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Research Article

Mannheim Offsets of Ruled Surfaces

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In a recent works Liu and Wang (2008; 2007) study the Mannheim partner curves in the three dimensional space. In this paper, we extend the theory of the Mannheim curves to ruled surfaces and define two ruled surfaces which are offset in the sense of Mannheim. It is shown that, every developable ruled surface have a Mannheim offset if and only if an equation should be satisfied between the geodesic curvature and the arc-length of spherical indicatrix of it. Moreover, we obtain that the Mannheim offset of developable ruled surface is constant distance from it. Finally, examples are also given.

1. Introduction

A surface is said to be "ruled" if it is generated by moving a straight line continuously in Euclidean space $\mathbb{E}^3$. Ruled surfaces are one of the simplest objects in geometric modeling.

One important fact about ruled surfaces is that they can be generated by straight lines. One would never know this from looking at the surface or its usual equation in terms of x, y, and z coordinates, but ruled surfaces can all be rewritten to highlights the generating lines. A practical application of ruled surfaces is that they are used in civil engineering. Since building materials such as wood are straight, they can be thought of as straight lines. The result is that if engineers are planning to construct something with curvature, they can use a ruled surface since all the lines are straight.

Among ruled surfaces, developable surfaces form an important subclass since they are useful in sheet metal design and processing. Every developable surface can be obtained as the envelope surface of a moving plane (under a one-parameter motion). Developable ruled surfaces are well-known and widely used in computer aided design and manufacture. A "developable" ruled surface is a surface that can be rolled on a plane, touching along

the entire surface as it rolls. Such a surface has a constant tangent plane for the whole length of each ruling. Parallel geodesic loops (in a direction perpendicular to the rulings) on closed developable ruled surfaces all have the same length; such surfaces are thus "constant perimeter" surfaces.

In the past, offsets of ruled surfaces have been the subject of some studies: Ravani and Ku [1], studied Bertrand offsets of ruled surfaces. Pottman et al. [2], presented classical and circular offsets of rational ruled surfaces.

In this paper, the Mannheim offsets of ruled surfaces are considered. It is shown that a theory similar to that of the Mannheim partner curves can be developed for ruled surfaces.

2. Mannheim Offset of a Curve

Offset curves play an important role in areas of CAD/CAM, robotics, cam design and many industrial applications, in particular in mathematical modeling of cutting paths milling machines. The classic work in this area is that of Bertrand [3], who studied curve pairs which have common principal normals. Such curves referred to as Bertrand curves and can be considered as offsets of one another. Another kind of associated curves is the Mannheim offsets.

In plane, a curve α rolls on a straight line, the center of curvature of its point of contact describes a curve β which is the Mannheim of α, [4].

The theory of the Mannheim curves has been extended in the three dimensional Euclidean space by Liu and Wang [5, 6].

Let C and C^* be two space curves. C is said to be a *Mannheim partner curve* of C^*, if there exists a one to one correspondence between their points such that the binormal vector of C is the principal normal vector of C^*. Such curves are referred to as *"Mannheim offsets,"* [5].

Let $C^* : \boldsymbol{\alpha} = \boldsymbol{\alpha}(s^*)$ be a Mannheim curve with the arc-length parameter s^*. Then $C : \boldsymbol{\beta} = \boldsymbol{\beta}(s)$ is the Mannheim partner curve of C^* if and only if the curvature κ and the torsion τ of C satisfy the following equation

$$\dot{\tau} = \frac{d\tau}{ds} = \frac{\kappa}{\lambda}\left(1 + \lambda^2\tau^2\right) \tag{2.1}$$

for some nonzero constant λ, [5].

The detailed discussion concerned with the Mannheim curves can be found in [5, 6].

3. Differential Geometry of Ruled Surfaces

A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation,

$$\varphi(s, v) = \boldsymbol{\alpha}(s) + v\mathbf{e}(s), \tag{3.1}$$

where $\boldsymbol{\alpha}(s)$ represents a space curve which is called *the base curve* and $\mathbf{e}$ is a unit vector representing the direction of a straight line.

The vector $\mathbf{e}$ traces a curve on the surface of unit sphere S^2 called *spherical indicatrix* of the ruled surface, [1].

The orthonormal system $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ is called *the geodesic Frenet thiedron* of the ruled surface φ such that $\mathbf{t} = \mathbf{e}_s/\|\mathbf{e}_s\|$ and $\mathbf{g} = (\mathbf{e} \times \mathbf{e}_s)/\|\mathbf{e}_s\|$ are *the central normal* and *the asymptotic normal direction* of φ, respectively.

For the geodesic Frenet vectors $\mathbf{e}$, $\mathbf{t}$ and $\mathbf{g}$, we can write

$$\begin{aligned} \mathbf{e}_q &= \mathbf{t} \\ \mathbf{t}_q &= \gamma \mathbf{g} - \mathbf{e} \\ \mathbf{g}_q &= -\gamma \mathbf{t}, \end{aligned} \tag{3.2}$$

where q and γ are the arc-length of spherical indicatrix $(\mathbf{e})$ and the geodesic curvature of $(\mathbf{e})$ with respect to S^2, respectively [1].

The striction point on a ruled surface φ is the foot of the common normal between two consecutive generators (or ruling). The set of striction points defines *the striction curve* given by

$$\mathbf{c}(s) = \boldsymbol{\alpha}(s) - \frac{\langle \boldsymbol{\alpha}_s, \mathbf{e}_s \rangle}{\langle \mathbf{e}_s, \mathbf{e}_s \rangle} \mathbf{e}(s). \tag{3.3}$$

If consecutive generators of a ruled surface intersect, then the surface is said to be *developable.* The spherical indicatrix, $\mathbf{e}$, of a developable surface is tangent of its striction curve, [1].

The distribution parameter of the ruled surface φ is defined by

$$P_e = \frac{\det(\boldsymbol{\alpha}_s, \mathbf{e}, \mathbf{e}_s)}{\|\mathbf{e}_s\|^2}. \tag{3.4}$$

The ruled surface is developable if and only if $P_e = 0$.

In this paper, the striction curve of the ruled surface φ will be taken as the base curve. In this case, for the parametric equation of φ, we can write

$$\varphi(s, v) = \mathbf{c}(s) + v\mathbf{e}(s). \tag{3.5}$$

4. Mannheim Offsets of Ruled Surfaces

The ruled surface φ^* is said to be *Mannheim offset* of the ruled surface φ if there exists a one to one correspondence between their rulings such that the asymptotic normal of φ is the central normal of φ^*. In this case, (φ, φ^*) is called *a pair of Mannheim ruled surface*.

Let φ and φ^* be two ruled surfaces which is given by

$$\begin{aligned} \varphi(s, v) &= \mathbf{c}(s) + v\mathbf{e}(s), \quad \|\mathbf{e}(s)\| = 1, \\ \varphi^*(s, v) &= \mathbf{c}^*(s) + v\mathbf{e}^*(s), \quad \|\mathbf{e}^*(s)\| = 1, \end{aligned} \tag{4.1}$$

where $(\mathbf{c})$ and $(\mathbf{c}^*)$ are the striction curves of φ and φ^*, respectively.

If φ^* is a Mannheim offset of φ, then we can write

$$\begin{bmatrix} \mathbf{e}^* \\ \mathbf{t}^* \\ \mathbf{g}^* \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ \sin\theta & -\cos\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{t} \\ \mathbf{g} \end{bmatrix}, \tag{4.2}$$

where $\{\mathbf{e}, \mathbf{t}, \mathbf{g}\}$ and $\{\mathbf{e}^*, \mathbf{t}^*, \mathbf{g}^*\}$ are the geodesic Frenet triplies at the point $\mathbf{c}(s)$ and $\mathbf{c}^*(s)$ of the striction curves of φ and φ^*, respectively.

The equation of φ^* in terms of φ can therefore be written as

$$\varphi^*(s, v) = \mathbf{c}(s) + R(s)\mathbf{g}(s) + v\left[\cos\theta\mathbf{e}(s) + \sin\theta\mathbf{t}(s)\right], \tag{4.3}$$

where $R = R(s)$ is distance between corresponding striction points and θ is the angle between corresponding rulings.

Let the ruled surface φ^* be Mannheim offset of the ruled surface φ. By definition,

$$\mathbf{t}^* = \mathbf{g}. \tag{4.4}$$

From the definition $\mathbf{t}^*$, we get $\mathbf{t}^* = \mathbf{e}_s^*/\|\mathbf{e}_s^*\|$.

Because of the last two equation, we have $\mathbf{e}_s^* = \lambda\mathbf{g}$ (λ a scalar). Since the base curve of φ^* is its striction curve, we get $\langle \mathbf{c}_s^*, \mathbf{e}_s^* \rangle = 0$.

From the equality $\mathbf{e}_s^* = \lambda\mathbf{g}$ it follows that $\langle (\mathbf{c} + R\mathbf{g})_s, \mathbf{g} \rangle = 0$ It therefore follows that $\|\mathbf{e}_s\|P_e + R_s = 0$. Thus we have the following theorem.

Theorem 4.1. *Let the ruled surface φ^* be Mannheim offset of the ruled surface φ. Then φ is developable if and only if R is a constant.*

Theorem 4.2. *Let the ruled surface φ^* be Mannheim offset of the developable ruled surface φ. Then φ^* is developable if and only if the following relationship can be written*

$$\sin\theta + R\gamma q_s \cos\theta = 0. \tag{4.5}$$

Proof. Suppose that φ^* is developable. Then we have

$$\mathbf{c}_s^* = \mu\mathbf{e}^* \quad (\mu \text{ a scalar}), \tag{4.6}$$

where s is the arc-length parameter of the striction curve ($\mathbf{c}$) of φ. Then we obtain

$$\mathbf{c}_s + Rq_s\mathbf{g}_q + R_s\mathbf{g} = \mu[\cos\theta\mathbf{e} + \sin\theta\mathbf{t}]. \tag{4.7}$$

From Theorem 4.1 and the realtion (3.2), we get

$$\mathbf{e} + Rq_s(-\gamma\mathbf{t}) = \mu\cos\theta\mathbf{e} + \mu\sin\theta\mathbf{t}. \tag{4.8}$$

The last equation implies that

$$\sin\theta + R\gamma q_s \cos\theta = 0. \tag{4.9}$$

Conversely, suppose that the equality

$$\sin\theta + R\gamma q_s \cos\theta = 0 \tag{4.10}$$

is satisfied. For the tangent of the striction curve of φ^*, we can write,

$$\begin{aligned} \mathbf{c}_s^* &= (\mathbf{c} + R\mathbf{g})_s \\ &= \mathbf{e} - R\gamma q_s \mathbf{t} \\ &= \frac{1}{\cos\theta}[\cos\theta\mathbf{e} + \sin\theta\mathbf{t}] \\ &= \frac{1}{\cos\theta}\mathbf{e}^*. \end{aligned} \tag{4.11}$$

Thus, φ^* is developable. □

Theorem 4.3. *Let φ be a developable ruled surface. The developable ruled surface φ^* is a Mannheim offset of the ruled surface φ if and only if the following relationship is satisfied:*

$$\gamma_s = \frac{d\gamma}{ds} = \frac{1}{R}(1 + R^2\gamma^2 q_s^2) - \frac{1}{q_s}\gamma q_{ss}. \tag{4.12}$$

Proof. Suppose that the developable ruled surface φ^* is a Mannheim offset of φ. Because of Theorem 4.2, we get

$$R\gamma q_s = -\tan\theta. \tag{4.13}$$

Using (4.2) and the chain rule of differentiation, we can write

$$\mathbf{e}_s^* = -\sin\theta(\theta_s + q_s)\mathbf{e} + \cos\theta(\theta_s + q_s)\mathbf{t} + \gamma q_s \sin\theta\mathbf{g}. \tag{4.14}$$

From (4.14) and definition of $\mathbf{t}^*$, we have

$$\theta_s = -q_s. \tag{4.15}$$

By taking the derivative of (4.13) with respect to arc s and using (4.15), we obtain

$$\gamma_s = \frac{d\gamma}{ds} = \frac{1}{R}(1 + R^2\gamma^2 q_s^2) - \frac{1}{q_s}\gamma q_{ss}. \tag{4.16}$$

Conversely, suppose that the equality $\gamma_s = d\gamma/ds = (1/\mathrm{R})(1 + R^2\gamma^2 q_s^2) - (1/q_s)\gamma q_{ss}$ is satisfied. For nonzero constant scalar R, we can define the ruled surface

$$\varphi^*(s,v) = \mathbf{c}^*(s) + v\mathbf{e}^*(s), \tag{4.17}$$

where $\mathbf{c}^*(s) = \mathbf{c}(s) + R\mathbf{g}(s)$.

We will prove that φ^* is a Mannheim offset of φ. Since φ^* is developable, we have

$$\mathbf{c}_s^* = \frac{ds^*}{ds}\mathbf{e}^*, \tag{4.18}$$

where s and s^* are the arc-length parameter of the striction curves $(\mathbf{c})$ and $(\mathbf{c}^*)$, respectively. From the equality $\mathbf{c}^*(s) = \mathbf{c}(s) + R\mathbf{g}(s)$ and (4.18), we get

$$\frac{ds^*}{ds}\mathbf{e}^* = \mathbf{e} - R\gamma q_s\mathbf{t}. \tag{4.19}$$

By taking the derivative of (4.19) with respect to arc s, we obtain

$$\frac{d^2s^*}{ds^2}\mathbf{e}^* + \frac{ds^*}{ds}\mathbf{e}_s^* = R\gamma q_s^2\mathbf{e} + (q_s - R\gamma_s q_s - R\gamma q_{ss})\mathbf{t} - R\gamma^2 q_s^2\mathbf{g}. \tag{4.20}$$

From the hypothesis and the definition of $\mathbf{t}^*$, we get

$$\frac{d^2s^*}{ds^2}\mathbf{e}^* + \frac{ds^*}{ds}\lambda\mathbf{t}^* = R\gamma q_s^2\mathbf{e} - R^2\gamma^2 q_s^3\mathbf{t} - R\gamma^2 q_s^2\mathbf{g}, \tag{4.21}$$

where λ is a scalar.

By taking the cross product of (4.19) with (4.21), we have

$$\left(\frac{ds^*}{ds}\right)^2\lambda\mathbf{g}^* = R^2\gamma^3 q_s^3\mathbf{e} + R\gamma^2 q_s^2\,\mathbf{t}. \tag{4.22}$$

Taking the cross product of (4.22) with (4.19), we obtain

$$\left(\frac{ds^*}{ds}\right)^3\lambda\mathbf{t}^* = -(R\gamma^2 q_s^2 + R^3\gamma^4 q_s^4)\mathbf{g}. \tag{4.23}$$

Thus, the developable ruled surface φ^* is a Mannheim offset of the ruled surface φ. □

Let the ruled surface φ^* be a Mannheim offset of the ruled surface φ. If the ruled surfaces which is generated by the vectors $\mathbf{t}^*$ and $\mathbf{g}^*$ of φ^* denote by $\varphi_{\mathbf{t}^*}$ and $\varphi_{\mathbf{g}^*}$, respectively, then we can write

$$\begin{aligned} \mathbf{e}_1^* &= \mathbf{g}, & \mathbf{t}_1^* &= \mp\mathbf{t}, & \mathbf{g}_1^* &= \pm\mathbf{e}, \\ \mathbf{e}_2^* &= \sin\theta\mathbf{e} - \cos\theta\mathbf{t}, & \mathbf{t}_2^* &= \mp\mathbf{g}, & \mathbf{g}_2^* &= \mp\cos\theta\mathbf{e} \pm \sin\theta\mathbf{t}, \end{aligned} \tag{4.24}$$

where $\{\mathbf{e}_1^*, \mathbf{t}_1^*, \mathbf{g}_1^*\}$ and $\{\mathbf{e}_2^*, \mathbf{t}_2^*, \mathbf{g}_2^*\}$ are the geodesic Frenet triplies of the striction curves of $\varphi_{\mathbf{t}^*}$ and $\varphi_{\mathbf{g}^*}$, respectively. Therefore, from (4.24) we have the following.

Corollary 4.4. (a) $\varphi_{\mathbf{t}^*}$ *is a Bertrand offset of* φ.

(b) $\varphi_{\mathbf{g}^*}$ *is a Mannheim offset of* φ.

Now, one will investigate developable of $\varphi_{\mathbf{t}^*}$ *and* $\varphi_{\mathbf{g}^*}$ *while* φ *is developable:*

Let the ruled surface φ^* *be a Mannheim offset of the developable ruled surface* φ. *From* (3.2), (3.4), *and* (4.2), *it is easy to see that,*

$$P_{\mathbf{t}^*} = \frac{1}{\gamma q_s}, \qquad P_{\mathbf{g}^*} = \frac{1}{\gamma q_s \cos\theta}(\cos\theta - R\gamma q_s \sin\theta). \tag{4.25}$$

As an immediate result we have the following.

Corollary 4.5. (a) $\varphi_{\mathbf{t}^*}$ *is nondevelopable while* φ *is developable.*

(b) $\varphi_{\mathbf{g}^*}$ *is developable while* φ *is developable if and only if the relationship* $\cos\theta - R\gamma q_s \sin\theta = 0$ *is satisfied.*

Example 4.6. The elliptic hyperboloid of one sheet is a ruled surface parametrized by

$$\varphi(s,v) = \left(\cos(s) - \frac{\sqrt{2}}{2}v\sin(s), \sin(s) + \frac{\sqrt{2}}{2}v\cos(s), \frac{\sqrt{2}}{2}v\right). \tag{4.26}$$

A Mannheim offset of this surface is

$$\begin{aligned}\varphi^*(s,v) = \Bigg(&\cos(s) - \frac{\sqrt{2}}{2}\sin(s)s - \left(1 + \frac{\sqrt{2}}{2}\right)v\cos(s)\sin(s),\\ &\sin(s) - \frac{\sqrt{2}}{2}\cos(s)s + \frac{\sqrt{2}}{2}v\cos^2(s) - v\sin^2(s),\\ &\frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2}v\cos(s)\Bigg),\end{aligned} \tag{4.27}$$

where $R = R(s) = s$.

Example 4.7. The surface

$$\varphi(s,v) = \left(\cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{\sqrt{2}}{2}v\sin\left(\frac{\sqrt{2}}{2}s\right), \sin\left(\frac{\sqrt{2}}{2}s\right) + \frac{\sqrt{2}}{2}v\cos\left(\frac{\sqrt{2}}{2}s\right), \frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2}v\right) \tag{4.28}$$

is a developable ruled surface.

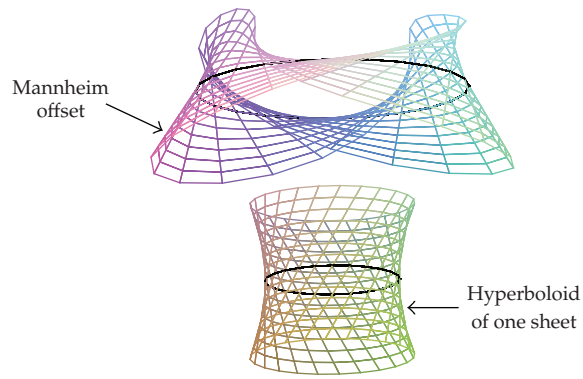

Figure 1: Hyperboloid of one sheet and its Mannheim offset.

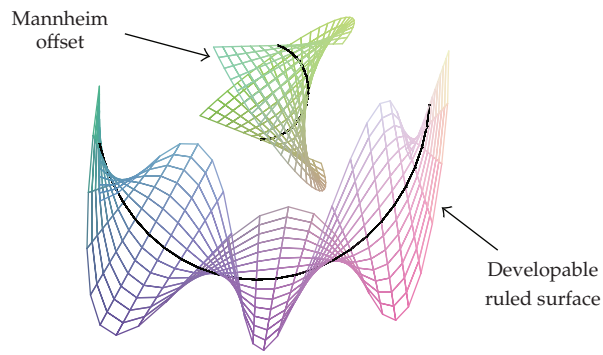

Figure 2: Developable ruled surface and its Mannheim offset.

A Mannheim offset of this surface is

$$\begin{aligned}\varphi^*(s,v) = \Bigg(&\cos\left(\frac{\sqrt{2}}{2}s\right) - \frac{5\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}s\right) - \left(1+\frac{\sqrt{2}}{4}\right)v\sin\left(\sqrt{2}s\right),\\ &\sin\left(\frac{\sqrt{2}}{2}s\right) - \frac{5\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}s\right) + \frac{\sqrt{2}}{2}v\cos^2\left(\frac{\sqrt{2}}{2}s\right) - v\sin^2\left(\frac{\sqrt{2}}{2}s\right),\\ &\frac{\sqrt{2}}{2}s + \frac{5\sqrt{2}}{2} + \frac{\sqrt{2}}{2}v\cos\left(\frac{\sqrt{2}}{2}s\right)\Bigg),\end{aligned} \tag{4.29}$$

where $R = R(s) = 5$. (See Figures 1 and 2.)

5. Conclusion

In this paper, a generalization of Mannheim offsets of curves for ruled surfaces has been developed. Interestingly, there are many similarities between the theory of Mannheim offsets in $\mathbb{E}^2$ and the theory of Mannheim offsets of ruled surfaces in $\mathbb{E}^3$. For instance, a ruled surface can have an infinity of Mannheim offsets in the some way as a plane curve can have an infinity of Mannheim mates. Furthermore, in analogy with three dimensional curves, a developable

ruled surface can have a developable Mannheim offset if a equation holds between the geodesic curvature and the arc-length of its spherical indicatrix.

Table of Symbols

$\mathbb{E}^3$:	Euclidean space of dimension three
κ:	curvature of a curve
τ:	torsion of a curve
s:	arc-length
s^*:	arc-length
S^2:	unit sphere
$\mathbf{e}$:	spherical indicatrix vector
$\mathbf{t}$:	central normal
$\mathbf{g}$:	asymptotic normal
$(\mathbf{e})$:	spherical indicatrix
γ:	geodesic curvature of $(\mathbf{e})$
q:	arc-length of $(\mathbf{e})$
$(\mathbf{c})$:	striction curve
P_e:	distribution parameter
$R = R(s)$:	function of distance
$\langle , \rangle$:	Riemannian metric

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