Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 346461, 9 pages
doi:10.1155/2009/346461

Research Article

New Exact Solutions of the Brusselator Reaction Diffusion Model Using the Exp-Function Method

F. Khani,[1] F. Samadi,[2] and S. Hamedi-Nezhad[1]

[1] *Department of Mathematics, Bakhtar Institute of Higher Education, P.O. Box 696, Ilam, Iran*
[2] *Department of Mathematics, Payam Noor University, Kermanshah, Iran*

Correspondence should be addressed to F. Khani, fkhani@razi.ac.ir

Received 2 December 2008; Revised 8 February 2009; Accepted 7 July 2009

Recommended by Katica R. (Stevanovic) Hedrih

Firstly, using a series of transformations, the Brusselator reaction diffusion model is reduced into a nonlinear reaction diffusion equation, and then through using Exp-function method, more new exact solutions are found which contain soliton solutions. The suggested algorithm is quite efficient and is practically well suited for use in these problems. The results show the reliability and efficiency of the proposed method.

1. Introduction

The Brusselator reaction model plays an important role both in biology and in chemistry. Since the model was put forward by Prigogine and Lefever in 1968, much attention had been paid to the model and many properties of it had been researched by marly people via using different methods [1–5]. In this paper, we mainly consider the model improved by Lefever et al. in 1977 [1] . The model is described as follows:

$$\begin{aligned} K\psi_{xx} - \psi_t + \psi^2\varphi - B\psi &= 0, \\ K\varphi_{xx} - \varphi_t - \psi^2\varphi + B\psi &= 0, \end{aligned} \tag{1.1}$$

where B is a constant, and K is the diffusion coefficient; the functions $\psi(x,t)$ and $\varphi(x,t)$ denote the concentrations. System (1.1) describes a biochemical model. Recently, many approaches have been suggested to solve the nonlinear equations, such as the variational iteration method [6–8], the homotopy perturbation method [9–11], the tanh-method [12], the extended tanh-method [13], the sinh-method [14], the homogeneous balance method [15, 16],

the F-expansion method [17], and the extended Fan's subequation method [18]. Recently, He and Wu [19] have proposed a straightforward method called Exp-function method to obtain the exact solutions of nonlinear evolution equations (NLEEs). It should be pointed out that the method is also valid for difference-different equations [20, 21]. The solution's procedure of this method is of utter simplicity, and this method has been successfully applied to many kinds of NLEEs [22–33]. The Exp-function method not only provides generalized solitonary solutions but also provides periodic solutions. Taking advantage of the generalized solitonary solutions, we can recover some known solutions obtained by the most existing methods such as decomposition method, tanh-function method, algebraic method, extended Jacobi elliptic function expansion method, F-expansion method, auxiliary equation method, and others [22–33].

2. Exp-Function Method and Exact Solutions

In this section we intend to find a solitary wave solution of (1.1). Therefore by using the following transformations:

$$\psi = v, \qquad \varphi = -v, \qquad x = \pm y, \qquad t = t, \tag{2.1}$$

the system (1.1) is reduced to a nonlinear reaction diffusion equation with respect to $v(y,t)$:

$$Kv_{yy} - v_t - Bv - v^3 = 0. \tag{2.2}$$

After that we use the transformation

$$v = \phi(\xi), \qquad \xi = \lambda(y - wt + \gamma), \tag{2.3}$$

where λ and w are constants to be determined later, and γ is arbitrary constant. Therefore (2.2) converts to

$$K\lambda^2 \frac{d^2\phi}{d\xi^2} + \lambda w \frac{d\phi}{d\xi} - B\phi - \phi^3 = 0. \tag{2.4}$$

By virtue of the Exp-function method [19], we assume that the solution of (2.4) is of the form

$$\phi(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_f \exp(f\xi) + \cdots + b_{-g} \exp(-g\xi)}, \tag{2.5}$$

where c, d, f, and g are unknown positive integers and to be determined later; a_n and b_m are constants.

By balancing the highest order of linear term ϕ'' with the highest order of nonlinear term ϕ^3, the values of c and f can be determined easily.

Since

$$\phi'' = \frac{c_1 \exp[(c+3f)\xi] + \cdots}{c_2 \exp[4f\xi] + \cdots},$$
$$\phi^3 = \frac{c_3 \exp[(3c+f)\xi] + \cdots}{c_4 \exp[4f\xi] + \cdots}, \tag{2.6}$$

setting

$$c + 3f = 3c + f \tag{2.7}$$

leads easily to $c = f$.

Similarly, to determine d and g, we balance the lowest-order linear term of Exp-function in (2.4):

$$\phi'' = \frac{\cdots + d_1 \exp[-(d+3g)\xi]}{\cdots + d_2 \exp[-3g\xi]},$$
$$\phi^3 = \frac{\cdots + d_3 \exp[-(3d+g)\xi]}{\cdots + d_4 \exp[-3g\xi]}. \tag{2.8}$$

This requires

$$-(d+3g) = -(3d+g), \tag{2.9}$$

which leads to $g = d$.

We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend on the choice of the values of c and d [19, 28]. Choosing $f = c = 1$ and $d = g = 1$ for simplicity causes the trial function (2.5) to become

$$\phi(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{2.10}$$

By substituting (2.10) into (2.4), we get

$$\frac{1}{Q}\left(k_3 \exp(3\xi) + k_2 \exp(2\xi) + k_1 \exp(\xi) + k_0 + k_{-1} \exp(-\xi) + k_{-2} \exp(-2\xi) + k_{-3} \exp(-3\xi)\right) = 0, \tag{2.11}$$

where $Q = (b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi))^3$; k_i $(i = -3, -2, \ldots, 2, 3)$ are constants. If the coefficients of $\exp(n\xi)$ are set to zero, we have

$$\begin{gathered} k_3 = 0, \qquad k_2 = 0, \qquad k_1 = 0, \\ k_0 = 0, \\ k_{-3} = 0, \qquad k_{-2} = 0, \qquad k_{-1} = 0. \end{gathered} \tag{2.12}$$

By solving this system of algebraic equations, we obtain the following sets of solutions.

Case A.

$$\begin{gathered} a_1 = 0, \qquad a_0 = a_0, \qquad a_{-1} = 0, \qquad \lambda^2 = \frac{B}{K}, \\ b_1 = b_1, \qquad b_0 = b_0, \qquad b_{-1} = -\frac{a_0^2}{8Bb_1}, \qquad w = 0. \end{gathered} \tag{2.13}$$

Case B.

$$\begin{gathered} a_1 = 0, \qquad a_0 = a_0, \qquad a_{-1} = \frac{Bb_{-1}}{\sqrt{-B}}, \qquad \lambda^2 = \frac{-2B}{K}, \\ b_1 = 0, \qquad b_0 = \frac{a_0}{\sqrt{-B}}, \qquad b_{-1} = b_{-1}, \qquad w = 0. \end{gathered} \tag{2.14}$$

Case C.

$$\begin{gathered} a_1 = a_1, \qquad a_0 = a_0, \qquad a_{-1} = \frac{Bb_0^2 + a_0^2}{4a_1}, \qquad \lambda^2 = \frac{-2B}{K}, \\ b_1 = \frac{a_1}{\sqrt{-B}}, \qquad b_0 = b_0, \qquad b_{-1} = \frac{1}{4} \frac{Ba_0b_0^2 + a_0^3 + b_0^3\left(B^2/\sqrt{-B}\right) + b_0\left(Ba_0^2/\sqrt{-B}\right)}{a_1B\left(a_0/\sqrt{-B} - b_0\right)}, \qquad w = 0. \end{gathered} \tag{2.15}$$

Case D.

$$\begin{gathered} a_0 = a_0, \qquad a_{-1} = 0, \qquad a_1 = 0 \qquad \lambda^2 = \frac{-B}{2K}, \\ b_0 = \frac{a_0}{\sqrt{-B}}, \qquad b_{-1} = b_{-1}, \qquad b_1 = 0, \qquad w = \frac{3B}{\sqrt{-2B/K}}. \end{gathered} \tag{2.16}$$

Case E.

$$a_0 = a_0, \qquad a_{-1} = 0, \qquad a_1 = a_1, \qquad \lambda^2 = \frac{-B}{2K},$$
$$b_0 = \frac{-a_0^2 + Ba_1b_{-1}/\sqrt{-B}}{Ba_0/\sqrt{-B}}, \qquad b_{-1} = b_{-1}, \qquad b_1 = \frac{a_1}{\sqrt{-B}}, \qquad w = \frac{3B}{\sqrt{-2B/K}}. \tag{2.17}$$

Substituting (2.13)–(2.17) into (2.10) gives the generalized solitonary solution

$$\phi(\xi) = \frac{a_0}{b_1 \exp(\xi) - \left(a_0^2/8Bb_1\right) \exp(-\xi)}, \tag{2.18}$$

where $\xi = \sqrt{B/K}(\pm x + \gamma)$ and the solutions

$$\phi(\xi) = \frac{a_0 + \left(Bb_{-1}/\sqrt{-B}\right) \exp(-\xi)}{a_0/\sqrt{-B} + b_{-1} \exp(-\xi)}, \tag{2.19}$$

where $\xi = \sqrt{-2B/K}(\pm x + \gamma)$, and

$$\phi(\xi) = \frac{a_1 \exp(\xi) + a_0 + \left(Bb_0^2 + a_0^2/4a_1\right) \exp(-\xi)}{\left(a_1/\sqrt{-B}\right) \exp(\xi) + b_0 + \mathfrak{S} \exp(-\xi)}, \tag{2.20}$$

where $\mathfrak{S}$ denotes $1/4((Ba_0b_0^2 + a_0^3 + b_0^3B^2/\sqrt{-B} + b_0Ba_0^2/\sqrt{-B})/(a_1B(a_0/\sqrt{-B} - b_0)))$, and $\xi = \sqrt{-2B/K}(\pm x + \gamma)$, and

$$\phi(\xi) = \frac{a_0}{a_0/\sqrt{-B} + b_{-1} \exp(-\xi)}, \tag{2.21}$$

where $\xi = \sqrt{-B/2K}(\pm x - (3B/\sqrt{-2B/K})t + \gamma)$, and

$$\phi(\xi) = \frac{a_0 + a_1 \exp(\xi)}{\left(a_1/\sqrt{-B}\right) \exp(\xi) + \left(-a_0^2 + Ba_1b_{-1}/\sqrt{-B}\right)/\left(Ba_0/\sqrt{-B}\right) + b_{-1} \exp(-\xi)}, \tag{2.22}$$

where $\xi = \sqrt{-B/2K}(\pm x - (3B/\sqrt{-2B/K})t + \gamma)$, respectively. The choice of $a_0 = 2\sqrt{-2B}$ and $b_1 = 1$ in our solution (2.18) gives the same bell solitary wave solution presented in [34] which was obtained on using the sine-cosine method

$$v(x,t) = \sqrt{-2B}\,\mathrm{sech}\sqrt{\frac{B}{K}}(\pm x + \gamma). \tag{2.23}$$

Also if we set

$$\begin{aligned} b_1 &= \pm 1, \qquad a_0 = 2\sqrt{2B}, \\ b_1 &= -1, \qquad a_0 = 4\sqrt{-B}, \end{aligned} \tag{2.24}$$

in (2.18), we obtain the new solutions

$$\begin{aligned} v(x,t) &= \pm\sqrt{2B}\,\mathrm{csch}\sqrt{\frac{B}{K}}(\pm x + \gamma), \\ v(x,t) &= \frac{4\sqrt{B}}{\cosh\sqrt{B/K}(\pm x + \gamma) - 3\sinh\sqrt{B/K}(\pm x + \gamma)}. \end{aligned} \tag{2.25}$$

Also, setting $a_0 = \sqrt{-B}$ and $b_{-1} = \pm 1$ causes (2.19) to lead to the new kink solitary wave solution

$$\begin{aligned} v(x,t) &= \sqrt{-B}\,\coth\sqrt{\frac{-B}{2K}}(\pm x + \gamma), \\ v(x,t) &= \sqrt{-B}\,\tanh\sqrt{\frac{-B}{2K}}(\pm x + \gamma). \end{aligned} \tag{2.26}$$

This solution is similar to the solution obtained in [34].

By setting

$$b_0 = 0, \qquad a_1 = \sqrt{-B}, \qquad a_0 = \pm\, 2\, i\sqrt{-B}, \tag{2.27}$$

in (2.20), we will have the solitary wave solution

$$v(x,t) = \sqrt{-B}\left(\tanh\sqrt{\frac{-2B}{K}}(\pm x + \gamma) \pm i\,\mathrm{sech}\sqrt{\frac{-2B}{K}}(\pm x + \gamma)\right), \tag{2.28}$$

where $i = \sqrt{-1}$. Now if we set $a_0 = \sqrt{-B}$ and $b_{-1} = 1$ in (2.21), we will have another kink solitary wave solution

$$v(x,t) = \frac{\sqrt{-B}}{2} + \frac{\sqrt{-B}}{2} \tanh\sqrt{\frac{-B}{8K}}\left(\pm x - \frac{3B}{\sqrt{-2B/K}}\, t + \gamma\right), \tag{2.29}$$

which is similar to the solution obtained in [34].

If we also set $b_{-1} = 1$ and $a_0 = -\sqrt{-B}$ in (2.21), we obtain the new kink solitary wave solution

$$v(x,t) = \frac{\sqrt{-B}}{2} + \frac{\sqrt{-B}}{2} \coth\sqrt{\frac{-B}{8K}}\left(\pm x - \frac{3B}{\sqrt{-2B/K}} t + \gamma\right). \tag{2.30}$$

Also, setting $b_{-1} = a_0$ causes (2.21) to lead to the new soliton solution

$$v(x,t) = \frac{1}{-\left(\sqrt{-B}/B\right) + \cosh\sqrt{-B/2K}\left(\pm x - \left(3B/\sqrt{-2B/K}\right)t + \gamma\right) - \mathfrak{H}}, \tag{2.31}$$

where $\mathfrak{H}$ denotes $\sinh\sqrt{-B/2K}(\pm x - (3B/\sqrt{-2B/K})t + \gamma)$.

By choosing $a_1 = \sqrt{-B}$, $a_{-1} = 1$, and $a_0 = i\sqrt{-B}$ in (2.22), we can find solitary wave solution

$$\begin{aligned} v(x,t) = \frac{\sqrt{-B}}{2}\Bigg(&1 + \tanh\sqrt{\frac{-B}{2K}}\left(\pm x - \frac{3B}{\sqrt{-2B/K}}\, t + \gamma\right) \\ &+ i\,\mathrm{sech}\sqrt{\frac{-B}{2K}}\left(\pm x - \frac{3B}{\sqrt{-2B/K}}\, t + \gamma\right)\Bigg), \end{aligned} \tag{2.32}$$

where $i = \sqrt{-1}$.

3. Conclusion

An investigation on the Brusselator reaction diffusion model was established by using the Exp-function method. During this procedure some new exact solitary wave solutions, mostly solitons and kinks solutions, were obtained as well as some special cases. In particular, Yan's solution [34] can be considered as a special case of our result, and our result can turn into kink, soliton, and bell solutions with a suitable choice of the parameters. The study reveals the power of the method.

References

[1] R. Lefever, M. Herschkowitz-Kaufman, and J. W. Turner, "Dissipative structures in a soluble nonlinear reaction-diffusion system," *Physics Letters A*, vol. 60, no. 5, pp. 389–391, 1977.

[2] P. K. Vani, G. A. Ramanujam, and P. Kaliappan, "Painleve analysis and particular solutions of a coupled nonlinear reaction diffusion system," *Journal of Physics A*, vol. 26, no. 3, pp. L97–L99, 1993.

[3] A. Pickering, "A new truncation in Painlevé analysis," *Journal of Physics A*, vol. 26, no. 17, pp. 4395–4405, 1993.

[4] I. Ndayirinde and W. Malfliet, "New special solutions of the "Brusselator" reaction model," *Journal of Physics A*, vol. 30, no. 14, pp. 5151–5157, 1997.

[5] A. L. Larsen, "Weiss approach to a pair of coupled nonlinear reaction-diffusion equations," *Physics Letters A*, vol. 179, no. 4-5, pp. 284–290, 1993.

[6] J.-H. He, "Variational iteration method: a kind of non-linear analytical technique: some examples," *International Journal of Non-Linear Mechanics*, vol. 34, no. 4, pp. 699–708, 1999.

[7] M. T. Darvishi, F. Khani, and A. A. Soliman, "The numerical simulation for stiff systems of ordinary differential equations," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 1055–1063, 2007.

[8] M. T. Darvishi and F. Khani, "Numerical and explicit solutions of the fifth-order Korteweg-de Vries equations," *Chaos, Solitons & Fractals*, vol. 39, no. 5, pp. 2484–2490, 2009.

[9] A. Molabahrami, F. Khani, and S. Hamedi-Nezhad, "Soliton solutions of the two-dimensional KdV-Burgers equation by homotopy perturbation method," *Physics Letters A*, vol. 370, no. 5-6, pp. 433–436, 2007.

[10] J.-H. He, "New interpretation of homotopy perturbation method," *International Journal of Modern Physics B*, vol. 20, no. 18, pp. 2561–2568, 2006.

[11] M. T. Darvishi and F. Khani, "Application of He's homotopy perturbation method to stiff systems of ordinary differential equations," *Zeitschrift fur Naturforschung. Section A*, vol. 63, no. 1-2, pp. 19–23, 2008.

[12] A.-M. Wazwaz, "The tanh method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations," *Chaos, Solitons & Fractals*, vol. 25, no. 1, pp. 55–63, 2005.

[13] C.-L. Bai and H. Zhao, "Generalized extended tanh-function method and its application," *Chaos, Solitons & Fractals*, vol. 27, no. 4, pp. 1026–1035, 2006.

[14] F. Pedit and H. Wu, "Discretizing constant curvature surfaces via loop group factorizations: the discrete sine- and sinh-Gordon equations," *Journal of Geometry and Physics*, vol. 17, no. 3, pp. 245–260, 1995.

[15] X. Zhao, L. Wang, and W. Sun, "The repeated homogeneous balance method and its application to nonlinear partial differential equations," *Chaos, Solitons & Fractals*, vol. 28, no. 2, pp. 448–453, 2006.

[16] J.-F. Zhang, "Homogeneous balance method and chaotic and fractal solutions for the Nizhnik-Novikov-Veselov equation," *Physics Letters A*, vol. 313, no. 5-6, pp. 401–407, 2003.

[17] S. Zhang, "New exact solutions of the KdV-Burgers-Kuramoto equation," *Physics Letters A*, vol. 358, no. 5-6, pp. 414–420, 2006.

[18] E. Yomba, "The modified extended Fan's sub-equation method and its application to (2 + 1)-dimensional dispersive long wave equation," *Chaos, Solitons & Fractals*, vol. 26, no. 3, pp. 785–794, 2005.

[19] J.-H. He and X.-H. Wu, "Exp-function method for nonlinear wave equations," *Chaos, Solitons & Fractals*, vol. 30, no. 3, pp. 700–708, 2006.

[20] S.-D. Zhu, "Exp-function method for the Hybrid-Lattice system," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 3, pp. 461–464, 2007.

[21] S.-D. Zhu, "Exp-function method for the discrete mKdV lattice," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 3, pp. 465–468, 2007.

[22] A. Bekir and A. Boz, "Exact solutions for a class of nonlinear partial differential equations using exp-function method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 8, no. 4, pp. 505–512, 2007.

[23] J.-H. He and L.-N. Zhang, "Generalized solitary solution and compacton-like solution of the Jaulent-Miodek equations using the Exp-function method," *Physics Letters A*, vol. 372, no. 7, pp. 1044–1047, 2008.

[24] F. Khani, S. Hamedi-Nezhad, and A. Molabahrami, "A reliable treatment for nonlinear Schrödinger equations," *Physics Letters A*, vol. 371, no. 3, pp. 234–240, 2007.

[25] F. Khani, S. Hamedi-Nezhad, M. T. Darvishi, and S.-W. Ryu, "New solitary wave and periodic solutions of the foam drainage equation using the Exp-function method," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 3, pp. 1904–1911, 2009.

[26] F. Khani, "Analytic study on the higher order Ito equations: new solitary wave solutions using the Exp-function method," *Chaos, Solitons & Fractals*, vol. 41, no. 4, pp. 2128–2134, 2009.

[27] F. Khani and S. Hamedi-Nezhad, "Some new exact solutions of the (2 + 1)-dimensional variable coefficient Broer-Kaup system using the Exp-function method," *Computers & Mathematics with Applications*. In press.

[28] J.-H. He and M. A. Abdou, "New periodic solutions for nonlinear evolution equations using Exp-function method," *Chaos, Solitons & Fractals*, vol. 34, no. 5, pp. 1421–1429, 2007.

[29] X.-H. Wu and J.-H. He, "Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 966–986, 2007.

[30] S. A. El-Wakil, M. A. Madkour, and M. A. Abdou, "New traveling wave solutions for nonlinear evolution equations," *Physics Letters A*, vol. 365, no. 5-6, pp. 429–438, 2007.

[31] M. A. Abdou, A. A. Soliman, and S. T. El-Basyony, "New application of Exp-function method for improved Boussinesq equation," *Physics Letters A*, vol. 369, no. 5-6, pp. 469–475, 2007.

[32] S. Zhang, "Application of Exp-function method to high-dimensional nonlinear evolution equation," *Chaos, Solitons & Fractals*, vol. 38, no. 1, pp. 270–276, 2008.

[33] A. Ebaid, "Exact solitary wave solutions for some nonlinear evolution equations via Exp-function method," *Physics Letters A*, vol. 365, no. 3, pp. 213–219, 2007.

[34] Z.-Y. Yan and H.-Q. Zhang, "Auto-Darboux transformation and exact solutions of the Brusselator reaction diffusion model," *Applied Mathematics and Mechanics*, vol. 22, no. 5, pp. 541–546, 2001.