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Research Article

Computation of Positive Realizations for Given Impulse Response Matrices of Discrete-Time Systems

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A method is presented for computation of positive realizations for given impulse response matrices of discrete-time systems. Sufficient conditions for the existence of positive realizations are established and a procedure for computation of positive realizations for given impulse response matrices is proposed.

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, and water and atmospheric pollution models. A variety of models having positive linear systems behavior can be found in engineering, management science, economics, social sciences, biology, medicine, and so forth.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [7, 11]. Recent developments in positive system theory and some new results are given in [12]. Realizations problem of positive linear systems without time delays has been considered in many papers and books [1–4, 6–11].

Recently, the reachability, controllability, and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [5, 18]. The realization problem for positive multivariable discrete-time systems and continuous-time systems with delays was formulated and solved in [13, 14, 16, 17]. The notion of cone realization of discrete-time systems without delays has been introduced in [15].

The main purpose of this paper is to present a method for computation of positive realizations for given impulse response matrices of discrete-time systems. Sufficient conditions for the existence of positive realizations will be established and a procedure for computation of positive realizations for given impulse response matrices will be proposed.

2. Preliminaries and problem formulation

Let $R^{n\times m}$ be the set of $n \times m$ real matrices and $R^m = R^{m\times 1}$.

Consider the discrete-time system

$$x_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ := \{0, 1, \dots\}, \tag{2.1a}$$

$$y_i = Cx_i + Du_i, \tag{2.1b}$$

where $x_i \in R^n$, $u_i \in R^m$, $y_i \in R^p$ are the state, input, and output vectors, respectively, and $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n},\ D \in R^{p\times m}$.

Let $R_+^{n\times m}$ be the set of $n \times m$ real matrices with nonnegative entries and $R_+^m = R_+^{m\times 1}$.

Definition 2.1 (see [7, 11]). The system (2.1a) and (2.1b) is called (internally) positive if for every $x_0 \in R_+^n$ and any input sequence $u_i \in R_+^m$, $x_i \in R_+^n$, and $y_i \in R_+^p$ for $i \in Z_+$.

Theorem 2.2 (see [11]). *The system* (2.1a) *and* (2.1b) *is positive if and only if*

$$A \in R_+^{n\times n}, \quad B \in R_+^{n\times m}, \quad C \in R_+^{p\times n}, \quad D \in R_+^{p\times m}. \tag{2.2}$$

The transfer matrix of (2.1a) *and* (2.1b) *is given by*

$$T(z) = C\left[I_n z - A\right]^{-1}B + D. \tag{2.3}$$

The impulse response matrix of (2.1a), (2.1b) *and the transfer matrix* (2.3) *are related by*

$$G = Z^{-1}[T(z)] = g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots, \tag{2.4}$$

where

$$g_i = \begin{cases} D, & \text{for } i = 0, \\ CA^{i-1}B, & \text{for } i = 1, 2, \dots. \end{cases} \tag{2.5}$$

If the system (2.1a) *and* (2.1b) *is positive, then [11]*

$$g_i \in R_+^{p\times m}, \quad \text{for } i \in Z_+. \tag{2.6}$$

It is well known [7, 11] that for positive asymptotically stable systems

$$g_{i+1} < g_i, \quad i = n, n+1, \dots, \tag{2.7a}$$

and for unstable systems

$$g_{i+1} \ge g_i, \quad i = n, n+1, \dots. \tag{2.7b}$$

The positive realization problem for a given impulse matrix can be stated as follows. Given an impulse response matrix (2.4) *satisfying the condition* (2.7a) *and* (2.7b), *find a positive realization* (2.2) *of the discrete-time system* (2.1a) *and* (2.1b).

In this paper, a procedure will be proposed for computation of a positive realization (2.2) for a given impulse response matrix (2.4).

3. Problem solution

To simplify the notation, we will consider the single-input single-output (SISO) discrete-time system with transfer function of the form

$$T(z) = \frac{b_{n-1}z^{n-1} + \cdots + b_1 z + b_0}{z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0} + D. \tag{3.1}$$

The transfer function (3.1) can be written in the form

$$T(z) = g_0 + g_1 z^{-1} + g_2 z^{-2} + \cdots. \tag{3.2}$$

From (3.1) and (3.2), we have

$$D = g_0 \tag{3.3}$$

and the strictly proper part of $T(z)$ is given by

$$T_{sp}(z) = T(z) - D = \frac{b_{n-1}z^{n-1} + \cdots + b_1 z + b_0}{z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0} = g_1 z^{-1} + g_2 z^{-2} + \cdots. \tag{3.4}$$

Knowing g_0 and using (3.3), we can find D. Therefore, the realization problem has been reduced to finding the matrices

$$A \in R_+^{n\times n}, \quad B \in R_+^{n\times m}, \quad C \in R_+^{p\times n} \tag{3.5}$$

for given $g_i \geq 0$ for $i = 1, 2, \ldots$.

In further considerations, it is assumed that the given sequence $g_i, i = 0, 1, \ldots$, satisfies the condition (2.7a) and(2.7b). Solution of the problem is based on the following lemmas.

Lemma 3.1. *The values $g_1, g_2, \ldots$ of the impulse response of the discrete-time system with transfer function* (3.4) *satisfy the equality*

$$a_0 g_k + a_1 g_{k+1} + \cdots + a_{n-1} g_{k+n-1} = g_{k+n}, \quad \text{for } k = 1, 2, \ldots. \tag{3.6}$$

Proof. From (3.4) we have

$$b_{n-1}z^{n-1} + \cdots + b_1 z + b_0 = \left(z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0\right)\left(g_1 z^{-1} + g_2 z^{-2} + \cdots\right). \tag{3.7}$$

Comparison of the coefficients at the same power z^{-k} of equality (3.7) yields (3.6). □

Using (3.6) for $k = 1, 2, \ldots, n$, we obtain the matrix equation

$$G_{nn} a = G_n, \tag{3.8}$$

where

$$G_{nn} = \begin{bmatrix} g_1 & g_2 & \cdots & g_n \\ g_2 & g_3 & \cdots & g_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ g_n & g_{n+1} & \cdots & g_{2n-1} \end{bmatrix}, \qquad a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}, \qquad G_n = \begin{bmatrix} g_{n+1} \\ g_{n+2} \\ \vdots \\ g_{2n} \end{bmatrix}. \tag{3.9}$$

From (3.4) and (3.6), it follows that

$$\operatorname{rank} G_{nn} = n. \tag{3.10}$$

Thus, solving (3.8) for given $g_1, g_2, \ldots, g_{2n}$, we can find the coefficients $a_0, a_1, \ldots, a_{n-1}$ of the denominator of the transfer function (3.4).

Lemma 3.2. *Let the given sequence $g_1, g_2, \ldots$ satisfy the condition* (2.7a) *and* (2.7b) *and*

$$G_{n+1,n+1} = \begin{bmatrix} G_{nn} & G_n \\ G_n^T & g_{2n+1} \end{bmatrix} \in R^{(n+1)\times(n+1)} \quad (T \text{ denotes the transpose}), \qquad G_n = \begin{bmatrix} g_{n+1} \\ g_{n+2} \\ \cdots \\ g_{2n} \end{bmatrix}. \tag{3.11}$$

Then,

$$\operatorname{rank} G_{n+1,n+1} = n \tag{3.12}$$

and the adjoint matrix

$$\overline{G}_{n+1,n+1} = \operatorname{Adj} G_{n+1,n+1} = \begin{bmatrix} \overline{G}_{nn} & \overline{G}_n \\ \overline{G}_n^T & \overline{g}_{n+1,n+1} \end{bmatrix}, \quad \overline{G}_{nn} \in R^{n\times n}, \ \overline{G}_n \in R^n, \tag{3.13}$$

has the following properties:

(i) *the columns $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_{n+1}$ of the matrix* (3.13) *are proportional, that is,*

$$\overline{G}_{n+1,n+1} = \begin{bmatrix} \overline{g}_1 & \overline{g}_2 & \cdots & \overline{g}_{n+1} \end{bmatrix} = \left[\overline{g}_1, k_1\overline{g}_1, \ldots, k_n\overline{g}_1\right], \tag{3.14}$$

$$k_i \begin{cases} > 0, & \text{for } i = 1, \ldots, n-1, \\ < 0, & \text{for } i = n; \end{cases} \tag{3.15}$$

(ii) *the square matrix $\overline{G}_{nn}$ has negative entries, the column matrix $\overline{G}_n$ has positive entries, and $\overline{g}_{n+1,n+1}$ is negative.*

Proof. The condition (3.12) follows from equality (3.6). The condition (3.12) implies that the dimension of the null subspace of $G_{n+1,n+1}$ is equal to 1, that is,

$$\dim \ker G_{n+1,n+1} = 1. \tag{3.16}$$

Thus the columns $\overline{g}_1, \overline{g}_2, \dots, \overline{g}_{n+1}$ of the matrix (3.13) are proportional. The adjoint matrix (3.13) is symmetrical since the matrix (3.11) is symmetrical. The symmetry of the matrix (3.13) implies the condition (3.15).

From definition of the adjoint matrix (3.13) and (3.12), we have

$$G_{n+1,n+1}\overline{G}_{n+1,n+1} = \overline{G}_{n+1,n+1}G_{n+1,n+1} = 0 \quad \text{(zero matrix)}. \tag{3.17}$$

From (3.17) and $g_i > 0$, $i = 1, 2, \dots$, it follows that every column and every row of the matrix (3.13) has at least one positive entry and at least one negative entry. Taking into account this fact and the symmetry of the matrix (3.13), it is easy to show that the condition (ii) holds. □

Theorem 3.3. *Let the given sequence $g_1, g_2, \dots$ satisfy the condition* (2.7a) *and* (2.7b) *and let there exist a natural number n such that*

$$\operatorname{rank} G_{n+1,n+1} = \operatorname{rank} G_{nn} = n. \tag{3.18}$$

Then, there exists a positive realization (3.5) *of the form*

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \tag{3.19}$$

or

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}, \qquad B = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{3.20}$$

Proof. Using the well-known Cramer rule for (3.8), we obtain

$$a_{i-1} = \frac{\det G_{ni}}{\det G_{nn}}, \quad \text{for } i = 1, \dots, n, \tag{3.21}$$

where G_{ni} is the $n \times n$ matrix obtained from the matrix G_{nn} by replacement of its ith column by the column G_n.

From the condition (ii) of Lemma 3.2 and (3.21), it follows that $a_{i-1} > 0$, for $i = 1, \dots, n$, since both $\det G_{ni}$ and $\det G_{nn}$ are negative. Hence, the matrices A of realizations (3.19) and (3.20) have nonnegative entries.

We will show that the matrices (3.19) and (3.20) are positive realizations of the transfer function (3.4). Using (3.19), we obtain

$$\begin{aligned} T(z) &= C\left[I_n z - A\right]^{-1}B = \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} z & 0 & \cdots & 0 & -a_0 \\ -1 & z & \cdots & 0 & -a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 & z - a_{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= \frac{\begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix}}{z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0} \begin{bmatrix} 1 & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \\ 0 & 1 & -a_{n-1} & \cdots & -a_3 & -a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \times \begin{bmatrix} z^{n-1} \\ z^{n-2} \\ \vdots \\ z \\ 1 \end{bmatrix}. \end{aligned} \tag{3.22}$$

Comparing the coefficients at the same power of z of equality (3.7), we obtain

$$b_{n-1} = g_1, \qquad b_{n-2} = g_2 - a_{n-1}g_1, \ldots, \qquad b_0 = g_n - a_{n-1}g_{n-1} - \cdots - a_1 g_1, \tag{3.23}$$

$$\begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} 1 & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \\ 0 & 1 & -a_{n-1} & \cdots & -a_3 & -a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 \end{bmatrix}. \tag{3.24}$$

Substitution of (3.24) into (3.22) yields

$$T(z) = \frac{\begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_0 \end{bmatrix}}{z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0} \begin{bmatrix} z^{n-1} \\ z^{n-2} \\ \vdots \\ z \\ 1 \end{bmatrix} = \frac{b_{n-1}z^{n-1} + \cdots + b_1 z + b_0}{z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0}. \tag{3.25}$$

The proof for the matrices (3.20) follows immediately from the equality

$$T(z) = [T(z)]^T = \left\{ C[I_n z - A]^{-1} B \right\}^T = B^T [I_n z - A^T]^{-1} C^T, \tag{3.26}$$

where upper index T denotes transpose. □

From the above considerations, we have the following procedure for finding a positive realization for given $g_i > 0,\ i = 0, 1, \ldots$.

Procedure 1.
Step 1. Knowing g_0 and using (3.3), find D.
Step 2. Using g_i, for $i = 1, \ldots, 2n$, find n satisfying the condition (3.18).
Step 3. Using (3.21), find the entries $a_0, a_1, \ldots, a_{n-1}$ of the matrix A.
Step 4. Find the matrices B and C of the positive desired realization.

4. Examples

The procedure will be illustrated by two numerical examples. In the first example, the system will be unstable and in the second one the system will be asymptotically stable.

Example 4.1. Find a positive realization (2.2) for the given values of the impulse function

$$g_0 = 2, \qquad g_1 = 1, \qquad g_2 = 3, \qquad g_3 = 5, \qquad g_4 = 11, \qquad g_5 = 25, \ldots. \tag{4.1}$$

Using Procedure 1, we obtain the following.

Step 1. Using (3.3), we get

$$D = g_0 = 2. \tag{4.2}$$

Step 2. In this case, $n = 2$ since

$$\operatorname{rank}\begin{bmatrix} g_1 & g_2 \\ g_2 & g_3 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} g_1 & g_2 & g_3 \\ g_2 & g_3 & g_4 \\ g_3 & g_4 & g_5 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 11 \\ 5 & 11 & 25 \end{bmatrix} = n = 2. \tag{4.3}$$

Step 3. Taking into account that

$$G_{22} = \begin{bmatrix} g_1 & g_2 \\ g_2 & g_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix} \tag{4.4}$$

and using (3.8), we obtain

$$\begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}, \tag{4.5}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{4.6}$$

Step 4. Using (3.19) and (4.6), we obtain

$$A = \begin{bmatrix} 0 & a_0 \\ 1 & a_1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} g_1 & g_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \end{bmatrix}. \tag{4.7}$$

The desired positive realization is given by (4.2) and (4.7).

The transfer function of the unstable system has the form

$$\begin{aligned} T(z) &= C[I_n z - A]^{-1}B + D = \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} z & -2 \\ -1 & z-1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 = \frac{2z^2 - z - 2}{z^2 - z - 2} \\ &= 2 + z^{-1} + 3z^{-2} + 5z^{-3} + 11z^{-4} + 25z^{-5} + \cdots. \end{aligned} \tag{4.8}$$

Example 4.2. Find a positive realization (2.2) for the values of the impulse function

$$g_0 = 0, \quad g_1 = 1, \quad g_2 = 1.1, \quad g_3 = 0.21, \quad g_4 = 0.131, \quad g_5 = 0.0341. \tag{4.9}$$

Using Procedure 1, we obtain the following.

Step 1. From (3.3), we have

$$D = g_0 = 0. \tag{4.10}$$

Step 2. Using (4.9) and (3.18), we obtain

$$\begin{aligned} \operatorname{rank} G_{22} &= \operatorname{rank}\begin{bmatrix} g_1 & g_2 \\ g_2 & g_3 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1.1 \\ 1.1 & 0.21 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} g_1 & g_2 & g_3 \\ g_2 & g_3 & g_4 \\ g_3 & g_4 & g_5 \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} 1 & 1.1 & 0.21 \\ 1.1 & 0.21 & 0.131 \\ 0.21 & 0.131 & 0.0341 \end{bmatrix} = n = 2. \end{aligned} \tag{4.11}$$

Step 3. In this case, (3.8) has the form

$$\begin{bmatrix} 1 & 1.1 \\ 1.1 & 0.21 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0.21 \\ 0.131 \end{bmatrix} \tag{4.12}$$

and its solution is

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1.1 \\ 1.1 & 0.21 \end{bmatrix}^{-1} \begin{bmatrix} 0.21 \\ 0.131 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}. \tag{4.13}$$

Step 4. Using (3.20) and (4.13), we obtain

$$A = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.1 & 0.1 \end{bmatrix}, \qquad B = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{4.14}$$

The desired positive realization is given by (4.10) and (4.14). The transfer function of the stable system has the form

$$\begin{aligned} T(z) = C\left[I_n z - A\right]^{-1}B + D &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z & -1 \\ -0.1 & z-0.1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1.1 \end{bmatrix} = \frac{z+1}{z^2 - 0.1\,z - 0.1} \\ &= z^{-1} + 1.1z^{-2} + 0.21z^{-3} + 0.131z^{-4} + 0.0341z^{-5} + \cdots. \end{aligned} \tag{4.15}$$

The above considerations can be easily extended for multi-input multi-output discrete-time systems.

5. Concluding remarks

A method for computation of positive realizations for given impulse response matrices of discrete-time systems has been proposed. The method is based on Lemma 3.2 and Theorem 3.3 that formulates sufficient conditions for the existence of positive realizations. A procedure for computation of positive realizations for given impulse responses has been proposed. The procedure has been illustrated by two numerical examples. The details of the proposed method have been given for SISO discrete-time systems but it can be easily extended for multi-input multi-output discrete-time and continuous-time linear systems.

An extension of the method is also possible for linear systems with delays and for 2D linear systems [11].

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