Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2008, Article ID 869614, 7 pages
doi:10.1155/2008/869614

Research Article

Applying He's Variational Iteration Method for Solving Differential-Difference Equation

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Received 31 January 2008; Revised 1 April 2008; Accepted 14 May 2008

Recommended by Oleg Gendelman

We extend He's variational iteration method (VIM) to find the approximate solutions for nonlinear differential-difference equation. Simple but typical examples are applied to illustrate the validity and great potential of the generalized variational iteration method in solving nonlinear differential-difference equation. The results reveal that the method is very effective and simple. We find the extended method for nonlinear differential-difference equation is of good accuracy.

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1. Introduction

In recent years, some promising approximate analytical solutions are proposed, such as exp-function method [1], homotopy perturbation method [2–11], and variational iteration method (VIM) [12–25]. The variational iteration method is the most effective and convenient one for both weakly and strongly nonlinear equations. This method has been shown to effectively, easily, and accurately solve a large class of nonlinear problems with component converging rapidly to accurate solutions.

Differential-difference equations (DDEs) have been the focus of many nonlinear studies. DDEs describe many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains, and so on. DDEs play important role in the study of modern physics and also play a crucial role in numerical simulations of nonlinear partial differential equations (NLPDEs), queueing problems, and discretization in solid state and quantum physics. At the same time, finding exact solutions of DDEs is extremely important in mathematical physics.

On the other hand, in order to find directly exact solutions to DDEs, some methods [16–26] for solving nonlinear differential equations are applied to DDEs. For example, Dehghan and Shakeri [16] have extended successfully multilinear variable separation approach to special DDEs. Baldwin et al. [26], Wang et al. [27] have applied homotopy analysis method (HAM) to DDEs. Dai and Zhang [28] have given a Jacobian elliptic function expansion method to solve the doubly periodic traveling wave solutions and kink-type tanh solitary solutions to some DDEs. There also have been some methods for nonlinear DDEs, such as Backlund transformation [29, 30], Hirota method [31, 32], Darboux transformation [33], and Adomian decomposition method [34].

2. He's variational iteration method

Now, to illustrate the basic concept of He's variational iteration method, we consider the following general nonlinear differential equation given in the form

$$Lu(t) + Nu(t) = g(t), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known analytical function. We can construct a correction functional according to the variational method as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\big(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\big)d\xi, \tag{2.2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via variational theory, the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation, namely $\delta\tilde{u}_n = 0$.

In the following example, we will illustrate the usefulness and effectiveness of the proposed technique.

3. Application to Volterra equation

Consider the following Volterra equation:

$$\frac{du_n}{dt} = u_n\big(u_{n+1} - u_{n-1}\big), \tag{3a}$$

with the initial condition

$$u_n(0) = n, \tag{3b}$$

whose exact solution can be written as

$$u_n(t) = \frac{n}{1-2t}. \tag{3.1}$$

We apply variational iteration method to the discussed problem. Using He's variational iteration method, the correction functional can be written in the form

$$u_{n,m+1}(t) = u_{n,m}(t) + \int_0^t \lambda(s)\left\{\frac{du_{n,m}(s)}{ds} - \big(u_{n,m}(s)\big)\big(u_{n+1,m}(s) - u_{n-1,m}(s)\big)\right\}ds. \tag{3.2}$$

The stationary conditions

$$1+\lambda = 0,\quad \lambda' = 0 \tag{3.3}$$

follow immediately. This in turn gives

$$\lambda = -1. \tag{3.4}$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (3.2) gives the iteration formula

$$u_{n,m+1}(t) = u_{n,m}(t) - \int_0^t \left\{ \frac{du_{n,m}(s)}{ds} - \left(u_{n,m}(s)\right)\left(u_{n+1,m}(s) - u_{n-1,m}(s)\right) \right\} ds. \tag{3.5}$$

We can start with $u_{n,0} = n$, and we obtain the following successive approximations:

$$\begin{aligned} u_{n,0}(t) &= n, \\ u_{n,1}(t) &= n + 2nt, \\ u_{n,2}(t) &= n + 2nt + 4nt^2, \\ u_{n,3}(t) &= n + 2nt + 4nt^2 + 8nt^3, \\ u_{n,4}(t) &= n + 2nt + 4nt^2 + 8nt^3 + 16nt^4. \end{aligned} \tag{3.6}$$

Hence, the solution series in general gives

$$u_n(t) = n + 2nt + 4nt^2 + 8nt^3 + 16nt^4 \dots, \tag{3.7}$$

$$u_n(t) = n\left(1 + 2t + 4t^2 + 8t^3 + 16t^4 \dots\right). \tag{3.8}$$

The closed form of the series (3.8) is $u_n(t) = n/(1-2t)$ which gives exact solution of problem.

4. Application to mKDV lattice equation

Consider the following discretized mKDV lattice equation:

$$\frac{du_n}{dt} = \left(1 - u_n^2\right)\left(u_{n+1} - u_{n-1}\right), \tag{16a}$$

with the initial condition

$$u_n(0) = \tanh(k)\tanh(kn). \tag{16b}$$

We apply variational iteration method to the discussed problem. Using He's variational iteration method, the correction functional can be written in the form

$$u_{n,m+1}(t) = u_{n,m}(t) + \int_0^t \lambda(s) \left\{ \frac{du_{n,m}(s)}{ds} - \left(1 - u_{n,m}^2(s)\right)\left(u_{n+1,m}(s) - u_{n-1,m}(s)\right) \right\} ds. \tag{4.1}$$

The stationary conditions

$$1+\lambda=0,\qquad \lambda'=0 \tag{4.2}$$

follow immediately. This in turn gives

$$\lambda=-1. \tag{4.3}$$

Substituting this value of the Lagrange multiplier $\lambda=-1$ into the functional (4.1) gives the iteration formula

$$u_{n,m+1}(t)=u_{n,m}(t)-\int_0^t\left\{\frac{du_{n,m}(s)}{ds}-\left(1-u_{n,m}^2(s)\right)\left(u_{n+1,m}(s)-u_{n-1,m}(s)\right)\right\}ds. \tag{4.4}$$

We can start with $u_{n,0}=\tanh(k)\tanh(kn)$, and we obtain the following successive approximations:

$$\begin{aligned}
u_{n,0}(t)&=\tanh(k)\tanh(kn),\\
u_{n,1}(t)&=\tanh(k)\tanh(kn)\\
&\quad+\Big[\tanh(k)\big(\tanh\left(k(n+1)\right)-\tanh\left(k(n-1)\right)\big)\\
&\qquad-\tanh^2(k)\tanh^2(kn)\big(\tanh(k)\big(\tanh\left(k(n+1)\right)-\tanh\left(k(n-1)\right)\big)\big)\Big]t,\\
u_{n,2}(t)&=\tanh(k)\tanh(kn)\\
&\quad+\Big[\tanh(k)\big(\tanh\left(k(n+1)\right)-\tanh\left(k(n-1)\right)\big)\\
&\qquad-\tanh^2(k)\tanh^2(kn)\big(\tanh(k)\big(\tanh\left(k(n+1)\right)-\tanh\left(k(n-1)\right)\big)\big)\Big]t\\
&\quad+\Big[\tanh(k)\tanh\left(k(n+2)\right)-2\tanh(k)\tanh(kn)\\
&\qquad-\tanh^2(k)\tanh^2\left(k(n+1)\right)\big(\tanh(k)\tanh\left(k(n+2)\right)\big)-\tanh(k)\tanh(kn)\\
&\qquad+\tanh(k)\tanh\left(k(n-2)\right)+\tanh^2(k)\tanh^2\left(k(n-1)\right)\big(\tanh(k)\tanh(kn)\big)\\
&\qquad-\tanh(k)\tanh\left(k(n-2)\right)-2\tanh(k)\tanh(kn)\tanh(k)\tanh\left(k(n+1)\right)\\
&\qquad-\tanh(k)\tanh\left(k(n-1)\right)\ \tanh(k)\tanh\left(k(n+1)\right)-\tanh(k)\tanh\left(k(n-1)\right)\\
&\qquad-\tanh^2(k)\tanh^2(kn)\ \tanh(k)\tanh\left(k(n+1)\right)-\tanh(k)\tanh\left(k(n-1)\right)\\
&\qquad-\tanh^2(k)\tanh^2(kn)\big(\tanh(k)\tanh\left(k(n+2)\right)-2\tanh(k)\tanh(kn)\\
&\qquad\qquad-\tanh^2(k)\tanh^2\left(k(n+1)\right)\big)\\
&\qquad\times\big(\tanh(k)\tanh\left(k(n+2)\right)-\tanh(k)\tanh(kn)\big)+\tanh(k)\tanh\left(k(n-2)\right)\\
&\qquad+\tanh^2(k)\tanh^2\left(k(n-1)\right)\ \tanh(k)\tanh(kn)-\tanh(k)\tanh\left(k(n-2)\right)\Big]0.5t^2.
\end{aligned} \tag{4.5}$$

The other components of $u_{n,m}(t)$ can be generated in a similar way. Generally speaking, it is possible to calculate more components via some calculation software such as Maple to improve the accuracy of the approximate solutions.

Table 1: For constant $k = 0.1$, and time $t = 0.5$.

n	ADM-u_6	VIM-u_2	Absolute error
–25	−0,09804197166	−0,09804373331	0.00000176165
–15	−0,08824837298	−0,08825728153	0,00000890855
–5	−0,03789706610	−0,03788612415	0,00001094195
0	0,009900946992	0,009933709149	0,000032762157
5	0,05350310282	0,05351081023	0,00000770741
15	0,091855587327	0,09184745591	0,00000813142
25	0,09857365542	0,09857205219	0,00000160323

Table 2: For constant $k = 0.1$, and time $t = 1.5$.

n	ADM-u_6	VIM-u_2	Absolute error
–25	−0,09725516662	−0,09730760788	0,00005244126
–15	−0,083118934180	−0,08337156677	0,00025263259
–5	−0,01977150813	−0,01938285277	0,00038865536
0	0,02894478018	0,02890112744	0,00004365274
5	0,06613063122	0,06625691101	0,00012627979
15	0,09435553904	0,09414208992	0,00021344912
25	0,09893218337	0,09889256453	0,00003961884

In order to verify numerically whether the proposed methodology leads to high accuracy, we evaluate the numerical solutions using only second-order approximation and compared it with Adomian decomposition solution (ADM) using six-term approximation [34]. Tables 1 and 2 show the absolute errors between ADM-u_6 and numerical solution (VIM-u_2) of (16a) with initial condition (16b).

Tables 1 and 2 show that the numerical approximate solution has a high degree of accuracy. As we know, the more terms added to the approximate solution, the more accurate it will be. Although we only considered second-order approximation, it achieves a high level of accuracy.

5. Conclusion

In this paper, by the variational iteration method, firstly, we obtain the exact solution of Volterra equation. Secondly, we obtain the approximate solution of mKDV lattice equation. The method is extremely simple, easy to use, and is very accurate for solving nonlinear differential-difference equation. Also, the method is a powerful tool to search for solutions of various linear/nonlinear problems. This variational iteration method will become a much more interesting method to solve nonlinear DDEs in science and engineering.

Acknowledgment

The author thanks The Scientific and Technological Research Council of Turkey (TÜB İ TAK) for their financial support.

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