basic element $(s^\alpha - \lambda)^{-j}$ where $j \in \mathbb{N},\ \alpha \in \mathbb{R}^+$, and $\lambda \in \mathbb{C}$. See also [10, 11] for recent developments on this subject.

In the literature, most studies on the infinite-dimensional LTI systems are focused on the stability of a class of distributed systems whose transfer functions involve $\sqrt{s}$ and/or $e^{-\sqrt{s}}$. The former group of studies relies on using Pontryagin's theory of quasipolynomials [12], coprime factorization together with Nyquist-like criterion [13], pseudodelay transformation [14], and Routh-like algorithm [15]. A numerical algorithm for stability testing of fractional delay systems can be found in [16].

The purpose of this paper is to present the necessary and sufficient conditions for the stability of certain class of fractional-order transfer functions. This class is identified by those multivalued transfer functions which are defined on a Riemann surface with limited number of Riemann sheets. It is shown in the paper that not only the poles but also the branch points are crucial in determining the stability. It concludes that not only a pole and/or a singularity originated from the characteristic equation, but the branch points in the numerator of a given fractional-order transfer function are also important for the stability analysis.

The rest of this paper is divided to three sections as follows. An overview of some important properties of the multivalued (complex) functions is presented in Section 2. Section 3 contains a theorem that provides the necessary and sufficient condition for the stability of multivalued transfer functions. Two illustrative examples are presented in Section 4. Finally, Section 5 concludes the paper.

2. Preliminaries and background

In this section, a review of the most important features of multivalued functions is presented, which will be instrumental in what follows. Most of the following results can be found in textbooks that study the concept of Riemann surface and multivalued functions (see i.e., [17, 18]).

The multivalued functions under consideration are defined on a Riemann surface which consists of finite number of Riemann sheets. These sheets are separated by branch cuts (BC) and each sheet has only one edge at a BC. Branch cuts are the boundaries of discontinuity for the multivalued function. It is also assumed that the transfer functions under consideration have two branch points (BPs) which belong to all Riemann sheets. The two BPs connect the two endings of a BC. It is a fact that BPs are unique but BCs are not. A BC can be any simple arc connecting the two BPs. It is often selected as the simplest curve which is a straight line connecting the two BPs. A BC on the $\mathbb{R}^-$ is usually selected when there is a BP at origin and another BP at infinity. Such a selection signifies the causality of the signal in the time domain.

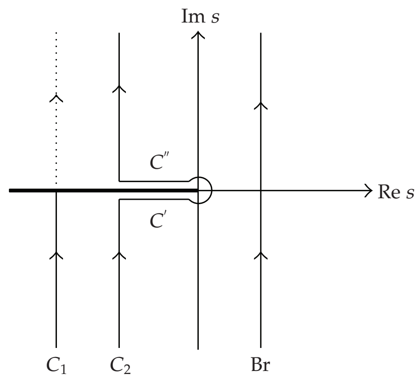

Figure 1: The complex plane with a BP at origin and a BC at $\mathbb{R}^-$.

It is a well-known fact that only the singularities of a multivalued function in the first Riemann sheet are of physical significance [19]. We also confine our study in this paper to the first Riemann sheet. The symbols C_+ and C_- stand for the closed RHP and the open RHP of the first Riemann sheet, respectively.

One important fact is that the integrations in opposite directions along opposite sides of a BC are not canceled due to the very discontinuity of the function on the BC. For example, Figure 1 shows a case where the BC is $\mathbb{R}^-$ and connects the two BPs at zero and at infinity. In such a case, the integration

$$\int_{C'} + \int_{C''} \tag{2.1}$$

is not necessarily zero. The BP of the multivalued function $F(s)$ at infinity (if any) can be investigated by examining $F(1/s)$ at origin. For instance, $F(s) = s^{-1/2}$ has BPs of order one at $s = 0$ and $s = \infty$.

The best way to locate the BPs of a multivalued function is to use the property that a multivalued function has fewer values at a BP than at other points. For instance, the function

$$F(s) = \left[s + \left(s^2 - 1\right)^{1/2}\right]^{1/2} \tag{2.2}$$

is four-valued, in general, but it reduces to a double-valued at $s = \pm 1$. Thus, the points $s = \pm 1$ are BPs depending on which sheet they are in.

In multivalued functions, a point at which the function becomes infinite is not necessarily a pole. For example, consider $F(s)$ as

$$F(s) = \frac{1}{s^{1/2}}, \tag{2.3}$$

which has a BP at $s = 0$. Although this function becomes infinite at the BP, this is not a pole. To find the reason, consider the integral

$$\int_C \frac{1}{s^{1/2}} ds, \tag{2.4}$$

where C is a counterclockwise closed curve encircling the origin once. For simplicity, consider the curve as a circle of radius ρ. Assuming $s = \rho e^{i\phi}$, it follows that

$$\int_C \frac{1}{s^{1/2}} ds = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\rho} e^{i\phi/2}} \left(i\rho e^{i\phi}\right) d\phi = i4\sqrt{\rho}. \tag{2.5}$$

It is observed that the integral around BP approaches zero as the radius of integration approaches zero. This would not be true for integration around a pole. We will use the following definition to deal with the singularities at origin.

Definition 2.1. The multivalued transfer function $F(s)$ is said to have a pole at origin if $\lim_{s\to 0} F(s) = \infty$.

Let $h(t)$ denote the impulse response of an LTI causal system. Then its Laplace transform $H(s)$ (the system transfer function) is defined as

$$H(s) \triangleq \int_0^{\infty} h(t) e^{-st}\, dt. \tag{2.6}$$

The set of all points on the fist Riemann sheet for which the Laplace integral (2.6) is absolutely convergent is called the region of convergence (ROC), that is, $s = \sigma + i\omega$ belongs to ROC if

$$\int_0^{\infty} |h(t) e^{-st}| dt = \int_0^{\infty} |h(t)| e^{-\sigma t}\, dt < \infty. \tag{2.7}$$

It is obvious that the ROC of (2.6) is a half-plane to right of the abscissa of convergence σ_c. The left-hand boundary of ROC is a line parallel to the imaginary axis.

Cares must be taken when calculating the inverse Laplace transform of a given multivalued function $F(s)$. The ROC (by definition in the first Riemann sheet) must be chosen equal to the right half-plane of the rightmost singularity (either pole or BP). For example, the ROC of

$$F(s) = \frac{e^{-\sqrt{s-1}}}{s+2} \tag{2.8}$$

must be chosen as $\Re\{s\} > 1$. After determining the ROC of $F(s)$, its inverse Laplace transform $f(t)$, can be calculated from

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\mathrm{Br}} F(s) e^{\mathrm{st}} ds, \tag{2.9}$$

where Br is the Bromwich contour considered in the ROC. Note that choosing the Br contour in the ROC (as described before) guarantees the zero result for $t < 0$. Figure 1 shows the Br contour of a multivalued function the rightmost singularity of which is a BP at origin. We are not authorized to use a contour that intersects the BC because such a contour enters to the next Riemann sheet and has a discontinuity at the BC (see C_1 in Figure 1).

3. Extension of the concept of stability

The following theorem presents the necessary and sufficient condition needed for the stability of a multivalued transfer function.

Theorem 3.1. *A given multivalued transfer function is stable if and only if it has no pole in C_+ and no BP in C_-.*

Proof. Assume the class of bounded input signals $u \in L_\infty$, that is, $\max_t\{|u(t)|\} < \infty$. The system is stable if for every input $u \in L_\infty$, the output $y(t) = u(t)*h(t) = \int_0^\infty h(\tau)u(t-\tau)d\tau$ is also bounded, that is, $y \in L_\infty$. It is then easy to prove that for a causal LTI system with impulse response $h(t)$ to be BIBO stable (as defined above), the necessary and sufficient condition is that $h \in L_1$ [20], that is,

$$\int_0^\infty |h(t)|dt < \infty. \tag{3.1}$$

Comparing to (2.7), $h(t)$ corresponds to a stable system if and only if the ROC of $H(s)$ includes the imaginary axis. According to the previous discussions, it will be the case if and only if $H(s)$ has no pole in C_+ and no BP in C_- (because, else the Laplace integral will not be convergent). This completes the proof. □

Note that this theorem cannot deal with the stability of the general case of the fractional-order systems. Interesting outcome of Theorem 3.1 is that an LTI system without any unstable pole may be unstable because of the unstable BP. It also implies that in dealing with fractional-order transfer functions, not only the denominator but also the numerator is important from the stability point of view. In fact, an LTI system can be unstable just because of an unstable BP appeared in the numerator term. It is concluded from Theorem 3.1 that the instabilities caused by unstable BPs cannot be removed by feedback because the transfer function of the resulted system will unavoidably have the same unstable BP. For that reason, such instabilities can be called *essential instabilities*.

Most of the practical fractional-order transfer functions have BPs at the origin (and infinity) but unstable BPs are not trivial. In the following, we study the propagation of an electric signal through a special electric line of length l. We show that such a system has an essential instability, that is, an unstable BP. Figure 2 shows the model of the (unbalanced) transmission line under consideration which applies the negative resistor. Per unit of length, the inductance is L, the capacity is C, and the conductance is G. It can also be easily verified that per unit of length the resistance is $R - R_N$. Kirschoff's laws read

$$\begin{aligned} L\frac{\partial i}{\partial t} &= -(R-R_N)i - \frac{\partial v}{\partial x}, \\ C\frac{\partial v}{\partial t} &= -\frac{\partial i}{\partial x} - Gv, \end{aligned} \tag{3.2}$$

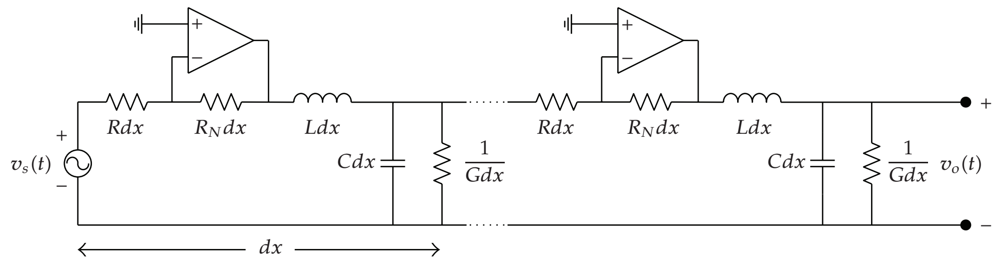

Figure 2: A transmission line with negative resistor.

where $0 \leq x \leq l,\ t \geq 0$, and the boundary conditions are

$$v(0,t) = v_s, \quad i(l,t) = 0. \tag{3.3}$$

Straight calculations yield

$$\frac{V_o(s)}{V_s(s)} = \frac{1}{\cosh\left(l\sqrt{(R - R_N + Ls)(G + Cs)}\right)}. \tag{3.4}$$

The above transfer function has BPs at $s_1 = (R_N - R)/L$ and $s_2 = -G/C$. Obviously s_1 corresponds to an unstable BP if $R_N > R$.

4. Examples

In this section, two numerical examples are presented to confirm the validity of Theorem 3.1. The effect of an unstable BP in the numerator term on system instability is studied in the following examples.

Example 4.1. Consider a system with transfer function

$$H(s) = \frac{(s-1)^{1/2}}{s^2 + 2s + 2}, \tag{4.1}$$

the domain of definition for which is a Riemann surface with two Riemann sheets where $s = 1$ is a BP of order one. $H(s)$ has four poles: $s_1 = \sqrt{2}e^{-j3\pi/4}$, $s_2 = \sqrt{2}e^{j3\pi/4}$, $s_3 = \sqrt{2}e^{j5\pi/4}$, and $s_4 = \sqrt{2}e^{j11\pi/4}$ where s_1 and s_2 are on the first Riemann sheet and s_3 and s_4 belong to the second Riemann sheet. Although there is no pole in C_+, it is concluded from Theorem 3.1 that $H(s)$ is unstable because of a BP in C_-. In the following, the instability caused by the BP at $s = 1$ is verified by direct calculation of the system impulse response.

In order to calculate the inverse Laplace transform of (4.1), consider the contour Γ shown in Figure 3. It is concluded from the residue theorem that

$$\oint_\Gamma H(s)e^{st}ds = \int_{\mathrm{Br}} + \int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{c_4} \tag{4.2}$$

$$= 2\pi i(\mathrm{Res}_{s=s_1} + \mathrm{Res}_{s=s_2}). \tag{4.3}$$

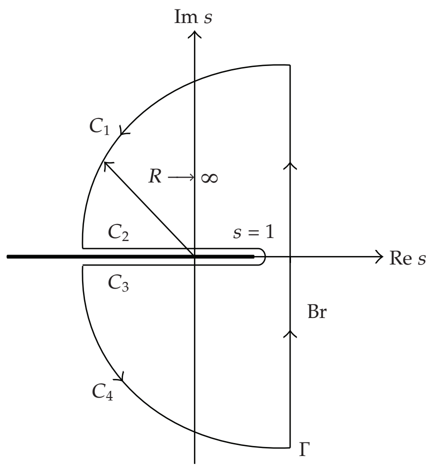

Figure 3: Integration path in Example 4.1.

It can easily be verified that

$$\int_{c_1} H(s)e^{st}\,ds = \int_{c_4} H(s)e^{st}\,ds = 0. \tag{4.4}$$

Some algebra yields

$$\int_{c_2} + \int_{c_3} = \int_0^\infty \frac{i2r^{1/2}e^{(1-r)t}}{r^2 - 4r + 5}dr, \tag{4.5}$$

$$\mathrm{Res}_{s=s_1} + \mathrm{Res}_{s=s_2} = \sqrt{5}\cos(t - 0.5\arctan 0.5)e^{-t} \tag{4.6}$$

$$= \sqrt{5}\cos(t - 13.3^\circ)e^{-t}. \tag{4.7}$$

Now, it is concluded from (4.3), (4.4), (4.5), and (4.7) that

$$h(t) = \sqrt{5}\cos(t - 13.3^\circ)e^{-t} - \frac{1}{\pi}\int_0^\infty \frac{r^{1/2}e^{(1-r)t}}{r^2 - 4r + 5}dr. \tag{4.8}$$

The term $\sqrt{5}\cos(t - 13.3^\circ)e^{-t}$ represents a damped oscillation but the integral

$$\int_0^\infty \frac{r^{1/2}e^{(1-r)t}}{r^2 - 4r + 5}dr \tag{4.9}$$

corresponds to an unstable function of time. To find out the reason of instability, the integrand is plotted in Figure 4 as a function of r for several values of t. As it is observed, the area under the plot is increased unboundedly by increasing time. Figure 5 shows the system impulse response which is given by (4.8).

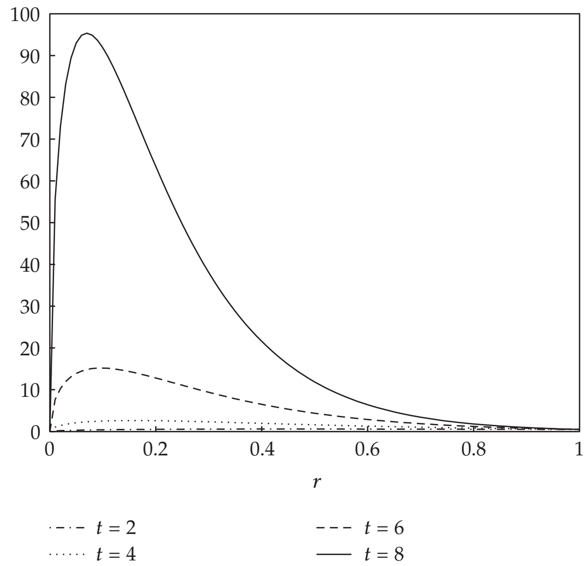

Figure 4: The integrand of (4.9) plotted versus r for various values of t.

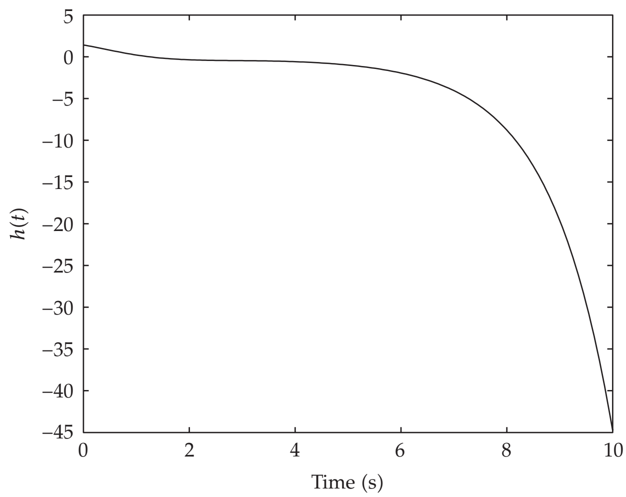

Figure 5: The impulse response of Example 4.1.

Example 4.2. Consider the LTI system governed by the integrodifferential equation

$$(y'' + 4y' + 4y) \star (y'' + 4y' + 4y) = (u' - u) \star (u' + u), \tag{4.10}$$

where $\star$ is the convolution operator. The above equation corresponds to the transfer function

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\sqrt{(s-1)(s+1)}}{(s+2)^2}. \tag{4.11}$$

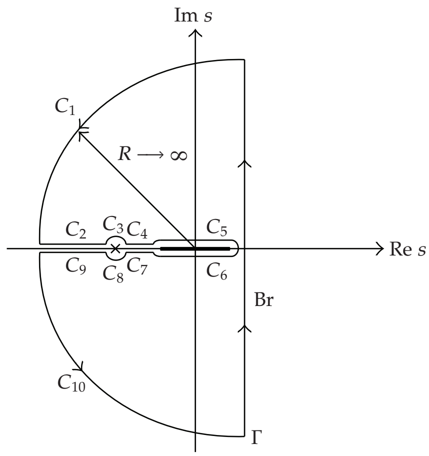

Figure 6: Integration path in Example 4.2.

The above transfer function has a stable BP at $s = -1$ and an unstable BP at $s = 1$. The BC is assumed to be the straight line connecting two BPs as depicted in Figure 6. Such a system is unstable according to Theorem 3.1. In order to calculate the system impulse response, consider the contour Γ shown in Figure 6. Since there is no pole inside the contour, it is concluded from the residue theorem that

$$\oint_{\Gamma} H(s)e^{st}ds = \int_{\mathrm{Br}} + \int_{C_1} + \cdots + \int_{C_{10}} = 0. \tag{4.12}$$

It can easily be verified that

$$\int_{C_1} = \int_{C_{10}} = 0. \tag{4.13}$$

Since there is no cut on $\Re\{s\} < -1$, we can write

$$\begin{gathered}\int_{C_2} + \int_{C_9} = 0, \\ \int_{C_4} + \int_{C_7} = 0.\end{gathered} \tag{4.14}$$

The residue theorem implies that

$$\int_{C_3} + \int_{C_8} = 2\pi i \mathrm{Res}\left[\frac{\sqrt{(s-1)(s+1)}}{(s+2)^2},\ s = -2\right] = 0. \tag{4.15}$$

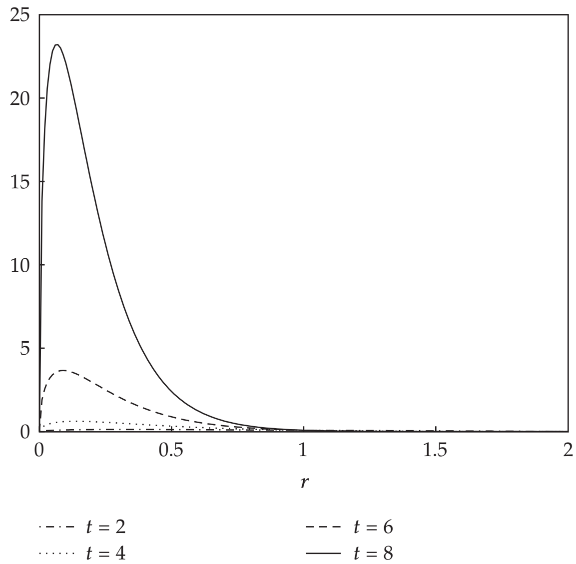

Figure 7: The integrand of (4.19) plotted versus r for various values of t.

In order to do the integration along C_5 and C_6, let us do the parameterization $s = re^{\pm i\pi} \pm i\delta + 1$ where positive and negative signs correspond to C_5 and C_6, respectively, $r \in (0, 2)$ and δ is a small positive number which tends to zero. Some algebra yields

$$\int_{C_5} + \int_{C_6} = \int_{r=2}^{0} \frac{i\sqrt{r(2-r)}}{(3-r)^2} e^{(1-r)t}(-dr) + \int_{r=0}^{2} \frac{-i\sqrt{r(2-r)}}{(3-r)^2} e^{(1-r)t}(-dr), \tag{4.16}$$

or equivalently

$$\int_{C_5} + \int_{C_6} = \int_{r=0}^{2} \frac{i2\sqrt{r(2-r)}}{(3-r)^2} e^{(1-r)t} dr. \tag{4.17}$$

It is concluded from (4.12), (4.13), (4.14), (4.15), and (4.17) that

$$h(t) = \frac{1}{i2\pi} \int_{\mathrm{Br}} H(s) e^{st}\, ds \tag{4.18}$$

$$= -\int_{r=0}^{2} \frac{\sqrt{r(2-r)}}{\pi(3-r)^2} e^{(1-r)t}\, dr. \tag{4.19}$$

The integrand of (4.19) is plotted in Figure 7 as a function of r for various values of t which indicates that the integrand is an unbounded function of time. The system impulse response is shown in Figure 8 which exhibits instability as expected.

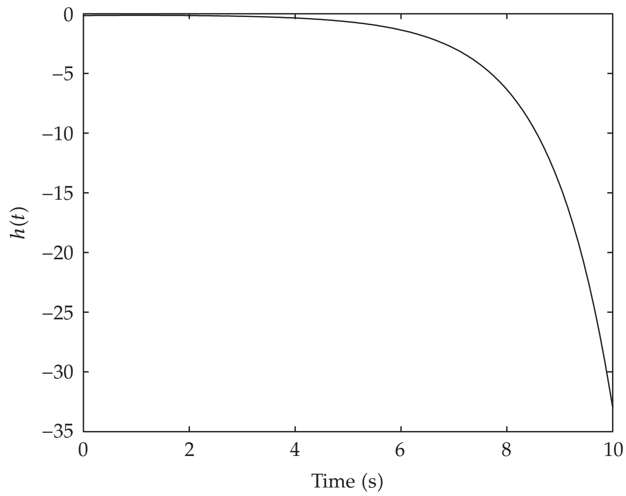

Figure 8: The impulse response of Example 4.2.

5. Conclusion

The stability of a certain class of fractional-order (multivalued) transfer functions is studied. A stability theorem is proved which is applicable for open-loop stability testing of many fractional-order transfer functions. It is shown that in dealing with the stability of multivalued transfer functions, the numerator term is as important as the characteristic function. The effect of unstable BPs is studied and two illustrative examples are presented which are unstable because of the unstable BP appeared in numerator. A family of challenging instabilities is introduced which are called essential instabilities in this paper.

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