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Research Article

Stability Results for Switched Linear Systems with Constant Discrete Delays

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This paper investigates the stability properties of switched systems possessing several parameterizations (or configurations) while being subject to internal constant point delays. Some of the stability results are formulated based on Gronwall's lemma for global exponential stability, and they are either dependent on or independent of the delay size but they depend on the switching law through the requirement of a minimum residence time. Another set of results concerned with the weaker property of global asymptotic stability is also obtained as being independent of the switching law, but still either dependent on or independent of the delay size, since they are based on the existence of a common Krasovsky-Lyapunov functional for all the above-mentioned configurations. Extensions to a class of polytopic systems and to a class of regular time-varying systems are also discussed.

1. Introduction

Time-delay systems are gaining important attention in the last years. The reason is that they offer a very significant modelling tool for dynamic systems since a wide variety of physical systems possess delays either in the state (internal delays) or in the input or output (external delays). Examples of time-delay systems are war/peace models, biological systems, like, for instance, the sunflower equation, Minorsky's effect in tank ships, transmission systems, teleoperated systems, some kinds of neural networks, and so forth (see, e.g., [1–12]). Time-delay models are useful for modelling both linear systems (see, e.g., [1–4, 13]) and certain nonlinear physical systems (see, e.g., [4, 7–9, 14]). A subject of major interest in time-delay systems, as it is in other areas of control theory, is the investigation of the stability as well as the closed-loop stabilization of unstable systems, [2–4, 6–9, 13–16] either with delay-free controllers or by using delayed controllers. Dynamic systems subject to internal

delays are infinite—dimensional by nature so that they have infinitely many characteristic zeros. Therefore, the differential equations describing their dynamics are functional rather than ordinary. Recent research on time-delay systems is devoted to numerical stability tests, to stochastic time-delay systems, diffusive time-delayed systems, medical and biological applications, [17–20], and characterization of minimal state-space realizations, [21]. Another research field of recent growing interest is the investigation in switched systems including their stability and stabilization properties. A general insight to this problem is given in [22–24]. Switched systems are related to the fact that a system possesses several distinct parameterizations and commutes in between them through time according a certain switching rule. The problem is relevant in applications since the corresponding models are useful to describe changing operating points or relevant to synthesize different controllers which can adjust to operate on a given plant according to situation of changing parameters, dynamics and so forth. Specific related problems are the following.

(1) The nominal order of the dynamics changes according to the frequency content of the control signal since fast modes are excited with fast input where they are not excited under slow controls. This circumstance can imply the need to use different controllers through time.

(2) The systems parameters are changing so that the operation point changes. Thus, a switched model being adjusted to several operation points may be useful, [22–24].

(3) The adaptation transient has a bad performance due to a poor estimates initialization due to very imprecise knowledge of the true parameters. In this case, a multiparameterized adaptive controller, whose parameterization varies through time governed by a parallel multiestimation scheme, might improve the whole system performance. For this purpose, the parallel multiestimation scheme selects trough time, via a judicious supervision rule, the particular estimator associated with either the best identification, tracking or mixed identification/tracking objectives. Such strategies can improve the switched system performance compared to the use of a single estimator/controller pair [5, 25]. The asymptotic stability of switched system has been investigated exhaustively along the last decade (see, e.g., [22–24] and references there in). However, parallel general results for switched time-delay systems are not abundant in the literature. Stability results of time-delay switched systems have been obtained recently for the case of one single delay by decomposition of the dynamics into a sum of a linear ordinary differential equation and a linear delay differential equation, [12]. Related results have also been obtained by adapting switching rules for ordinary differential equations to those describing time-delay systems, [11]. Some further recent related research investigates time-varying systems under point delays. In that context, switches are considered to produce an impulsive time derivative of the matrix of dynamics, [26]. This paper is devoted to the investigation of the stability properties of switched systems subject to internal constant point delays. A first package of results is concerned with the global exponential stability of switched systems either independent of or dependent on the delay sizes based on the use of Gronwall's lemma, [27]. Exponential stability independent of the delay size is proved under the restriction of small dynamics (characterized in terms of norms) of the delayed dynamics provided that the current delay free matrix of dynamics is stable. Exponential stability dependent on the delay size is proved under the restriction of sufficiently small delays provided that the delay-free system is exponentially stable. Note that it is well known that both of them have to be stable for any linear time-invariant configuration in order that the corresponding time-delay system may be asymptotically stable, [1, 4, 13]. It has to be pointed out that Gronwall's lemma has been chosen as elementary analysis tool for exponential stability of switched systems since Lyapunov techniques are direct only for discussing the weaker property of asymptotic stability in the case of time-delay

systems since either the Lyapunov functional, its time-derivative, or both depend on the state-squared norm on a time interval depending on the delay size rather than on each current time instant. A minimum residence time at each particular configuration, which depends on the parameterization, is required in order to guarantee exponential stability for this first set of stability results. A second package of weaker stability results concerning asymptotic stability is also discussed based on the use of "ad hoc" Krasovsky-Lyapunov's functionals, [1]. Contrarily to the first package of results, those of the second one do not require the maintenance of a minimum residence time between any two consecutive switching instants since it is assumed that all the parameterizations possess a common Krasovsky-Lyapunov functional. Basically, four types of systems are included in different sections of this paper, namely: (a) all the configurations are time invariant with identical delays, (b) the various distinct parametrical configurations can eventually posses distinct delays, (c) the system has a polytopic structure with the vertices being associated with limit configurations, and (d) the various system parameters vary continuously through time under the restriction that the dynamics possesses time derivative almost everywhere and there are finite parametrical jumps of sufficiently small sizes at isolated time instants. In this case, asymptotic stability is guaranteed if the jumps occur at sufficiently large intervals compared to available bounds depending on other parameters of the dynamic system establishing worst-case situations for the stability degree.

2. Stability results based on the use of Gronwall's lemma and matrix measures

Consider the nth linear and time-invariant dynamic system with internal (i.e., in the state) delayed dynamics with constant discrete (or point) delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \tag{2.1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector and $A_i \in \mathbf{R}^{n\times n}$ $(i = 0,1)$ are matrices of delay-free and delayed dynamics, respectively, for delay $h \in [0,\overline{h}]$, either for some admissible delay upper-bound $0 \le \overline{h} < \infty$ or for all $h \in [0,\infty)$. The first characterization will lead to stability dependent on the delay size results while the second one will lead to results about stability independent of the delay size. The initial condition is defined by any piece-wise absolutely continuous function $\varphi : [-h,0] \to \mathbf{R}^n$ with $\varphi(0) = x(0) = x_0$. The generalization of potential results for (2.1) for more general systems involving q delays $0 = h_0 < h_{i-1} < h_i$ $(i \in \overline{q} := \{1,2,\ldots,q\})$ of the form

$$\dot{x}(t) = \sum_{i=0}^{q} A_i x\left(t-h_i\right) \tag{2.2}$$

is direct, and some related results will be discussed in Section 3. Thus, the attention is focused on the behavior and properties of (2.1). The solution of (2.1) over $\mathbf{R}_{0+} := \{\mathbf{R} \ni t \ge 0\}$ is unique for each given such a function of initial conditions from the well-known Picard-Lindeloff uniqueness theorem [27]. One expression of such a solution may be calculated via the superposition principle for linear systems with the unforced solution calculated as associated with the C_0-semigroup of infinitesimal generator A_0 with the particular correction due to the forcing term $A_1 x(t-h)$. If $h = 0$, then (2.1) becomes $\dot{x}(t) = (A_0 + A_1)x(t)$, so that

(2.1) is delay free. Norm-upper bounds of the solution of (2.1) through time can be computed by adapting the well-known Gronwall's lemma [27] to calculate a norm upper-bound for the solution of (2.1). Note thatGronwall's lemma is a useful tool to prove exponential stability either under small delays or for the case of small norms of the matrices of delayed dynamics provided that the delay-free system counterpart or the system without delay dynamics matrices, respectively, are stable. Exponential stability is a stronger property than asymptotic stability which is the one usually proved for time-delay systems with Lyapunov stability tools. The following results are proved in Appendix A and extend their weaker previous statement for asymptotic stability (see, e.g., [1]). They are of interest for the subsequent study related to results on exponential stability under switching among distinct parameterizations of time-delay systems with point delays.

Theorem 2.1. *Assume that* (A_0+A_1) *is a stable matrix (i.e., all its eigenvalues are in* $\operatorname{Re} s < 0$*). Then, it exists a maximum allowable delay* $\overline{h} \in \mathbf{R}_+ := \{\mathbf{R} \ni t \geq 0\}$ *such that* (2.1) *is globally exponentially stable for any delay* $h \in [0, \overline{h})$.

Theorem 2.2. *Assume that* A_0 *is a stable matrix. Then,* (2.1) *is globally exponentially stable independent of the delay (i.e., for any delay* $h \in \mathbf{R}_{0+}$ *provided that* $\|A_1\|$ *is sufficiently small compared to the stability abscissa of* A_0*). For* ℓ_2 *(spectral)-norms, a related specific testable condition is* $\|A_1\|_2 < \rho_0$*, for any real constant* $\rho_0 \in \mathbf{R}_+$ *satisfying* $\rho_0 \in (0, |\mu_2(A_0)|)$*, where* $\mu_2(A_0) := (1/2)\max_{1\leq i\leq \xi} \operatorname{Re} \lambda_i(A_0 + A_0^T)$ *is the matrix measure of* A_0 *with respect to the* ℓ_2*-norm, and* $\lambda_i(A_0 + A_0^T)$ $(i \in \overline{\xi} := \{1, 2, \ldots, \xi\}, \xi \leq n)$ *denote the distinct eigenvalues of such a matrix.*

If the condition of Theorem 2.2 is relaxed to $\|A_1\|_2 \leq \rho_0$, then the time-delay system is guaranteed to be globally Lyapunov's stable since $-\rho_0 < 0$. Theorem 2.3(i) below states that stability independent of delay implies that A is a stable matrix, related to the case of arbitrarily large delay while $(A_0 + A_1)$ is also a stable matrix which is related to the case of zero delay [1]. Theorem 2.3(ii) is to some extent the converse of Theorem 2.2. It is proved that if $\|A_1\|_2 \geq |\mu_2(A_0)|$, with $\mu_2(A_0) := (1/2)\max_{1\leq i\leq \xi} \operatorname{Re} \lambda_i(A_0 + A_0^T) < 0$ being the matrix measure of A_0 with respect to the ℓ_2-norm, then some dynamic system of the same structure as (2.1), eventually (2.1) itself, and some matrix of delayed dynamics of the same ℓ_2-norm as that of A_1 becomes at least critically stable, or even instable, for small delay size. As a result, it cannot be globally exponentially stable even if the delay-free system matrix A_0 is a stable matrix.

Theorem 2.3. *The following properties hold.*

(i) *If* $\mu_2(A_0) < 0$ *and* $\|A_1\|_2 < |\mu_2(A_0)|$*, then* (2.1) *is globally exponentially stable independent of the delay size and both* A_0 *and* $(A_0 + A_1)$ *are stable matrices.*

(ii) *If* A_0 *is a stable matrix and* $\|A_1\|_2 \geq |\mu_2(A_0)|$*, then there exists a system* (2.1) *with a matrix* A_1^* *fulfilling* $\|A_1\|_2 = \|A_1^*\|_2$ *(so that* $\mu_2(A_0) + \|A_1^*\|_2 \geq 0$*) such that the system* $\dot{x}(t) = A_0 x(t) + A_1^* x(t-h)$ *is either critically stable or unstable.*

3. Switching among distinct parameterizations

Consider the nth linear and time-invariant dynamic *switched system* with internal delayed dynamics:

$$\dot{x}(t) = A_{0\sigma(t)}(t)x(t) + A_{1\sigma(t)}(t)x(t-h), \tag{3.1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector and $A_{i\sigma} : \mathbf{R}_{0+} \to \{A_{i1}, \ldots, A_{ip}\} \subset \mathbf{M}(\mathbf{R}^{n\times n})$; $i = 0, 1$; $j \in \overline{p} := \{1, 2, \ldots, p\}$ are matrix functions of piecewise constant entries of delay-free and delayed dynamics, respectively, for some delay $h \in [0, \overline{h}]$ and some admissible delay upper-bound $\overline{h} \in \mathbf{R}_+$, or for $h \in [0, \infty)$, where $\mathbf{M}(\mathbf{R}^{n\times n})$ denotes the set of real-square matrices of n order. In other words, $A_{i\sigma(t)} \in \{A_{i1}, \ldots, A_{ip}\} \subset \mathbf{M}(\mathbf{R}^{n\times n})$; $i = 0, 1$. The piecewise constant real function $\sigma : \mathbf{R}_{0+} \to \overline{p} \subset \mathbf{N}$ is the *switching law* defined by $\sigma(t) = j \in \overline{p}$ for all $t \in [t_k, t_{k+1})$, and $\{t_k \in \mathbf{R}_{0+}\}_{k\in\mathbf{N}_s}$ is a strictly increasing real sequence of *switching instants* and $\mathbf{N}_s \subset \mathbf{N}$ is the *switching indicator* of either finite cardinal (i.e., the switching process stops in finite time) or infinity cardinal (i.e., the switching process never ends). The initial condition function is defined by any given piece-wise absolutely continuous function $\varphi : [-h, 0] \to \mathbf{R}^n$ with $\varphi(0) = x(0) = x_0$.

3.1. Constrained switching

Some results of Section 3, obtained for a single parameterization, are directly extendable to the case of switched systems for constrained switching between different parameterization of the same structure as that of (2.1). The switching law is required to respect a minimum residence (or dwelling) time between any two consecutive switches. The following result, proved in Appendix B, extends Theorem 2.1 for sufficiently small delay.

Theorem 3.1. *Assume that $(A_{0j} + A_{1j})$ are stable matrices for all $j \in \overline{p}$. Thus, there exists a maximum allowable delay $\overline{h} \in \mathbf{R}_+ := \{\mathbf{R} \ni t \geq 0\}$ such that* (3.1) *is globally exponentially stable for any delay $h \in [0, \overline{h})$ for any switching law $\sigma : \mathbf{R}_{0+} \to \overline{p} \in \mathbf{N}$ fulfilling $\sigma(t) = j \in \overline{p}$ for all $t \in [t_k, t_k + \overline{h} + T)$ with $T \in \mathbf{R}_+$ being a minimum residence time which depends on the set $\{A_{i1}, \ldots, A_{ip}\}$; $i = 0, 1$; with $\{t_k\}_{k\in\mathbf{N}_s}$ being the sequence of switching instants.*

Note that Theorem 3.1 holds also *if there are many infinite parameterizations* with $(A_{0j} + A_{1j})$ being stable matrices for all $j \in \mathbf{N}$ with a switching law $\sigma(t) = j \in \mathbf{N}$. Theorem 2.2 is now extended to the switched system (3.1) possessing internal delay of arbitrary size. Its proof is given in Appendix B. It is stated that global exponential stability holds if all the parameterizations are stable and an appropriate minimum residence time in between any two consecutive distinct parameterizations is respected. A minimum size of the residence time required for stability is calculated in the proof.

Theorem 3.2. *Assume that A_{0j} are stable matrices for all $j \in \overline{p}$. Thus,* (3.1) *is globally exponentially stable independent of the delay provided that the following hold.*

(1) *$\|A_{1j}\|$ for all $j \in \overline{p}$ are sufficiently small compared to the maximum stability abscissa of A_{0j} for all $j \in \overline{p}$.*

(2) *The switching law $\sigma : \mathbf{R}_{0+} \to \overline{p} \in \mathbf{N}$ which generates the switching instants fulfils $\sigma(t) = j \in \overline{p}$ for all $t \in [t_k, t_k + h + T)$ with $T \in \mathbf{R}_+$ being a minimum residence (or dwelling) time which depends on the set $\{A_{i1}, \ldots, A_{ip}\}$; $i = 0, 1$, with $\{t_k\}_{k\in\mathbf{N}_s}$ being the switching instants.*

For ℓ_2 (spectral)-norms, a related specific testable condition in Theorem 3.2 is $\max_{1\leq j\leq p}(-\rho_{0j} + \|A_{1j}\|_2) < 0$ for any real constant $\rho_{0j} \in \mathbf{R}_+$ satisfying $\rho_{0j} \in (0, |\mu_2(A_{0j})|)$ for all $j \in \overline{p}$, where $\mu_2(A_{0j}) := (1/2)\max_{1\leq i\leq \xi} \operatorname{Re} \lambda_i(A_{0j} + A_{0j}^T)$ for all $j \in \overline{p}$ is the measure of the matrix A_{0j} with respect to the ℓ_2-norm, and $\lambda_i(A_{0j} + A_{0j}^T)$ $(i \in \overline{\xi} := \{1, 2, \ldots, \xi\},\ j \in \overline{p})$ denote the distinct eigenvalues of such a matrix.

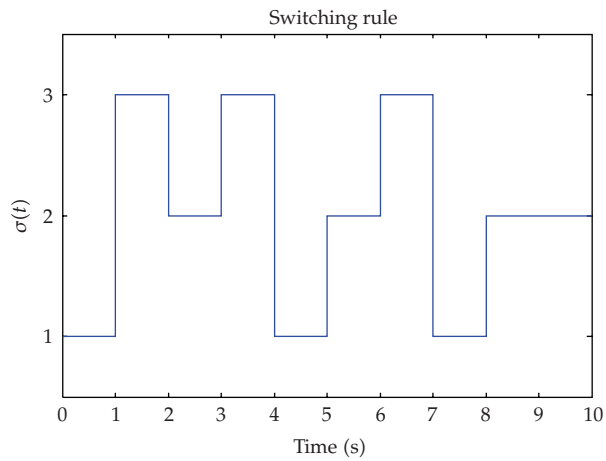

Figure 1: Switching rule evolution.

3.1.1. Simulation example 1

Consider the linear switched system (3.1) with $h = 0.75$ second, and three parameterizations:

$$A_{01} = \begin{pmatrix} -7 & -1 \\ 2 & -4 \end{pmatrix}, \quad A_{02} = \begin{pmatrix} -5.5 & 1 \\ -2 & -8 \end{pmatrix}, \quad A_{03} = \begin{pmatrix} -7 & 1 \\ -1.5 & -9 \end{pmatrix}, \quad A_{11} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix},$$
$$A_{12} = \begin{pmatrix} -2 & 1 \\ -1 & 1 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} -3 & 1 \\ -1 & -2 \end{pmatrix}. \tag{3.2}$$

According to Theorem 3.2, all the matrices A_{0j} are stable for $j = 1, 2, 3$, and $\mu_2(A_{01} + \|A_{11}\|_2 < -1.61 < 0$, $\mu_2(A_{02}) + \|A_{12}\|_2 < -2.8 < 0$, and $\mu_2(A_{03}) + \|A_{13}\|_2 < -3.7 < 0$. Also, note that the matrices A_{11} and A_{12} are not stable since it is not required in Theorem 3.2. The initial conditions are $\varphi_1(t) = 12500$, $\varphi_2(t) = 13500$, $t \in [-0.75, 0]$. They are taken of large amplitudes to visualize the exponential behavior of the solution under eight switches of this exponentially stable system with high stability degree. The residence time is given by $T = 1$ second. The simulated results are shown in Figure 2 for the switching rule of Figure 1.

3.2. Arbitrary switching

It is well known for the case of time-delay time-invariant systems that if all the parameterizations possess a common Lyapunov function, then the switching law can be arbitrary (i.e., without requiring a minimum residence time in contrast with Theorems 3.1, 3.2) while keeping global asymptotic stability. It has been also proved that the various parameterizations possess a common Lyapunov function independently of their stability abscissas if and only if all the matrices of dynamics pair-wise commute [2]. A stronger sufficient condition for existence of a common Lyapunov function which does not require pair-wise commutation is that the norm deviations among the various parameterizations be sufficiently small related to any of the stability abscissas. A generalization for the switched time-delay system (3.1) under sufficiently small delay is given below. Its proof is provided

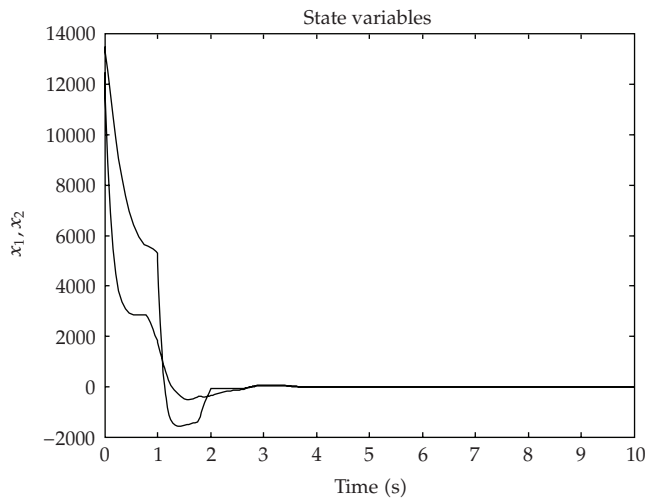

Figure 2: State variables evolution.

in Appendix B under the assumption that a sum of any number of matrices parameterizing the switched system (3.1) is a stable matrix and pair-wise commutes with its transpose. The global exponential stability of the switched system (3.1) for an arbitrary switching law is also guaranteed under the less restrictive assumption that all the above matrices pair-wise commute while being stable matrices.

Theorem 3.3. *The following properties hold.*

(1) *The switched system* (3.1) *is globally exponentially stable for any arbitrary switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p} \in \mathbf{N}$, *provided that the delay is sufficiently small if* $A_{0j} + A_{1j}$ *are all stable matrices such that* $(A_{0j} + A_{1j})$ *and* $(A_{0j} + A_{1j})^T$ *pair-wise commute for all* $j \in \overline{p}$.

(2) *The switched system* (3.1) *is globally exponentially stable for any arbitrary switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p} \in \mathbf{N}$, *provided that the delay is sufficiently small if* $(A_{0j} + A_{1j})$ *are all stable matrices and pair-wise commute for all* $j \in \overline{p}$.

An alternative condition for the existence of common Lyapunov functions follows by direct calculation for arbitrarily close parameterizations of the switching system. Define $\Delta A_{ikj} := A_{ik} - A_{ij}$ $(i = 0,1)$ for any $j \in \overline{p}$ and for all $k(\neq j) \in \overline{p}(\Delta A_{ijj} = 0$ $(i = 0,1$ for all $j \in \overline{p}))$ and let $\rho_j < 0$ be any real constant to the right of the stability abscissa of $(A_{0j} + A_{1j})$ and close to it. Since $(A_{0j} + A_{1j})$ is a stable matrix, for any positive definite square real n-matrix Q, $P = \int_0^\infty e^{(A_{0j}+A_{1j})^T\tau} Q e^{(A_{0j}+A_{1j})\tau} d\tau$ is a positive definite symmetric solution matrix to the Lyapunov equation $(A_{0j} + A_{1j})^T P + P(A_{0j} + A_{1j}) = -Q$ for all $j \in \overline{p}$. For all $k(\neq j) \in \overline{p}$,

$$\left(A_{0k} + A_{1k}\right)^T P + P\left(A_{0k} + A_{1k}\right) = -Q + \left(\Delta A_{0k} + \Delta A_{1k}\right)^T P + P\left(\Delta A_{0k} + \Delta A_{1k}\right). \tag{3.3}$$

The right-hand side of the above equation is guaranteed to be negative definite if

$$\lambda_{\min}(Q) > \frac{K_j \lambda_{\max}(Q)}{\rho_j} \left\| \Delta A_{0k} + \Delta A_{1k} \right\|_2 \geq 2\lambda_{\max}(P) \left\| \Delta A_{0k} + \Delta A_{1k} \right\|_2 \tag{3.4}$$

for all $k(\neq j) \in \overline{p}$, with $K_j \geq 1$ being such that $\|e^{(A_{0j}+A_{1j})t}\|_2 \leq K_j e^{-\rho_j t}$ for all $t \in \mathbf{R}_{0+}$, $\|\cdot\|_2$ denoting the ℓ_2 vector and corresponding (induced) matrix norm, and $\lambda_{\min}(M)$, $\lambda_{\max}(M)$ denoting the minimum and maximum eigenvalues of the real symmetric matrix $M = P, Q$. The last constraint is identical to

$$\lambda_{\min}(Q) > \frac{K_j \lambda_{\max}(Q)}{\rho_j} \left\| \Delta A_{0k} + \Delta A_{1k} \right\|_2 = \frac{K_j \lambda_{\max}(Q)}{\rho_j} \lambda_{\max}^{1/2}\left(\left(\Delta A_{0k} + \Delta A_{1k} \right)^T \left(\Delta A_{0k} + \Delta A_{1k} \right) \right) \tag{3.5}$$

for all $k(\neq j) \in \overline{p}$ which guarantees the existence of (at least) a common Lyapunov function for all the parameterizations of the switched system. More than one common Lyapunov function may exist if all the parameterizations are close to each other in terms of small norm deviations. The above constraint implies a physical one on the stability abscissas of the various parameterizations. Note that the choice $Q = I_n$ can be made and $K_j = 1$ in the better case. Thus, a common Lyapunov function for zero delay exists, even if not all the p parameterizations pair-wise commute, provided that

$$\rho_j > \max_{i \neq k \in \overline{p}} \lambda_{\max}^{1/2}\left(\left(\Delta A_{0k} + \Delta A_{1k} \right)^T \left(\Delta A_{0k} + \Delta A_{1k} \right) \right) \quad \forall j \in \overline{p}. \tag{3.6}$$

Two simple consequences of Theorem 3.3(i) follow as specific results.

Corollary 3.4. *The switched system* (3.1) *is globally exponentially stable for any arbitrary switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p} \in \mathbf{N}$ *provided that the delay is sufficiently small if* $(A_{0j} + A_{1j})$ *are all stable matrices for all* $j \in \overline{p}$ *and all the matrices in the set* $\{A_{0i}, A_{1i};\ i \in \overline{p}\}$ *pair-wise commute.*

The above result is direct since the commutation condition implies that of Theorem 3.3. The subsequent results refer to the feature that any two matrices of product compatible orders commute if and only if any one of them is a matrix function of the other, [2].

Corollary 3.5. *The switched system* (3.1) *is globally exponentially stable for any arbitrary switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p} \in \mathbf{N}$ *if and only if* $A_{0j} + A_{1j} = f_{ji}(A_{0i} + A_{1i})$ *for all* $j \in \overline{p}$, *and any given* $i \in \overline{p}$ *(i.e., all the matrices are function matrices on any particular one in the set) provided that the delay is sufficiently small.*

Estimations for the maximum sizes of the delay which guarantee global exponential stability are given in the proofs of the results. It has been proved formerly that two matrices of the same order commute if and only if any of them is a matrix function of the other [2, 24]. For any arbitrary switching law, the switching instants may be arbitrarily close and the switching from each current parameterization can occur to any other in the set of p parameterizations. Subsequently, sufficient-type conditions for exponential stability independent of the delay for arbitrary switching are discussed by using Krasovsky Lyapunov functionals [1] for stability testing. Denote that $x_t = x(t+\theta)$ for all $\theta \in [-h, 0]$. The subsequent notations $M > 0$, $M \geq 0$, $M < 0$, $M \leq 0$ mean that the real square matrix M is positive definite, positive semidefinite, negative definite, and negative semidefinite, respectively. Consider the Krasovsky-Lyapunov functional candidate defined with matrices $P = P^T > 0$,

$S = S^T > 0$ [1]:

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(\tau)Sx(\tau)d\tau. \tag{3.7}$$

Theorem 3.6. *The following properties hold.*

(1) *The switched system* (3.1) *is globally asymptotically stable independent of the delay size for any switching law* $\sigma : \mathbf{R}_0^+ \to \overline{p}$ *and each admissible function of initial conditions if*

$$Q_i := \begin{bmatrix} A_{0i}^T P + PA_{0i} + S & PA_{1i} \\ A_{1i}^T P & -S \end{bmatrix} < 0 \quad \forall i \in \overline{p}, \tag{3.8}$$

for some $\mathbf{R}^{n\times n} \ni P = P^T > 0$, $\mathbf{R}^{n\times n} \ni S = S^T > 0$. *For such a system, the functional* (3.7) *is a common Krasovsky-Lyapunov functional parameterization for its* p *distinct parameterizations.*

(2) *Consider the general polytopic switched system*

$$\dot{x}(t) = A_{0\sigma(t)}(t)x(t) + \sum_{j=1}^{q} \omega_{j\sigma(t)} A_{j\sigma(t)}(t)x(t-h_j), \tag{3.9}$$

where $x(t) \in \mathbf{R}^n$ *is the state vector,* $A_{i\sigma} : \mathbf{R}_{0+} \to \{A_{i1}, \ldots, A_{ij}\} \subset \mathbf{M}(\mathbf{R}^{n\times n})$; $i \in \overline{q} \cup \{0\}$; $j \in \overline{p}$ *are matrix functions defining the switched dynamics and the* h_j $(j \in \overline{q})$ *are the* q *delays (being eventually distinct) for each of the* p *distinct parameterizations, and* ω_{ji} *are either zero or unity scalars while* ω_{ji} *is unity for at least one* $i \in \overline{q}$ *for each* $j \in \overline{p}$. *Thus, if*

$$\overline{Q}_i := \begin{bmatrix} A_{0i}^T P + PA_{0i} + \sum_{i=1}^{q} S_i & \omega_{1i}PA_{1i} & \cdots & \omega_{qi}PA_{qi} \\ \omega_{1i}A_{1i}^T P & -S_1 & 0\cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ & \vdots & & \\ \omega_{qi}A_{qi}^T P & 0 & \cdots & -S_q \end{bmatrix} < 0 \quad \forall i \in \overline{p} \tag{3.10}$$

for some $\mathbf{R}^{n\times n} \ni P = P^T > 0$, $\mathbf{R}^{n\times n} \ni S_i = S_i^T > 0$ *(for all* $i \in \overline{q}$*), then the switched system* (3.9) *is globally asymptotically stable independent of the delays for any switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$ *and each admissible function of initial conditions. Also,*

$$\overline{V}(t, x_t) = x^T(t)Px(t) + \sum_{i=1}^{q} \int_{t-h_i}^{t} x^T(\tau)S_i x(\tau)d\tau, \tag{3.11}$$

with x_t *redefined by* $x(t) = x(t+\theta)$ *for all* $\theta \in [-\widehat{h}, 0]$ *and* $\widehat{h} := \max_{1\le i\le q}(h_i)$ *is a common Krasovsky-Lyapunov functional parameterization for its* p *distinct parameterizations.*

The proof of Theorem 3.6(1) follows straightforwardly by direct calculation noting since the time-derivative of (3.7) along any trajectory solution of (3.1) is $\dot{V}(t) = \overline{x}(t)Q_{\sigma(t)}\overline{x}(t) \leq -\eta\|\overline{x}(t)\|_2^2 < 0$ for some $\eta \in \mathbf{R}_+$, provided that $\overline{x}(t) := (x^T(t), x^T(t-h))^T \neq 0$, so that it is a negative quadratic form in $\mathbf{R}^{2n}$ since $Q_i < 0$ for all $i \in \overline{p}$. Theorem 3.6(2) follows since $\dot{\overline{V}}(t) = \overline{x}(t)\overline{Q}_{\sigma(t)}\overline{x}(t) \leq -\overline{\eta}\|\overline{x}(t)\|_2^2 < 0$ if $\overline{x}(t) := (x^T(t), x^T(t-h_1), \ldots, x^T(t-h_p))^T \neq 0$ for some $\overline{\eta} \in \mathbf{R}_+$, so that it is a negative quadratic form in $\mathbf{R}^{qn}$ since $\overline{Q}_i < 0$ for all $i \in \overline{p}$ from (3.10). Thus, $\overline{V}(t, x_t)$ (3.11) is a Krasovsky-Lyapunov functional for the switched system (3.9).

Remark 3.7. Theorem 3.6(2) applies when each of the p-switched parameterizations has a unique delay and all of them are distinct, that is, $q = p$, $\omega_{ii} = 1$, $\omega_{ij} = 0$, $h_i \neq h_j$ for all $j, i(\neq j) \in \overline{q}$. This situation means that switching implies modifications of the delays and that of the matrices of dynamics. Another particular case of Theorem 3.6(ii) is that dealt with in Theorem 3.6(1), that is, there is a single delay h for all the parameterizations, that is, $q = p$, $\omega_{ii} = 1$, $\omega_{ij} = 0$, $h = h_i$, $S = S_i$, for all $j, i(\neq j) \in \overline{p}$.

Remark 3.8. Note that Theorem 3.6 does not involve necessary stability conditions. The reason is that in the delay-free case of linear and time-invariant systems, the solvability Lyapunov matrix equation, and then the existence of its associate Lyapunov function is a necessary and sufficient condition for global asymptotic stability (which also implies and is implied for exponential stability) but the property does not extend to functional equations and Krasovsky-Lyapunov functionals like (3.7) or (3.11).

Theorem 3.6 extends for stability dependent on the delay size as follows.

Theorem 3.9. *The following properties hold.*

(1) *The switched system* (3.1) *is globally asymptotically stable for any delay* $h \in [0, \overline{h}]$ *and for any switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$ *and each admissible function of initial conditions if*

$$\widehat{Q}_i := \begin{bmatrix} (A_{0i}^T + A_{1i}^T)P + P(A_{0i} + A_{1i}) + \overline{h}(S_0 + S_1) & \overline{h}PA_{1i}M_i \\ \overline{h}A_{1i}^T PM_i & -\overline{h}S \end{bmatrix} < 0 \quad \forall i \in \overline{p}, \tag{3.12}$$

$$\begin{aligned} M_i &:= [A_{0i}, \ldots, A_{1i}] \in \mathbf{R}^{n\times 2n} \quad \forall i \in \overline{p}, \\ S &:= \operatorname{diag}(S_0, S_1) \in \mathbf{R}^{2n\times 2n} \end{aligned} \tag{3.13}$$

for some $\mathbf{R}^{n\times n} \ni P = P^T > 0$, $\mathbf{R}^{n\times n} \ni S_j = S_j^T > 0$ $(j = 1, 2)$. *The real functional,*

$$\widehat{V}(t, x_t) = x^T(t)Px(t) + \sum_{j=1}^{2} \int_{-jh}^{-(j-1)h} \int_{t+\theta}^{t} x^T(\tau)S_j x(\tau)d\tau\, d\theta, \tag{3.14}$$

is a common Krasovsky-Lyapunov functional for its p *distinct parameterizations.*

(2) *The switched system* (3.9) *is globally asymptotically stable independent of the delays for any switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$ *and all delays* $h_i \in [0, \widehat{h}_i]$; *(for all* $i \in \overline{q}$*)if*

$$\widehat{\overline{Q}}_i := \begin{bmatrix} \left(\sum_{j=0}^{q} A_{ji}^T\right)P + P\left(\sum_{j=0}^{q} A_{ji}\right) + \sum_{i=1}^{q}\sum_{j=0}^{q} \widehat{h}_i S_{ij} & \omega_{1i}\widehat{h}_1 P A_{1i} M_1 & \cdots & \omega_{qi}\widehat{h}_q P A_{qi} M_q \\ \omega_{1i}\widehat{h}_1 M_1^T A_{1i}^T P & -R_1 & 0\cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ & \vdots & & \\ \omega_{qi}\widehat{h}_q M_q^T A_{qi}^T P & 0 & \cdots & -R_q \end{bmatrix} < 0 \quad \forall i \in \overline{p}, \tag{3.15}$$

$$\begin{aligned} M_i &:= \left[A_{0i}, \ldots, A_{qi}\right] \in \mathbf{R}^{n\times(q+1)n}, \\ R_i &:= \operatorname{diag}\left(S_{i0}, S_{i1}, \ldots, S_{iq}\right) \in \mathbf{R}^{(q+1)n\times(q+1)n} \quad \forall i \in \overline{p}, \end{aligned} \tag{3.16}$$

for some $\mathbf{R}^{n\times n} \ni P = P^T > 0$, $\mathbf{R}^{n\times n} \ni S_{ij} = S_{ij}^T > 0$ *(for all* $i \in \overline{q}$*, for all* $j \in \overline{q} \cup \{0\}$*). The real functional,*

$$\widehat{\overline{V}}(t, x_t) = x^T(t)Px(t) + \sum_{i=1}^{q}\sum_{j=0}^{q}\int_{-h_i-h_j}^{-h_j}\int_{t+\theta}^{t} x^T(\tau)S_{ij}x(\tau)d\tau\, d\theta, \tag{3.17}$$

with $\mathbf{R}^{n\times n} \ni P = P^T > 0$, $\mathbf{R}^{n\times n} \ni S_{ij} = S_{ij}^T > 0$ *(for all* $i \in \overline{p}$*, for all* $j \in \overline{q} \cup \{0\}$*) is a common Krasovsky-Lyapunov functional for its* p *distinct parameterizations with* x_t *being redefined by* $x(t) = x(t+\theta)$ *for all* $\theta \in [-\widehat{h}, 0]$ *and* $\widehat{h} := \max_{1\le i\le q}(\widehat{h}_i)$.

The proof is direct since it is very close to that of Theorem 3.6. $\dot{\widehat{V}}(t) = \overline{x}(t)\widehat{Q}_{\sigma(t)}\overline{x}(t) \le -\widehat{\eta}\|\overline{x}(t)\|_2^2 < 0$ if $\overline{x}(t) := (x^T(t), x^T(t-h_1), \ldots, x^T(t-h_q))^T \neq 0$ for Theorem 3.9(1) and $\dot{\widehat{\overline{V}}}(t) = \overline{x}(t)\widehat{\overline{Q}}_{\sigma(t)}\overline{x}(t) \le -\widehat{\overline{\eta}}\|\overline{x}(t)\|_2^2 < 0$ for any nonzero $\overline{x}(t)$ for Theorem 3.9(2).

Remark 3.10. Note that if the conditions (3.8), (3.10), (3.12), and (3.15) of Theorems 3.6(1), (2) and 3.9(1), (2) are relaxed to $Q_i < 0$, $\overline{Q}_i < 0$, $\widehat{Q}_i < 0$, and $\widehat{\overline{Q}}_i < 0$ for all $i \in \overline{p}$, respectively, then the corresponding switched system (3.1) or (3.9) is guaranteed to be globally Lyapunov's stable.

A generalization of interest of the polytopic switched system (3.9) is

$$\dot{x}(t) = \sum_{i=1}^{\vartheta}\lambda_i(t)\left(A_{0\sigma(t)}(t)x(t) + \sum_{j=1}^{q}\omega_{j\sigma(t)}A_{j\sigma(t)}(t)x(t-h_j)\right) \tag{3.18}$$

with $\sigma : \mathbf{R}_{0+} \to \overline{p}$, ω_{ij} being either zero or unity scalars and at least one $\omega_{ji} = 1$ for $j \in \overline{q}$ for each $i \in \overline{p}$ and $\lambda_i : \mathbf{R}_{0+} \to [0,1]$ subject to $\sum_{i=1}^{\vartheta}\lambda_i(t) = 1$. The results of Theorems 3.6(1), (2) and 3.9(1), (2) correspond to switches through time among the various parameterizations of (3.18) at any of the ϑ vertices of the polytope of parameters all of them being stable [3, 4].

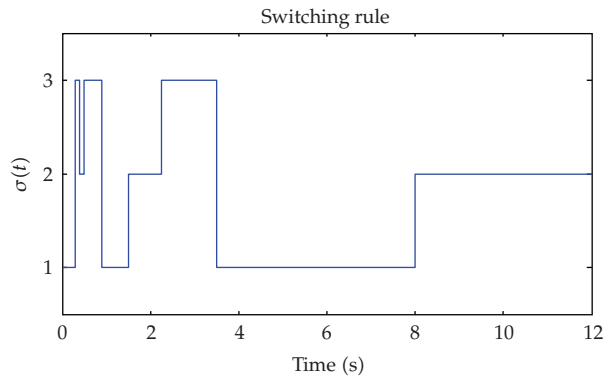

Figure 3: Switching law.

The switched system (3.18) is more general since any parameterization built with the convex hull of the sets of matrices A_{ji} for all $(j,i) \in (\overline{q} \cup \{0\}) \times \overline{p}$ is admissible. The following result extends directly Theorems 3.6(2) and 3.9(2) to the switched polytopic system (3.18).

Theorem 3.11. *The following properties hold.*

(1) *Assume that* $\overline{Q}_i < 0$ (3.10) *for all* $i \in \overline{p}$. *Then, the switched system* (3.18) *is globally asymptotically stable independent of the delays (i.e., for all* $h_i \in [0,\infty)$*; for all* $i \in \overline{p}$*) irrespective of the switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$. *The real functional* (3.11) *is a Krasovsky-Lyapunov functional for the switched polytopic system* (3.18).

(2) *If* $\widehat{\overline{Q}}_i < 0$, (3.15) *for all* $i \in \overline{p}$, *then the switched system* (3.18) *is globally asymptotically stable for any delays* $h_i \in [0, \widehat{h}_i]$ *for all* $i \in \overline{p}$, *irrespective of the switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$. *The real functional* (3.17) *is a Krasovsky-Lyapunov functional for the switched polytopic system* (3.18).

The proof of Theorem 3.11 is direct since all the real functions $\lambda_i : \mathbf{R}_{0+} \to [0,1]$ are nonnegative and they cannot be simultaneously zero at any time instant. This leads to functional time derivatives $\dot{\overline{V}}(t) = \sum_{i=1}^{\vartheta} \lambda_i(t)\overline{x}(t)\overline{Q}_{\sigma(t)}\overline{x}(t) < 0$ and $\dot{\widehat{\overline{V}}}(t) = \sum_{i=1}^{\vartheta} \lambda_i(t)\overline{x}(t)\widehat{\overline{Q}}_{\sigma(t)}\overline{x}(t) < 0$ for all nonzero $\overline{x}(t)$ along any nonzero state trajectory solution, respectively. The subsequent direct refusal of the conditions of Theorem 3.11, supported by "ad-hoc" Lyapunov's instability theorems, leads to situations where asymptotic stability of the switched system is impossible.

Theorem 3.12. *The following properties hold.*

(i) *Assume that* $Q_i \geq 0$ (3.8) *for all* $i \in \overline{p}$. *Then, the switched system* (3.1) *is not globally asymptotically stable independent of the delay size for any switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$ *but it can be still globally Lyapunov's stable for some switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$. *Those properties also hold for any particular delay. If* $Q_i > 0$ *for all* $i \in \overline{p}$, *then the switched system* (3.1) *is instable for any delay and any given switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$ *for any given admissible function of initial conditions.*

(ii) *Assume that* $\overline{Q}_i \geq 0$, (3.10) *for all* $i \in \overline{p}$. *Then, neither the switched system* (3.9) *nor any parameterization of the polytopic switched system* (3.18) *are globally asymptotically stable independent of the delay size for any switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$ *but it can be still globally Lyapunov's stable for some switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$. *Those properties also hold for any particular delay. If*

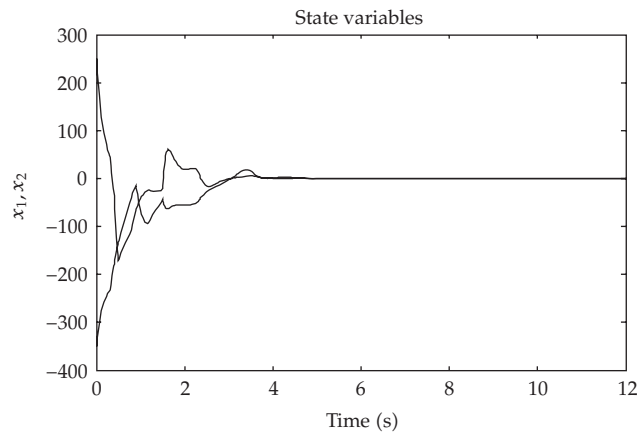

Figure 4: State-space variables evolution.

$\overline{Q}_i > 0$ *for all* $i \in \overline{p}$, *then the switched systems* (3.9) *and* (3.18) *are both instable for any delay and any given switching law* $\sigma : \mathbf{R}_{0+} \to \overline{p}$ *for any given admissible function of initial conditions.*

(iii) *Assume that* $Q_i < 0$ *for at least some* $i \in \overline{p}$. *Then, there are infinitely many switching laws* $\sigma : \mathbf{R}_{0+} \to \overline{p}$ *for which the switched system* (3.1) *is globally asymptotically stable and infinitely many for which it is globally Lyapunov's stable. The same property holds for the switched system* (3.9) *and infinitely many (but not all) parameterizations of the polytopic switched system* (3.18) *provided that* $\overline{Q}_i < 0$ *for at least some* $i \in \overline{p}$.

3.2.1. Simulation example 2

Consider the linear switched system (3.1) with $h = 0.75$ seconds and parameterizations

$$A_{01} = \begin{pmatrix} -7 & -1 \\ 2 & -4 \end{pmatrix}, \quad A_{02} = \begin{pmatrix} -15 & -1 \\ 3 & -2 \end{pmatrix}, \quad A_{03} = \begin{pmatrix} -5 & 2 \\ 1 & -0.5 \end{pmatrix},$$
$$A_{11} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} -12 & 1 \\ -3.5 & 1 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} -4 & -2 \\ -1 & -0.5 \end{pmatrix}. \tag{3.19}$$

According to Theorem 3.3, all the matrices $A_{0j} + A_{1j}$ are stable and pair-wise commute for $j = 1, 2, 3$. The initial conditions are given by $\varphi_1(t) = 250$, $\varphi_2(t) = -350$, $t \in [-0.75, 0]$. The arbitrary switching is defined by the switching law of Figure 3 where the residence time between consecutive switching time instants ranges from a small to a larger one. Figure 4 shows that the state variables tend asymptotically to zero.

3.3. A more general time-varying switched system

In the following, global asymptotic stability independent of the delay is stated for the linear time-varying system

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h), \tag{3.20}$$

which is a generalization of (3.1) since, in general, parametrical switches are admitted at sufficiently large separated time instants. It is assumed that the $A(t)$ matrix is a stable matrix for all time which is time differentiable almost everywhere, namely, except at the time instants where its parameters have bounded discontinuities at some of its entries. Those time instants may be considered as switching instants for the system parameterization. The proof is given in Appendix B.

Theorem 3.13. *Consider* (3.1) *under the following assumptions.*

(1) *Each value $A_0(t)$ of the matrix function $A_0 : \mathbf{R}_{0+} \to \mathbf{R}^{n\times n}$ is a stable matrix satisfying* $\operatorname{Re}(\lambda_i(A_0(t))) \le -\rho_0 < 0$ *for all $t \in \mathbf{R}_{0+}$.*

(2) *$A_0(t)$, $A_1(t)$ has uniformly bounded entries for all time.*

(3) *$\dot{A}_0(t)$ exists and it is bounded almost everywhere in $\mathbf{R}_{0+}$ with sufficiently small upper-bound $\gamma_{dA} \ge \|\dot{A}_0(t)\|_2$, where $\dot{A}_0(t)$ in $\mathbf{R}_{0+}$ exists.*

(4) *There exist positive definite real n-matrix functions $P(t)$ and $S(t)$ satisfying*

$$\begin{bmatrix} S(t) + \left(\dfrac{K^2}{2\rho^2}\gamma_{dA} - 1\right) I_n & P(t)A_1(t) \\ A_1^T(t)P(t) & -S(t-h) \end{bmatrix} < 0 \quad \forall t \in \mathbf{R}_{0+}. \tag{3.21}$$

(5) *The time interval between any two consecutive time instants at which $\dot{A}_0(t)$ does not exist is sufficiently large according to the amplitude-bound K_A (in terms of ℓ_2-norm) of the corresponding bounded discontinuities of $A_0(t)$, the absolute value of the stability abscissa ρ, γ_{dA}, and $P(t)$ and $S(t)$.*

Then, (3.20) *is globally asymptotically stable independent of the delay.*

However, (3.20) is a special case of switched system, the switches being of arbitrary but sufficiently small size at sufficiently large separated time instants. The dynamics are time-varying and one of the matrices is almost everywhere time differentiable with bounded norm time derivative. An alternative characterization which does not require time differentiability of the matrix entries can be developed directly for the case that the matrix function $A(t)$, which is not required to be either continuous or time differentiable, is locally deviated in norm for all time with respect to a comparison stability constant matrix with sufficiently large absolute stability abscissa with respect to the constant matrix. Stability results based on Gronwall's lemma are direct even if the matrix entries have sufficiently small bounded jumps, related to the absolute stability abscissa of the constant comparison matrix, through time.

Remark 3.14. In the light of Theorem 3.13, Theorems 3.6–3.11 can be extended for the case of time-varying systems possessing also eventual jumps in the entries of the matrix functions defining their dynamics. The extensions apply to more general systems under the structures of (3.1), (3.9), and (3.18) but where the parameters are uniformly bounded for all time and almost everywhere continuous and almost everywhere time-differentiable functions. A common property is that the stability is not independent of the switching law consisting of the rule for selecting the time instants where parametrical jumps appear. The conditions are a "mutatis-mutandis" modification of those of Theorem 3.13 by using the extended quadratic

form of the time derivatives of the corresponding borrowed Krasovsky-Lyapunov candidates and extended directly from those in Theorems 3.6–3.11. Therefore, such conditions are not made explicit. In particular, note that the following hold.

(a) Theorem 3.13 is an extension of Theorem 3.6(1) if (3.1) is generalized to (3.20).

(b) In the same way, Theorem 3.6(2) may be generalized for asymptotic stability independent of the delay if the polytopic system (3.9) is generalized to one possessing a similar structure based on the linear time-varying system (3.20).

(c) Theorem 3.9 is extendable with the ideas of Theorem 3.13 for global asymptotic stability dependent on the delay size. Also, Theorem 3.9(1) is extendable for the extended system (3.20) from (3.1) for stability dependent on the delay provided that $(A_0(t) + A_1(t))$ is a bounded stable matrix with a prescribed stability abscissa which is almost everywhere time differentiable with bounded derivative and which possess jumps of sufficiently small amplitudes at sufficiently separated time instants according to their sizes and the remaining system parameters. Theorem 3.9(2) may be extended for the generalization of (3.9) to (3.20).

(d) Theorem 3.11(1) is extendable for stability independent of the delay directly to the generalization of the polytopic system (3.18) to a similar polytopic form with $A_0(t)$ being a matrix function under Theorem 3.13. However, Theorem 3.11(2) is extendable for stability dependent on the delay size to such an extended polytopic system provided that $(A_0(t) + A_1(t))$ is bounded and stable with prescribed abscissa and almost every where time-differentiable entries with eventual jumps at sufficiently large time-intervals.

Remark 3.15. Note that the various systems dealt with through this manuscript might be reformulated to consider *asynchronous switching laws* in the sense that the matrices associated with each delay may belong to its own set of matrices each governed by an independent switching law. Consider, for instance, (3.1) extended to the more general one:

$$\dot{x}(t) = A_{0\sigma_0(t)}(t)x(t) + A_{1\sigma_1(t)}(t)x(t-h), \tag{3.22}$$

where $\sigma_j : \mathbf{R}_{0+} \to \overline{p}$ select for all time matrices $A_{j\sigma_j(t)} \in \{A_{j1}, \ldots, A_{jp}\}$ for all $j \in \{0,1\}$, for all $t \in \mathbf{R}_{0+}$. Also, (3.9) extended to the more general one:

$$\dot{x}(t) = A_{0\sigma_0(t)}(t)x(t) + \sum_{j=1}^{q} \omega_{j\sigma_j(t)} A_{j\sigma_j(t)}(t)x(t-h_j), \tag{3.23}$$

where $\sigma_j : \mathbf{R}_{0+} \to \overline{p}$ select for all time matrices $A_{j\sigma_j(t)} \in \{A_{j1}, \ldots, A_{jp}\}$ for all $j \in \overline{q} \cup \{0\}$, for all $t \in \mathbf{R}_{0+}$. A close extension is valid for the polytopic switched system (3.20). The various given stability results either for arbitrary switching laws or those requiring a minimum residence time extend directly to these more general systems if one considers the combined natural synchronous switching law $\sigma(t)$ defined with the whole set of asynchronous ones by a $(q+1)$-tuple of positive integers $\sigma(t) := (\sigma_0(t), \sigma_0(t), \ldots, \sigma_q(t))$ which is a vector mapping on a Cartesian product

$$\sigma : \mathbf{R}_{0+} \longrightarrow \overbrace{\overline{p} \times \cdots \times \overline{p}}^{q}. \tag{3.24}$$

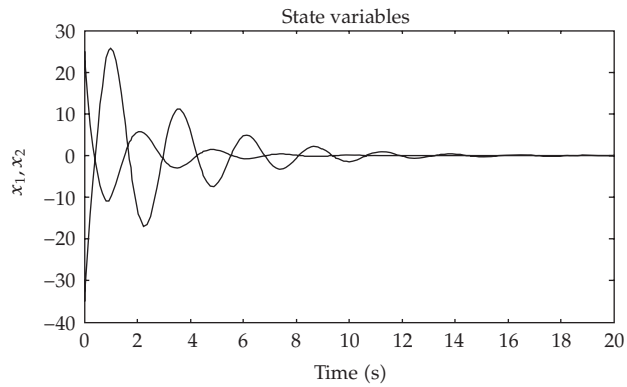

Figure 5: State-variables evolution.

The above to a synchronous switching law has at most p^{q+1} distinct parameterizations of the switched system formed with all the combinations of matrices associated with each particular delay with $q = 1$ for the switched system (3.1). If the set of matrices for some particular delay is less than p, the total number of distinct parameterizations decreases accordingly.

3.3.1. Simulation example 3

Consider now the time-varying system given by (3.20) with $h = 0.75$ sec., $\varphi_1(t) = 25$, $\varphi_2(t) = -35$, $t \in [-0.75, 0]$, $A_0(t) = \begin{pmatrix} -1 & a_0(t) \\ 0 & -1 \end{pmatrix}$, and $A_1(t) = \begin{pmatrix} -1 & 1/(2+t) \\ 0 & -2 \end{pmatrix}$ with

$$a_0(t) = \begin{cases} \dfrac{1}{1+t^n}, & n-1 < t \leq n;\ n = 1, 2, \ldots, 6, \\ \dfrac{1}{1+t^7}, & t > 6, \end{cases} \tag{3.25}$$

and $a_0(0) = 1$. Note that $a_0(t)$ is a discontinuous function implying that $A_0(t)$ has impulsive behavior at integer times. The solution of (3.20) with the above parameterization is depicted in Figure 5.

4. Concluding remarks

This paper has discussed the global exponential stability and the global asymptotic stability of standard classes of switched linear system subject to internal point time delays. The considered classes of systems have been those involving a set of time-invariant parameterizations, those having a polytopic structure, and those being time varying for all time with potential parametrical switches at certain times. The properties of global exponential and global asymptotic stability dependent on and independent of the delay size have been investigated for arbitrary switching laws and for switching laws subject to minimum residence time among any two consecutive switches. As in the delay-free case, stability under arbitrary switching requires commutation conditions on the system matrices of the various parameterizations. Otherwise, minimum residence times which depend on the parameterizations are required to keep the stability of the switched system.

Appendices

A. Proofs of the results of Section 2

A.1. Proof of Theorem 2.1

It holds that (2.1) may be equivalently rewritten as

$$\dot{x}(t) = (A_0 + A_1)x(t) - hA_1\dot{x}(t) + (x^T(t), x^T(t-h))^T o(h), \tag{A.1}$$

where the standard Landau's "small o" and " big O" notations are used; that is, $f(h) = o(h) \Leftrightarrow f(h) \to 0$ as $h \to 0$ provided that $f(h) = O(h) \Leftrightarrow |f(h)| \leq \chi_1 h + \chi_2$ with $\chi_i \in \mathbf{R}_{0+}$. It is obvious that for all $h \in [0, \overline{h})$ and some sufficiently small $\overline{h} \in \mathbf{R}_+ := \mathbf{R}_{0+} \setminus \{0\}$, $(I_n + hA_1)^{-1} = I_n - hA_1 + o(h)$ exists where I_n is the nth identity matrix. Thus,

$$\begin{aligned}\dot{x}(t) &= (I_n + hA_1)^{-1}(A_0 + A_1)x(t) + x(t)o(h) + x(t-h)o(h)\\ &= (A_0 + A_1)x(t) - hA_1(A_0 + A_1)x(t) + x(t)o(h) + x(t-h)o(h) \quad \forall h \in [0, \overline{h})\end{aligned} \tag{A.2}$$

from (A.1). The unique solution of (A.2) over $\mathbf{R}_{0+}$ by any piece-wise absolutely continuous function $\varphi : [-h, 0] \to \mathbf{R}^n$ with $\varphi(0) = x(0) = x_0$, which is the same as that of (2.1) for all $h \in [0, \overline{h})$, some sufficiently small $\overline{h} \in \mathbf{R}_+$, is

$$x(t) = e^{(A_0+A_1)t}x_0 + \int_0^t e^{(A_0+A_1)(t-\tau)}\left[x(\tau)o(h) + x(\tau-h)o(h) - hA_1(A_0 + A_1)x(\tau)\right]d\tau, \tag{A.3}$$

so that

$$\begin{aligned}\|x(t')\| &= \operatorname*{Sup}_{t-h\leq\tau\leq t}\|x(\tau)\| \leq \|e^{(A_0+A_1)t'}x_0\|\\ &\quad + \left\|\int_0^h e^{(A_0+A_1)(t'-\tau)}\left[(|o(h)| + h\|A_1(A_0 + A_1)\|)\operatorname*{Sup}_{\tau-h\leq\tau'\leq\tau}\|x(\tau')\|\right]d\tau\right\|\\ &\quad + \left\|\int_h^{t'} e^{(A_0+A_1)(t-\tau)}\left[(|o(h)| + h\|A_1(A_0 + A_1)\|)\operatorname*{Sup}_{\tau-h\leq\tau'\leq\tau}\|x(\tau')\|\right]d\tau\right\|,\end{aligned} \tag{A.4}$$

where

$$\begin{aligned}t' = t'(t) := \max\Big(\tau \in [t-h, t] : \|x(t')\| &= \operatorname*{Sup}_{t-h\leq\tau\leq t}\|x(\tau)\| \wedge \|x(\tau)\| \leq \|x(t')\|\\ &\qquad \forall\tau(\neq t') \in [t-h, t]\Big)\end{aligned} \tag{A.5}$$

is the largest time instant of maximum state norm within $[t-h,t]$. Since (A_0+A_1) is a stable matrix, there exist real constants $K \geq 1$ (being norm-dependent) and $\rho \in \mathbf{R}_+$ such that $\|e^{(A_0+A_1)(t-\tau)}\| \leq Ke^{-\rho(t-\tau)}$. Then, one gets from (A.4) via Gronwall's lemma [27]

$$\begin{aligned}\operatorname*{Sup}_{t-h\leq\tau\leq t}\|x(\tau)\| \leq K\left(1+\left(|o(h)|+h\|A_1(A_0+A_1)\|\right)\frac{e^{\rho h}-1}{\rho}\right)\operatorname*{Sup}_{-h\leq\tau\leq 0}\|\varphi(\tau)\| \\ \times e^{-(\rho-K(|o(h)|+h\|A_1(A_0+A_1)\|))t'}.\end{aligned} \tag{A.6}$$

It follows from (A.6) that there exists a sufficiently small $\overline{h} \in \mathbf{R}_+$ fulfilling $\overline{h} < \rho/\|A_1(A_0+A_1)\|$ such that $\infty > \operatorname{Sup}_{t-h\leq\tau\leq t}\|x(\tau)\| \to 0$ exponentially as $t \to \infty$ for all $h \in [0,\overline{h})$ since $t'(t) \to \infty$ as $t \to \infty$. Thus, $\|x\| \in L_\infty$ and $\|x(t)\| \to 0$ exponentially as $t \to \infty$.

A.2. Proof of Theorem 2.2

The solution of (2.1) is

$$x(t) = e^{A_0 t}\left(x_0 + \int_0^h e^{-A_0\tau}A_1\varphi(\tau-h)d\tau + \int_h^t e^{-A_0\tau}A_1 x(\tau-h)d\tau\right). \tag{A.7}$$

Since A_0 is a stable matrix, there exist real constants $K_0 \geq 1$ (being norm-dependent) and $\rho_0 \in \mathbf{R}_+$ such that $\|e^{A_0(t-\tau)}\| \leq Ke^{-\rho_0(t-\tau)}$. Then, if

$$\begin{aligned}t' = t'(t) := \max\Big(\tau \in [t-h,t] : \|x(t')\| = \operatorname*{Sup}_{t-h\leq\tau\leq t}\|x(\tau)\| \wedge \|x(\tau)\| \\ \leq \|x(t')\| \ \forall\tau(\neq t') \in [t-h,t]\Big),\end{aligned} \tag{A.8}$$

then one gets from (A.7) with the application of Gronwall's lemma for $v(t) := e^{\rho_0 t}\|x(t)\|$:

$$\begin{aligned}\|x(t)\| &\leq \operatorname*{Sup}_{t-h\leq\tau\leq t}\|x(\tau)\| \\ &\leq K_0\left(1+\frac{e^{\rho_0 h}-1}{\rho_0}\|A_1\|\right)\left(\sup_{-h\leq\tau\leq 0}\|\varphi(\tau)\|\right)e^{-(\rho_0-K_0\|A_1\|)t'}.\end{aligned} \tag{A.9}$$

If $\rho_0 > K_0\|A_1\|$, then $\infty > \operatorname{Sup}_{t-h\leq\tau\leq t}\|x(\tau)\| \to 0$ exponentially as $t \to \infty$ for all $h \in \mathbf{R}_{0+}$ since $t'(t) \to \infty$ as $t \to \infty$. Thus, $\|x\| \in L_\infty$ and $\|x(t)\| \to 0$ exponentially as $t \to \infty$. It turns out that for $t \geq t_0 := \ln K_0/(\rho_0-\rho_0')+h$ and any real constant $\rho_0' \in (0,\rho_0)$ or, equivalently, for $t \geq t_0$, and any (dependent on t_0) real constant $\rho_0' \in (0,\rho_0-\ln K_0/(t_0-h))$, for any given sufficiently large finite $t_0 > h$, it follows from (A.9) that

$$\begin{aligned}\|x(t)\| \leq \operatorname*{Sup}_{t-h\leq\tau\leq t}\|x(\tau)\| \leq \left(1+\frac{e^{\rho_0 h}-1}{\rho_0}\|A_1\|\right)\left(\operatorname*{Sup}_{-h\leq\tau\leq 0}\|\varphi(\tau)\|\right) \\ \times e^{-(\rho_0'-\|A_1\|)t'} \quad \forall t \geq t'+h \geq t_0 := \frac{\ln K_0}{\rho_0-\rho_0'}+h,\end{aligned} \tag{A.10}$$

since $K_0 e^{-\rho_0 t'} < e^{-\rho_0' t'}$. If ℓ_2-(spectral) norms $\|\cdot\|_2$ are used for vectors and matrices, then the following choices of parameters are made in (A.10):

$$0 < \rho_0' < \rho_0 \leq \left|\mu_2(A_0)\right| = \frac{1}{2}\max_{1\leq i\leq \xi}\left|\operatorname{Re}\lambda_i\left(A_0 + A_0^T\right)\right| = \frac{1}{2}\max_{1\leq i\leq \xi}\left|\lambda_i\left(A_0 + A_0^T\right)\right|,$$

$$\mu_2\left(e^{A_0 t}\right) \leq \left\|e^{A_0 t}\right\|_2 := \max_{1\leq i\leq \xi}\lambda_i^{1/2}\left(e^{A_0^T t}e^{A_0 t}\right) = \overline{\sigma}\left(e^{A_0 t}\right) \leq e^{-|\mu_2(A_0)t|} \leq e^{-\rho_0' t} \quad \forall t \geq \frac{\ln K_0}{\rho_0 - \rho_0'} + h,$$

$$\left\|A_1\right\|_2 := \max_{1\leq i\leq \xi}\lambda_i^{1/2}\left(A_1^T A_1\right) = \overline{\sigma}\left(A_1\right), \tag{A.11}$$

where $\mu_2(M) := \lim_{\delta\to 0^+}((\|I_n + \delta M\|_2 - \|I_n\|_2)/\delta)$, with $\|I_n\|_2 = 1$, denotes the matrix measure with respect to the ℓ_2-norm of the square matrix M; and $\lambda_i(M)$ $(i \in \overline{\xi} := \{1, 2, \ldots, \xi\})$ and $\overline{\sigma}(M)$ denote, respectively, the distinct eigenvalues and the maximum singular value of such a matrix. Note that $\mu_2(A_0) := (1/2)\max_{1\leq i\leq \xi}\operatorname{Re}\lambda_i(A_0 + A_0^T) < 0$ since A_0 is a stable matrix. The inequality $\rho_0 < |\mu_2(A_0)|$ is strict if the maximum eigenvalue of $A_0 + A_0^T$ is multiple. The matrix measure with respect to the ℓ_2-norm of the matrix M is identical for all similar matrices to Mwhich follows directly from the definition of such a matrix measure. Since the stability properties of linear time-invariant systems are independent of the chosen state-space representation, it follows directly from (A.10) that (2.1) is globally exponentially stable independent of the delay if $\|A_1\|_2 < \rho_0$ for some real constant $0 < \rho_0 < -\mu_2(A_0)$.

A.3. Proof of Theorem 2.3

Note that $\mu_2(A_0) + \|A_1\|_2 < 0 \Rightarrow \mu_2(A_0) < 0$. From the definition of the matrix measure with respect to the ℓ_2-norm, all the eigenvalues of $A_0 + A_0^T$ and also those of A_0 are in $\operatorname{Re} s < 0$ so that A_0 is a stable matrix. Since A_0 is a stable matrix, Theorem 2.2 applies so that (2.1) is globally exponentially stable independent of the delay since $\mu_2(A_0) + \|A_1\|_2 < 0$. Since the system is globally exponentially stable independent of the delay, it is globally exponentially stable for zero delay so that $(A_0 + A_1)$ is a stable matrix. The first part of the result has been proved. The second part is proved by finding by construction a system which is not exponentially stable of delayed dynamics being of identical ℓ_2-norm to that of A_1 as follows. Construct $A_1^* = -kA_0$ with some $k \in \mathbf{R}_+$ to be specified later so that $\|A_1\|_2 = \|A_1^*\|_2 = |k|\|A_0\|_2$. Since A_0 is a stable matrix, A_1^* is, by construction, an antistable matrix (i.e., with all its eigenvalues in $\operatorname{Re} s > 0$). From the definitions of spectral norms and associate matrix measures and their properties, one has $\mu_2(A_1^*) = \|A_1^*\|_2 = \mu_2(-kA_0) = -k\mu_2(A_0) = k|\mu_2(A_0)| > 0$, and then

$$\begin{aligned}
\mu_2\left((1-k)A_0\right) &= (1-k)\mu_2\left(A_0\right) \\
&= \mu_2\left(A_0\right) - k\mu_2\left(A_0\right) \\
&= \mu_2\left(A_0\right) + \mu_2\left(A_1^*\right) \\
&= (k-1)\left|\mu_2(A_0)\right|
\end{aligned} \tag{A.12}$$

so that for $k \geq 1$, $\mu_2(A_0 + A_1^*) \geq 0$ since

$$\begin{aligned} 0 &\leq \mu_2(A_0) + \|A_1^*\|_2 \\ &= \mu_2(A_0) + \|A_1\|_2 \\ &= \mu_2(A_0) + \mu_2(A_1^*) \\ &= \mu_2(A_0 + A_1^*) \\ &= (k-1)|\mu_2(A_0)|. \end{aligned} \tag{A.13}$$

As a result, $(A_0 + A_1^*)$ is not a stable matrix and the system $\dot{x}(t) = A_0 x(t) + A_1^* x(t-h)$ is not globally exponentially stable independent of the delay size while being either critically stable or unstable for zero delay.

B. Proofs related to Section 3

B.1. Proof of Theorem 3.1

Note that $\|e^{(A_{0i}+A_{1i})(t-\tau)}\| \leq K_i e^{-\rho_i(t-\tau)} \leq K e^{-\rho_m(t-\tau)}$ for all $(t,\tau) \in \mathbf{R}_{0+} \times \mathbf{R}_{0+}$ for pairs $(K_i, \rho_i) \in \mathbf{R}_+ \times \mathbf{R}_+$ since the matrices $(A_{0i} + A_{1i})$ are all stable matrices for all $i \in \overline{p}$; with $K \geq \max(K_i : i \in \overline{p})$ and $0 < \rho_m \leq \min(\rho_i : i \in \overline{p})$. After defining $\rho_M \in \mathbf{R}_+$, so that $\rho_m \leq \min(\rho_i : i \in \overline{p}) \leq \max(\rho_i : i \in \overline{p}) \leq \rho_M$, (A.6) in the proof of Theorem 2.1 is now replaced on the interval $[t_i, t_{i+1})$ between two consecutive switching instants with

$$\begin{aligned} \operatorname*{Sup}_{t_{i+1}-h \leq \tau \leq t_{i+1}} \|x(\tau)\| &\leq K\left(1 + (|o(h)| + h\|A_{1\sigma(t_i)}(A_{0\sigma(t_i)} + A_{1\sigma(t_i)})\|)\frac{e^{\rho_M h} - 1}{\rho_m}\right) \operatorname*{Sup}_{t_i - h \leq \tau \leq t_i} \|x(\tau)\| \\ &\quad \times e^{-(\rho_m - K(|o(h)| + h\|A_{1\sigma(t_i)}(A_{0\sigma(t_i)} + A_{1\sigma(t_i)})\|))(t_{i+1} - t_i - h)} \\ &\leq \delta \operatorname*{Sup}_{t_i - h \leq \tau \leq t_i} \|x(\tau)\| < \operatorname*{Sup}_{t_i - h \leq \tau \leq t_i} \|x(\tau)\| \end{aligned} \tag{B.1}$$

with $\mathbf{R}_+ \ni \delta \in (0,1)$ if $t_{i+1} \geq t_i + T - h$ and $T \in \mathbf{R}_+$ being a sufficiently large minimum residence time provided that the delay $h \in [0, \overline{h})$ for sufficiently small $\overline{h} \in \mathbf{R}_+$. Assume that the switching law satisfies $\sigma(t) = j \in \overline{p}$ for all $t \in [t_k, t_{k+1}) \supset [t_k, t_k + T)$, for all $k \in \mathbf{N}_s$ and that (B.1) holds for any $k \in \mathbf{N}_{\alpha s} \subset \mathbf{N}_s$ with $\mathbf{N}_{\alpha s}$ being a proper subset of $\mathbf{N}_s$ of finite cardinal $\alpha \in \mathbf{N}$. Assume that $\operatorname{Sup}_{t_j - h \leq \tau \leq t_j} \|x(\tau)\| \leq \delta^j \operatorname{Sup}_{t_{j-1} - h \leq \tau \leq t_{j-1}} \|x(\tau)\|$ for all $j \in \overline{\alpha}$ adopting the convention $0 = t_0 \notin \mathbf{N}_s$. From (B.1), taking initial conditions at t_j and final ones at t_{j+1}, it follows that $\operatorname{Sup}_{t_{j+1} - h \leq \tau \leq t_{j+1}} \|x(\tau)\| \leq \delta^j \operatorname{Sup}_{t_j - h \leq \tau \leq t_j} \|x(\tau)\| < \infty$ for all $j \in \overline{\alpha}$ since the minimum residence time in between consecutive switches is respected for all switches. Thus, by complete induction $\operatorname{Sup}_{t_{j+1} - h \leq \tau \leq t_{j+1}} \|x(\tau)\| \to 0$ is exponentially fast and all the sequence is uniformly bounded provided that $\mathbf{N}_s$ has infinite cardinal. Otherwise, t' and t are kept without modification from (A.6) for $t \geq t_{n_s} + h$ with initial conditions at finite t_{n_s} if $\operatorname{card}(\mathbf{N}_s) = n_s < \infty$. Thus, the proof still follows from (A.9) with $\operatorname{Sup}_{t_j - h \leq \tau \leq t_j} \|x(\tau)\| \leq \delta^j \operatorname{Sup}_{t_{j-1} - h \leq \tau \leq t_{j-1}} \|x(\tau)\| < \infty$ for all $j \in \overline{n}_s$, and $\lim_{\mathbf{N} \ni i \to \infty} \operatorname{Sup}_{t_{n_s} + iT - h \leq \tau \leq t_{n_s} + iT} \|x(\tau)\| \leq \lim_{\mathbf{N} \ni i \to \infty} \delta^{n_s + i} \operatorname{Sup}_{t_{n_s} + (i-1)T - h \leq \tau \leq t_{n_s} + (i-1)T} \|x(\tau)\| = 0$.

B.2. Proof of Theorem 3.2

Note that $\|e^{A_{0i}(t-\tau)}\| \le K_{0i}e^{-\rho_{0i}(t-\tau)} \le K_0 e^{-\rho_{0m}(t-\tau)}$ for all $(t,\tau) \in \mathbf{R}_{0+} \times \mathbf{R}_{0+}$, for pairs $(K_{0i}, \rho_{0i}) \in \mathbf{R}_+ \times \mathbf{R}_+$, since the matrices A_{0i} are all stable matrices for all $i \in \overline{p}$; with $K_0 \ge \max(K_{0i} : i \in \overline{p})$ and $0 < \rho_{0m} \le \min(\rho_{0i} : i \in \overline{p})$. After defining $\rho_{0M} \in \mathbf{R}_+$, so that $\rho_{0m} \le \min(\rho_{0i} : i \in \overline{p}) \le \max(\rho_{0i} : i \in \overline{p}) \le \rho_{0M}$, (A.9) in the proof of Theorem 2.2 is now modified on the time interval $[t_i, t_{i+1})$ in between two consecutive switching instants $t_i,\ t_{i+1}$, as follows after using Gronwall's lemma for the real function $v(t) := e^{\rho_0 t}\|x(t)\|$:

$$\frac{\operatorname{Sup}_{t_{i+1}-h\le\tau\le t_{i+1}} \|x(\tau)\|}{\sup_{t_i-h\le\tau\le t_i} \|x(\tau)\|} \le K_0\left(1 + \frac{e^{\rho_{0m}h}-1}{\rho_{0M}} \max_{1\le i\le p} \|A_{1i}\|\right) e^{-(\rho_{0m}-K_0 \max_{1\le i\le p}\|A_{1i}\|)T} \le \delta < 1 \tag{B.2}$$

if $t_{i+1} \ge t_i + T - h$ and $T \in \mathbf{R}_+$ being a sufficiently large minimum residence time satisfying:

$$T = \frac{\ln K_0 + \ln\left(1 + \left(\left(e^{\rho_{0m}h}-1\right)/\rho_{0M}\right)\max_{1\le i\le p}\|A_{1i}\|\right) + |\ln\delta|}{\rho_{0m} - K_0 \max_{1\le i\le p} \|A_{1i}\|}, \tag{B.3}$$

and being identical for any switching instant provided that $\max_{1\le i\le p} \|A_{1i}\| < \rho_{0m}/K_0$ for any prefixed $\mathbf{R}_+ \ni \delta \in (0,1)$. An alternative weaker condition than (B.2) is to switch with $t_{i+1} \ge t_i + T_{\sigma(t_i)} + h$ subject to a switching-dependent minimum residence time:

$$T_{\sigma(t_i)} = \frac{\ln K_{0i} + \ln\left(1 + \left(\left(e^{\rho_{0i}h}-1\right)/\rho_{0i}\right)\|A_{1i}\|\right) + |\ln\delta|}{\rho_{0i} - K_{0i}\|A_{1i}\|}, \tag{B.4}$$

provided that $\|A_{1i}\| < \rho_{0i}/K_{0i}$ for any prefixed $\mathbf{R}_+ \ni \delta \in (0,1)$ and any $i \in \overline{p}$ provided that $\sigma(t_i) = i \in \overline{p}$. Both constraints lead to $\lim_{\mathbf{N}\ni i\to\infty} (\operatorname{Sup}_{t_{i+1}-h\le\tau\le t_{i+1}} \|x(\tau)\|/\sup_{t_i-h\le\tau\le t_i} \|x(\tau)\|) \le \lim_{\mathbf{N}\ni i\to\infty} \delta^i = 0$ if switching never stops. Under a maximum finite switching instant t_{n_s} and $\sigma(t_{n_s}) = i \in \overline{p}$, then

$$\infty > \frac{\operatorname{Sup}_{t-h\le\tau\le t} \|x(\tau)\|}{\sup_{t_{n_s}-h\le\tau\le t_{n_s}} \|x(\tau)\|} \le K_{0i}\left(1 + \frac{e^{\rho_{0i}h}-1}{\rho_{0i}}\|A_{1i}\|\right) e^{-(\rho_{0i}-K_{0i}\|A_{1i}\|)(t_i-t_{n_s}-h)} \longrightarrow 0 \tag{B.5}$$

as $t \to \infty$ with exponential convergence rate. It switching never stops, then

$$\infty > \frac{\operatorname{Sup}_{t-h\le\tau\le t} \|x(\tau)\|}{\sup_{-h\le\tau\le 0} \|x(\tau)\|} \longrightarrow 0 \Longrightarrow \lim_{\mathbf{R}\ni t\to\infty} \operatorname{Sup}_{t-h\le\tau\le t} \|x(\tau)\| = 0 \tag{B.6}$$

with exponential convergence rate. Thus, global exponential stability is guaranteed by any switching law respecting a minimum residence time in between any two consecutive switching instants. If ℓ_2-norms are used, then the particular stability result follows directly from the above general proof and the last part of the proof of Theorem 2.2. The proof is omitted.

B.3. Proof of Theorem 3.3

(i) Take $t \in [t_i, t_{i+1})$ with $\{t_i\}_{i\in\mathbf{N}_s}$ being the switching instants generated by the switching law $\sigma(t)$ and define the extended indicator $\overline{\mathbf{N}}_s := \mathbf{N}_s \cup \{0\}$ of switching instants with $t_0 = 0$ and interswitching periods $T_i := t_{i+1} - t_i$ for all $i \in \overline{\mathbf{N}}_s$. Thus, the switched system (3.1) for zero delay becomes

$$\begin{aligned}
\|x(t_{i+1})\|_2^2 &= x^T(t_i)e^{(A_{0\sigma(t_i)}+A_{1\sigma(t_i)})T_i}e^{(A_{0\sigma(t_i)}+A_{1\sigma(t_i)})^T T_i}x(t_i) \\
&= z^T(t_i)e^{\Lambda_{\sigma(t_i)}t}z(t_i) \\
&= x^T(0)\left(e^{\sum_{\overline{\mathbf{N}}_s\ni j\le i}(A_{0\sigma(t_j)}+A_{1\sigma(t_j)})T_j+(A_{0\sigma(t_j)}+A_{1\sigma(t_j)})^T T_j}\right)x(0) \\
&= x^T(0)\left(\prod_{\overline{\mathbf{N}}_s j\le i} e^{(A_{0\sigma(t_j)}+A_{1\sigma(t_j)})T_j+(A_{0\sigma(t_j)}+A_{1\sigma(t_j)})^T T_j}\right)x(0) \\
&= z^T(0)e^{\sum_{\overline{\mathbf{N}}_s\ni j\le i}\Lambda_{\sigma(t_j)}T_j}z(0) \\
&= z^T(0)\left(\prod_{\overline{\mathbf{N}}_s\ni j\le i} e^{\Lambda_{\sigma(t_j)}T_j}\right)z(0),
\end{aligned} \tag{B.7}$$

where $Q^T e^{(A_{0j}+A_{1j})t+(A_{0j}+A_{1j})^T t}Q = e^{\Lambda_i t}$ for all $j \in \overline{p}$ is a real diagonal matrix with real positive eigenvalues less than unity since $(A_{0j} + A_{1j})$ are all stable matrices. The real nonsingular matrix $Q \in \mathbf{R}^{n\times n}$ defining the state transformation $x(t_i) = Qz(t_i)$ is orthogonal since $e^{(A_{0\sigma(t_i)}+A_{1\sigma(t_i)})t+(A_{0\sigma(t_i)}+A_{1\sigma(t_i)})^T t}$ are all symmetric and identical for $\sigma(t_i) = j$ any $j \in \overline{p}$ (i.e., independent on the current $\sigma(t_i) = j \in \overline{p}$) and for all $t \in \mathbf{R}$, since $(A_{0j} + A_{1j})$; for all $j \in \overline{p}$ pair-wise commuting with its transpose implies that $e^{(A_{0i}+A_{1i})t+(A_{0i}+A_{1i})^T t}$ pair-wise commute; for all $i \in \overline{p}$, for all $t \in \mathbf{R}$, [28]. Thus, $\prod_{\overline{\mathbf{N}}_s\ni j\le i} e^{\Lambda_{\sigma(t_j)}t} = Q^T(\prod_{\overline{\mathbf{N}}_s\ni j\le i} e^{(A_{o\sigma(t_j)}+A_{1\sigma(t_j)})t+(A_{o\sigma(t_j)}+A_{1\sigma(t_j)})^T t})Q$ is real diagonal with eigenvalues less than unity for all $t \in \mathbf{R}_+$. During the interswitching periods,

$$\|x(t_{i+1}+\tau)\|_2^2 = z^T(0)e^{\Lambda_{\sigma(t_{n_s})}\tau}\left(\prod_{\overline{\mathbf{N}}_s\ni j\le i} e^{\Lambda_{\sigma(t_j)}T_j}\right)z(0); \quad \tau \in [0, T_{i+1})\ \forall i \in \mathbf{N}_s. \tag{B.8}$$

If $n_s < \infty$ is the cardinal of $\mathbf{N}_s$, then

$$\|x(t)\|_2^2 = z^T(0)e^{\Lambda_{\sigma(t_{n_s})}(t-t_{n_s})}\left(\prod_{\overline{\mathbf{N}}_s\ni j\le n_s} e^{\Lambda_{\sigma(t_i)}T_j}\right)z(0); \quad \mathbf{R}_+ \ni t \ge t_{n_s}. \tag{B.9}$$

Therefore, (B.7)–(B.9) imply the following in terms of matrix measures:

$$\|x(t)\|_2^2 = e^{-2(\mu(A_{o\sigma(t_i)}+A_{1\sigma(t_i)})(t-t_i)+\sum_{\overline{\mathbf{N}}_s\ni j\le i-1}\mu(A_{o\sigma(t_j)}+A_{1\sigma(t_j)})T_j)} \le e^{-2\rho t}\|z(0)\|_2^2 \tag{B.10}$$

for all $t \in [t_i, t_{i+1})$ or for all $t \geq t_{n_s}$ if $i = n_s$ is the finite cardinal of $\mathbf{N}_s$, where $\rho \geq (1/2)|\min_{1\leq j\leq \xi_i, 1\leq i\leq p} \lambda_j(A_{0i} + A_{1i} + A_{0i}^T + A_{1i}^T)| \in \mathbf{R}_+$ with ς_i being the number of distinct eigenvalues of $A_{0i} + A_{1i}$, $i \in \overline{p}$. Thus, $\|x(t)\|_2^2$ tends exponentially to zero irrespective of the switching law $\sigma(t)$ provided that the delay is zero. The switched system (3.1) is identical to

$$\dot{x}(t) = \left(A_{o\sigma(t)} + A_{1\sigma(t)}\right)x(t) + A_{1\sigma(t)}\left(x(t-h) - x(t)\right) \tag{B.11}$$

for any $t \in \mathbf{R}_{0+}$ and any delay $h \in [0, \overline{h}]$. In view of (B.10) for the case of zero delay, the solution of (B.11) for any admissible vector function of initial conditions is subject to

$$\begin{aligned}\|x(t+\tau)\|_2 \leq{}& \left\| e^{(A_{0\sigma(t)}+A_{1\sigma(t)})t}\left(x_0 + \int_0^h e^{-(A_{0\sigma(\tau)}+A_{1\sigma(\tau)})\tau} A_{1\sigma(\tau)}\left(\varphi(\tau-h) - \varphi(\tau)\right)\right)d\tau\right\|_2 \\ &+ \left\| \int_h^t e^{(A_{0\sigma(\tau)}+A_{1\sigma(\tau)})(t-\tau)} A_{1\sigma(\tau)}\left(x(\tau-h) - x(\tau)\right)d\tau\right\|_2,\end{aligned} \tag{B.12}$$

so that according to the proof of Theorem 2.1(A.4), one has for sufficiently small delay $h \in [0, \overline{h}]$ by using Gronwall's lemma:

$$\operatorname*{Sup}_{t-h\leq\tau\leq t} \|x(\tau)\|_2 \leq \left(1 + 2\frac{e^{\rho\overline{h}} - 1}{\rho}\|A_{1\sigma(0)}\|_2\right) e^{-(\rho - \overline{h}\max_{1\leq j\leq p}\|A_{1j}(A_{0j}+A_{1j})\|_2 - |o(\overline{h})|)t} \operatorname*{Sup}_{-\overline{h}\leq\tau\leq 0} \|\varphi(\tau)\|_2. \tag{B.13}$$

It follows from (B.18) that for any sufficiently small $\overline{h}$, $\lim_{t\to\infty} \operatorname{Sup}_{t-h\leq\tau\leq h} \|x(\tau)\|_2 = 0$ at exponential rate for any arbitrary switching law $\sigma : \mathbf{R}_{0+} \to \overline{p}$ for all $h \in [0, \overline{h}]$ since the above property holds if

$$\left(1 + 2\frac{e^{\rho\overline{h}} - 1}{\rho}\|A_{1\sigma(0)}\|_2\right) e^{-(\rho - 2\overline{h}\max_{1\leq j\leq p}\|A_{1j}(A_{0j}+A_{1j})\|_2)(t-\overline{h})} < 1 \tag{B.14}$$

for $t > \overline{h}$. By fixing $\overline{\varepsilon} = \overline{\varepsilon}(\overline{h}) = \max(2((e^{\rho\overline{h}} - 1)/\rho)\max_{1\leq j\leq p}\|A_{1j}\|_2, 2\overline{h}\max_{1\leq j\leq p}\|A_{1j}(A_{0j} + A_{1j})\|_2)$, the constraint (B.14) is guaranteed for all real $\varepsilon \in [0, \overline{\varepsilon}]$ with 0 and $\overline{\varepsilon}$ being the zeros of the convex parabola $g(\overline{\varepsilon}) = \overline{\varepsilon}^2 - \rho\overline{\varepsilon} - \ln(1 + \overline{\varepsilon})$. Thus, (B.14) holds for all $h \in [0, \overline{h})$ and $\overline{h} = \min((1/\rho)\ln(1 + (\rho\overline{\varepsilon}/2\max_{1\leq j\leq p}\|A_{1j}\|_2)), (1/2\max_{1\leq j\leq p}\|A_{1j}(A_{0j} + A_{1j})\|_2))$ independent of the switching law $\sigma : \mathbf{R}_{0+} \to \overline{p}$.

(ii) Let $\{t_i\}_{i\in N_s}$ be the switching instants generated by the switching law $\sigma(t)$ and define the extended indicator $\overline{N}_s = N_s \cup \{0\}$ of switching instants with $t_0 = 0$ and interswitching periods $T_i^{(\sigma(t_i))} := t_{i+1} - t_i$ for all $i \in N_s$. The superscript in $T_i^{(\sigma(t_i))}$ indicates that the switched system is parameterized by $A_{i\sigma(t_i)}$ for all $t \in [t_i, t_{i+1})$ if t_i is not the maximum

switching instant. Otherwise, $\lambda \in \mathbf{R}_{0+}$ so that $t \in [t_i, \infty)$. Thus, the solution of the switched system (3.1) for zero delay becomes

$$x(t) = x(t_{i+1} + \tau) = \prod_{j=0}^{i+1} \left(e^{(A_{0\sigma(t_j)} + A_{1\sigma(t_j)})\tau} e^{(A_{0\sigma(t_j)} + A_{1\sigma(t_j)})T_i^{(\sigma(t_j))}} \right) x_0$$
$$\forall t \in [t_i, t_{i+1}), \quad \forall \tau \in [0, T_{i+1}) \quad \text{for } x_0 = x(0). \tag{B.15}$$

The above product of matrices is defined from right to left as the running index increases. Since $(A_{0j} + A_{1j})$ pair-wise commute for all $j \in \overline{p}$, the above product can be rearranged regardless the switching rule to obtain

$$x(t) = x(t_{i+1} + \tau) = e^{(A_{0\sigma(t_{i+1})} + A_{1\sigma(t_{i+1})})\tau} \left[\prod_{j=1}^{p} \left(e^{(A_{0j} + A_{1j})(\sum_{k \in N_{sj}} T_k^{(j)})} \right) \right] x_0 \quad \forall t \in [t_i, t_{i+1}), \ \forall \tau \in [0, T_{i+1}), \tag{B.16}$$

where $\overline{N}_s \supset N_{sj} := \{i \in \overline{N}_s : \sigma(t_i) = j \in \overline{p}\}$; for all $j \in \overline{p}$, so that one has for any norm, $\|x(t)\| \leq \prod_{i=1}^{p} [K_i] e^{-\rho_{\sigma(t_{i+1})}\tau} [\prod_{j=1}^{p} (e^{-\rho_j (\sum_{k \in N_{sj}} T_k^{(j)})})] \|x_0\| \leq K e^{-\rho t} \|x_0\|$ for all $t \in [t_i, t_{i+1})$, for all $\tau \in [0, T_{i+1})$ with $\|e^{(A_{0i} + A_{1i})t}\| \leq K_i e^{-\rho_i t}$, some $\mathbf{R} \ni K_i \geq 1$, $\rho_i \in \mathbf{R}_+$, for all $i \in \overline{p}$ (since $(A_{0i} + A_{1i})$ are stable matrices; for all $i \in \overline{p}$); $K := \prod_{i=1}^{p} [K_i]$, and

$$\rho := \min_{i \in \overline{p}} (\rho_i) \geq \frac{1}{2} \left| \min_{1 \leq j \leq \xi_i, 1 \leq i \leq p} \lambda_j (A_{0i} + A_{1i} + A_{0i}^T + A_{1i}^T) \right| \in \mathbf{R}_+, \tag{B.17}$$

where ξ_i denotes the number of distinct eigenvalues of $(A_{0i} + A_{1i})$ for all $i \in \overline{p}$ for any switching time $\{t_i\}_{i \in N_s}$. Thus, $\|x(t)\|$ tends exponentially to zero as time tends to infinity irrespectively of the switching law $\sigma(t)$ for zero delay. The remaining part of the proof is similar to that of property (i).

B.4. Outline of proof of Theorem 3.12

The properties (i)-(ii) follow since the functionals (3.7) and (3.11) are nonnegative for any nonzero state trajectory solution with time-derivative non-less than zero so that the equilibrium is not asymptotically reachable by any of those trajectories under any switching law. If, in additions, the matrices referred to are positive definite, then those functionals are unbounded as time increases so that (3.1), (3.9), (3.18), respectively, are instable from "ad-hoc" Lyapunov's instability theorems since any nontrivial state trajectories are unbounded with time. Property (ii) can be proved by constructing some appropriate switching laws. If one parameterization $i \in \overline{p}$ is stable, any switching law $\sigma : \mathbf{R}_{0+} \to \overline{p}$ leading to such a parameterization in finite time and staying at it later on for all time leads to global exponential stability of the corresponding switched system (3.1), (3.9), or (3.18). Any switching law $\sigma : \mathbf{R}_{0+} \to \overline{p}$ which leads to such a parameterization and stays at it during a sufficiently large residence time after the last time the same parameterization was switched off guarantees global Lyapunov's stability.

B.5. Proof of Theorem 3.13

Let the Krasovsky-Lyapunov functional candidate for the switched system (3.1) be

$$V(t,x_t) = x^T(t)P(t)x(t) + \int_{-h}^{0} x^T(t+\tau)S(t+\tau)x(t+\tau)d\tau \quad \forall t \in \mathbf{R}_{0+} \tag{B.18}$$

(see [1]), for some $P(t) = P^T(t) > 0$, $S(t) = S^T(t) > 0$ for all $t \in \mathbf{R}_{0+}$. Since $\mathrm{Re}\,(\lambda_i(A(t))) \leq -\rho_0 < 0$ (for all $i \in \overline{n}$ including eventual eigenvalues with the same real parts), then the Lyapunov equation $A_0^T(t)P(t) + P(t)A_0(t) = -I_n$ has a unique symmetric positive definite solution $P(t) = \int_0^t e^{A_0^T(t)\tau}e^{A_0(t)\tau}d\tau$ with uniformly bounded norm $\|P(t)\|_2 \leq K_0/2\rho_0$ for some $\mathbf{R}_+ \ni K_0 \geq 1$. The time derivative of this Lyapunov function is a Lyapunov equation of the form $A_0^T(t)\dot{P}(t) + \dot{P}(t)A_0(t) = -(\dot{A}_0^T(t)P(t) + P(t)\dot{A}_0(t))$ having a unique solution

$$\dot{P}(t) = \int_0^t e^{A_0^T(t)\tau}\left(\dot{A}_0^T(t)P(t) + P(t)\dot{A}_0(t)\right)e^{A_0(t)\tau}d\tau \tag{B.19}$$

and, from (B.19), $\|\dot{P}(t)\|_2 \leq (K_0/\rho_0)\|P(t)\|_2\|\dot{A}(t)\|_2 \leq (K_0^2/2\rho_0^2)\|\dot{A}_0(t)\|_2$. Taking time derivatives in (B.18) and using the above norm constraints:

$$\begin{aligned}\dot{V}(t,x_t) &\leq -x^T(t)\left(\left(1-\frac{K_0^2}{2\rho^2}\left\|\dot{A}_0(t)\right\|_2\right)I_n - S(t)\right)x(t) - x^T(t-h)S(t-h)x(t-h)\\ &\quad + x^T(t)\left(P(t)A_1(t) + A_1^T(t)P(t)\right)x(t-h)\\ &\leq \overline{x}^T(t)\overline{Q}(t)\overline{x}(t) \quad \forall t \in \mathbf{R}_{0+},\end{aligned} \tag{B.20}$$

where $\overline{x}^T(t) := (x^T(t), x^T(t-h))^T$, and

$$\overline{Q}(t) := \begin{bmatrix} S(t) + \left(\dfrac{K_0^2}{2\rho^2}\gamma_{dA} - 1\right)I_n & P(t)A_1(t) \\ A_1^T(t)P(t) & -S(t-h)\end{bmatrix} \quad \forall t \in \mathbf{R}_{0+} \tag{B.21}$$

so that $\dot{V}(t,x_t) < 0$ for $\overline{x}(t) \neq 0$ if $\overline{Q}(t) < -0$ for all $t \in \mathbf{R}_{0+} \setminus ST_{\mathrm{imp}}$, where $\dot{A}_0(t)$ is not impulsive and satisfies $\|\dot{A}_0(t)\|_2 \leq \gamma_{dA}$ for some $\gamma_{dA} \in \mathbf{R}_{0+}$. A necessary condition is that the matrix blocks of the main diagonal of $\overline{Q}(t)$ be negative definite, that is, $0 < S(t) < (1-(K_0^2/2\rho_0^2)\gamma_{dA})I_n$ for all $t \in \mathbf{R}_{0+} \setminus ST_{\mathrm{imp}}$. ST_{imp} is the set of zero measure where $A(t)$ has (isolated) jump-bounded discontinuities, so that $\dot{A}_0(t)$ is, equivalently, impulsive at $t = t_\alpha$, then satisfying $\dot{A}_0(t_\alpha^+) - \dot{A}_0(t_\alpha^-) = K(t_\alpha)\delta(t-t_\alpha)$ so that

$$V(t,x_t)\big|_{t=t_\alpha^+} - V(t,x_t)\big|_{t=t_\alpha^-} \leq \frac{K_0^2K_A}{2\rho_0^2}\left\|x(t_\alpha^-)\right\|_2^2 \quad \forall t_\alpha \in ST_{\mathrm{imp}}, \tag{B.22}$$

where $\delta(t)$ is the Dirac distribution at $t = 0$ and K_A is an upper-bound for the ℓ_2-norm of all the impulsive matrices $\max_{t_\alpha \in ST_{\text{imp}}} \int_{t_\alpha^-}^{t_\alpha^+} \|\dot{A}_0(\tau)\|_2 d\tau \leq K_0^2 K_A / 2\rho_0^2$ for the set of impulses. On the other hand, note from (B.18), that

$$\begin{aligned} \frac{V(t,x_t)}{\lambda_{\min}(P(t))} &\geq \frac{V(t,x_t) - \int_{-h}^{0} x^T(t+\tau)S(t+\tau)x(t+\tau)d\tau}{\lambda_{\min}(P(t))} \geq \|x(t)\|_2^2 \\ &\geq \frac{V(t,x_t) - \int_{-h}^{0} x^T(t+\tau)S(t+\tau)x(t+\tau)d\tau}{\lambda_{\max}(P(t))} \quad \forall t \in \mathbf{R}_{0+}, \end{aligned} \tag{B.23}$$

$$\lambda_{\min}(P(t))\|x(t)\|_2^2 \leq \gamma_1 \operatorname*{Inf}_{-h\leq\tau\leq 0} \|x(t+\tau)\|_2^2 \leq V(t,x_t) \leq \gamma_2 \operatorname*{Sup}_{-h\leq\tau\leq 0} \|x(t+\tau)\|_2^2 \tag{B.24}$$

for some $\gamma_{1,2} \in \mathbf{R}_+$.

Then, from (B.20), (B.21), $V(t,x_t)$ is positive and monotonically decreasing along any nontrivial solution trajectory within any open impulse-free time-interval and within any semiopen (right-closed) time interval starting in a point of ST_{imp} for any two consecutive $t_{\alpha_i}, t_{\alpha_{i+1}} \in ST_{\text{imp}}$ since $\overline{Q}(t) < 0$ within such intervals. More precisely, $V(t,x_t) - V(t,x_t)|_{t=t_{\alpha_i}^+} < 0$, $V(t') - V(t'') < 0$ for all $t_{\alpha_i} \in ST_{\text{imp}}$, for all $t(> t_{\alpha_i}) \in \mathbf{R}_{0+} \setminus ST_{\text{imp}}$, for all $t'(> t'') \in \mathbf{R}_{0+} \setminus ST_{\text{imp}}$. Thus, global asymptotic stability holds if $V(t,x_t)|_{t=t_{\alpha_{i+1}}^+} - V(t,x_t)|_{t=t_{\alpha_i}^+} < 0$ for any two consecutive $t_{\alpha_i}, t_{\alpha_{i+1}} \in ST_{\text{imp}}$. One gets from (B.24) that $V(t,x_t)|_{t=t_{\alpha_{i+1}}^+} \leq (1 + K_0^2 K_A / 2\rho_0^2 \lambda_{\min}(P(t_{\alpha_i+1}^-)))V(t,x_t)|_{t=t_{\alpha_i+1}^-}$. From (B.20), (B.23), and (B.24),

$$\begin{aligned} &V(t,x_t)\big|_{t=t_{\alpha_{i+1}}^+} - V(t,x_t)\big|_{t=t_{\alpha_i}^+} \\ &\quad \leq \frac{K_0^2 K_A V(t,x_t)|_{t=t_{\alpha_{i+1}}^-}}{2\rho_0^2 \lambda_{\min}(P(t_{\alpha_{i+1}}^-))} - \int_0^{t_{\alpha_{i+1}}-t_{\alpha_i}} \left|\overline{x}^T(t_{\alpha_i}+\tau)\overline{Q}(\tau)\overline{x}(t_{\alpha_i}+\tau)\right| d\tau < 0, \end{aligned} \tag{B.25}$$

provided that

$$t_{\alpha_{i+1}} > t_{\alpha_i} + \frac{K_0^2 K_A V(t,x_t)\big|_{t=t_{\alpha_{i+1}}^-}}{2\rho_0^2 \lambda_{\min}(P(t_{\alpha_{i+1}}^-)) \operatorname{Inf}_{0\leq\tau\leq t_{\alpha_{i+1}}-t_{\alpha_i}} \left|\overline{x}^T(t_{\alpha_i}+\tau)\overline{Q}(\tau)\overline{x}(t_{\alpha_i}+\tau)\right|} \quad \forall t_{\alpha_i}, t_{\alpha_{i+1}} \in ST_{\text{imp}}. \tag{B.26}$$

Then, $V(t,x_t)$ is uniformly bounded and converges asymptotically to zero as $t \to \infty$. Thus, the maintenance of a minimum time interval

$$\overline{T}_{\text{imp}} \geq \operatorname*{Sup}_{t\in\mathbf{R}_{0+}} \frac{K_0^2 K_A V(t,x_t)\big|_{t=t_{\alpha_{i+1}}^-}}{2\rho_0^2 \lambda_{\min}(P(t_{\alpha_{i+1}}^-)) \operatorname{Inf}_{0\leq\tau\leq t_{\alpha_{i+1}}-t_{\alpha_i}} \left|\overline{x}^T(t_{\alpha_i}+\tau)\overline{Q}(\tau)\overline{x}(t_{\alpha_i}+\tau)\right|} \geq t_{\alpha_{i+1}} - t_{\alpha_i} \tag{B.27}$$

between any two consecutive impulses at times $t_{\alpha_i}, t_{\alpha_{i+1}} \in ST_{\text{imp}}$ guarantees the global asymptotic stability and the result is proved.

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