Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 358329, 13 pages
doi:10.1155/2009/358329

Research Article

Well Posedness for a Class of Flexible Structure in Hölder Spaces

Claudio Cuevas[1] and Carlos Lizama[2]

[1] *Departamento de Matemática, Universidade Federal de Pernambuco, Av. Prof. Luiz Freire, S/N, Recife-PE, CEP. 50540-740, Brazil*
[2] *Departamento de Matemática, Facultad de Ciencias, Universidad de Santiago de Chile, Casilla 307-Correo 2, Santiago-Chile, Chile*

Correspondence should be addressed to Carlos Lizama, carlos.lizama@usach.cl

Received 4 December 2008; Accepted 6 April 2009

Recommended by J. Rodellar

We characterize well-posedness in Hölder spaces for an abstract version of the equation $(*)\, u'' + \lambda u''' = c^2(\Delta u + \mu \Delta u') + f$ which model the *vibrations* of flexible structures possessing internal material damping and external force f. As a consequence, we show that in case of the Laplacian with Dirichlet boundary conditions, equation $(*)$ is always well-posed provided $0 < \lambda < \mu$.

Copyright © 2009 C. Cuevas and C. Lizama. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

During the last few decades, the use of flexible structural systems has steadily increased importance. The study of a flexible aerospace structure is a problem of dynamical system theory governed by partial differential equations.

We consider here the problem of characterize well posedness, for a mathematical model of a flexible space structure like a thin uniform rectangular panel. For example, a solar cell array or a spacecraft with flexible attachments. This problem is motivated by both engineering and mathematical considerations.

Such mechanical system was mathematically introduced in [1] and consists of a short rigid hub, connected to a flexible panel of length l. Control torque $Q(t)$ is applied to the hub. The panel is made of viscoelastic material with internal Voigt-type damping with coefficient μ, that is, an ideal dashpot damping which is directly proportional to the first derivative of the longitudinal displacement, and opposing the direction of motion. The equation of motion of the panel is given by

$$u'' = c^2\left(\Delta u + \mu \Delta u'\right), \tag{1.1}$$

where c is the velocity of longitudinal wave propagation, $c^2 = D_p/\rho J_p$, and D_p, ρ, J_p are, respectively, torsional rigidity, density and radius of gyration about the central axis of the panel. Initial position and deflection angle are known. In [1] exact controllability and boundary stabilization for the solution of (1.1) was analyzed and in [2, p. 188], the exact decay rate was obtained.

More generally, the study of *vibrations* of flexible structures possessing internal material damping is modeled by an equation of the form

$$u'' + \lambda u''' = c^2\left(\Delta u + \mu\Delta u'\right), \quad 0 < \lambda < \mu, \tag{1.2}$$

in a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary Γ, see [3, 4].

In [4] the explicit exponential energy decay rate was obtained for the solution of (1.2) subject to mixed boundary conditions. However, consideration of external forces interacting with the system, which lead us naturally with the well posedness for the nonhomogeneous version of (1.2), appears as an open problem.

In the first part of this paper we study well posedness of the following abstract version of (1.2):

$$u''(t) + \lambda u'''(t) = c^2 Au(t) + c^2\mu Au'(t) + f(t), \quad 0 < \lambda < \mu, \tag{1.3}$$

where A is a closed linear operator acting in a Banach space X and f is a X-valued function. We emphasize that when $A = \Delta$ in general one cannot expect that (1.3) is well posed due to the presence of the term u'''. In fact, it is well known that the abstract Cauchy problem associated with (1.3) is in general ill posed, see for example [5].

We are able to characterize well posedness, that is, temporal maximal regularity, of solutions of (1.3) solely in terms of boundedness of the resolvent set of A. This will be achieved in the Hölder spaces $C^\alpha(\mathbb{R}, X)$, where $0 < \alpha < 1$. The methods to obtain this goal are those incorporated in [6] where a similar problem in case of the first order abstract Cauchy problem has been studied.

2. Preliminaries

Let X, Y be Banach spaces, we write $\mathcal{B}(X, Y)$ for the space of bounded linear operators from X to Y and let $0 < \alpha < 1$. We denote by $\dot{C}^\alpha(\mathbb{R}, X)$ the spaces

$$\dot{C}^\alpha(\mathbb{R}, X) = \left\{f : \mathbb{R} \to X : f(0) = 0, \|f\|_\alpha < \infty\right\} \tag{2.1}$$

normed by

$$\|f\|_\alpha = \sup_{t\neq s} \frac{\|f(t) - f(s)\|}{\|t - s\|^\alpha}. \tag{2.2}$$

Let $\Omega \subset \mathbb{R}$ be an open set. By $C_c^\infty(\Omega)$ we denote the space of all C^∞-functions in $\Omega \subseteq \mathbb{R}$ having compact support in Ω.

We denote by $\mathcal{F}f$ or $\tilde{f}$ the Fourier transform, that is,

$$(\mathcal{F}f)(s) := \tilde{f}(s) := \int_{\mathbb{R}} e^{-ist} f(t)dt \tag{2.3}$$

$(s \in \mathbb{R}, f \in L^1(\mathbb{R}; X))$.

Definition 2.1. Let $M : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X, Y)$ be continuous. We say that M is a $\dot{C}^\alpha$-multiplier in $\mathcal{B}(X, Y)$ if there exists a mapping $L : \dot{C}^\alpha(\mathbb{R}, X) \to \dot{C}^\alpha(\mathbb{R}, Y)$ such that

$$\int_{\mathbb{R}} (Lf)(s)(\mathcal{F}\phi)(s)ds = \int_{\mathbb{R}} (\mathcal{F}(\phi \cdot M))(s)f(s)ds \tag{2.4}$$

for all $f \in C^\alpha(\mathbb{R}, X)$ and all $\phi \in C_c^\infty(\mathbb{R} \setminus \{0\})$.

Here $(\mathcal{F}(\phi \cdot M))(s) = \int_{\mathbb{R}} e^{-ist}\phi(t)M(t)dt \in \mathcal{B}(X, Y)$. Note that L is well defined, linear and continuous (cf. [6, Definition 5.2]).

Define the space $C^\alpha(\mathbb{R}, X)$ as the set

$$C^\alpha(\mathbb{R}, X) = \{f : \mathbb{R} \to X : \|f\|_{C^\alpha} < \infty\} \tag{2.5}$$

with the norm

$$\|f\|_{C^\alpha} = \|f\|_\alpha + \|f(0)\|. \tag{2.6}$$

Let $C^{\alpha+k}(\mathbb{R}, X)$ (where k is a positive integer) be the Banach space of all $u \in C^k(\mathbb{R}, X)$ such that $u^{(k)} \in C^\alpha(\mathbb{R}, X)$, equipped with the norm

$$\|u\|_{C^{\alpha+k}} = \left\|u^{(k)}\right\|_{C^\alpha} + \|u(0)\|. \tag{2.7}$$

Observe from Definition 2.1 and the relation

$$\int_{\mathbb{R}} (\mathcal{F}(\phi M))(s)ds = 2\pi(\phi M)(0) = 0, \tag{2.8}$$

that for $f \in C^\alpha(\mathbb{R}, X)$ we have $Lf \in C^\alpha(\mathbb{R}, X)$. Moreover, if $f \in C^\alpha(\mathbb{R}, X)$ is bounded then Lf is bounded as well (see [6, Remark 6.3]). The following multiplier theorem is due to Arendt, Batty and Bu [6, Theorem 5.3].

Theorem 2.2. *Let $M \in C^2(\mathbb{R} \setminus \{0\}, \mathcal{B}(X, Y))$ be such that*

$$\sup_{t\neq 0}\|M(t)\| + \sup_{t\neq 0}\left\|tM'(t)\right\| + \sup_{t\neq 0}\left\|t^2M''(t)\right\| < \infty. \tag{2.9}$$

Then M is a $\dot{C}^\alpha$-multiplier.

Remark 2.3. If X is B-convex, in particular if X is a UMD space, Theorem 2.2 remains valid if condition (2.2) is replaced by the following weaker condition:

$$\sup_{t\neq 0}\|M(t)\| + \sup_{t\neq 0}\left\|tM'(t)\right\| < \infty, \tag{2.10}$$

where $M\in C^1(\mathbb{R}\setminus\{0\},\mathcal{B}(X,Y))$ (cf. [6, Remark 5.5]).

3. A Characterization of Well Posedness in Hölder Spaces

In this section we characterize C^{α}-well posedness. Given $f \in C^{\alpha}(\mathbb{R},X)$, we consider in this section the linear problem

$$u''(t) + au'''(t) = bAu(t) + cAu'(t) + f(t), \quad t\in\mathbb{R}, \tag{3.1}$$

where A is a closed linear operator in X and $a,b,c > 0$. Note that the solution of (3.1) does not have to satisfy any initial condition. In the case $a = 0$, solutions of (3.1) with periodic boundary conditions has been recently studied in [7]. On the other hand, well posedness of the homogeneous abstract Cauchy problem has been observed recently in [8] for $a = 0$ and all $b \in \mathbb{C}$ under certain assumptions on A. See also [9] for related maximal regularity results in the case of a damped wave equation.

We denote by $[D(A)]$ the domain of A considered as a Banach space with the graph norm.

Definition 3.1. We say that (3.1) is C^{α}-well posed if for each $f \in C^{\alpha}(\mathbb{R},X)$ there is a unique function $u \in C^{\alpha+3}(\mathbb{R},X)\cap C^{\alpha+1}(\mathbb{R},[D(A)])\cap C^{\alpha}(\mathbb{R},[D(A)])$ such that (3.1) is satisfied.

In the next proposition, as usual we denote by $\rho(T)$, $R(\lambda,T)$ the resolvent set and resolvent of the operator T, respectively.

Proposition 3.2. *Assume that* (3.1) *is C^{α}-well-posed. Then*

(i) $l(\eta) := -\eta^2((1+ia\eta)/(b+ic\eta)) \in \rho(A)$ *for all* $\eta\in\mathbb{R}$ *and,*

(ii) $\sup_{\eta\in\mathbb{R}}\|(\eta^3/(b+ic\eta))R(l(\eta),A)\| < \infty$.

Proof. Denote by $L : C^{\alpha}(\mathbb{R},X) \to C^{\alpha+3}(\mathbb{R},X)$ the bounded operator which associates to each $f\in C^{\alpha}(\mathbb{R},X)$ the unique solution u of (3.1). Let $\eta\in\mathbb{R}$. Let $x\in D(A)$ be such that $Ax - l(\eta)x = 0$. Define $u(t) = e^{i\eta t}x$. Then it is not difficult to see that u is a solution of (3.1) with $f \equiv 0$. Hence, by uniqueness, $x = 0$.

Let $y\in X$ and define $f(t) = e^{i\eta t}y$. Let $u = Lf$. For fixed $s\in\mathbb{R}$ we define

$$v_1(t) = u(t+s), \quad v_2(t) = e^{i\eta s}u(t). \tag{3.2}$$

Then is easy to check that v_1 and v_2 are both solutions of (3.1) with f replaced by $e^{i\eta s}f$. By uniqueness, $u(t+s) = e^{i\eta s}u(t)$ for all $t, s \in \mathbb{R}$. In particular, it follows that $u(s) = e^{i\eta s}u(0)$ for all $s \in \mathbb{R}$. Let $x = u(0) \in D(A)$. Replacing $u(t) = e^{i\eta t}x$ in (3.1) we obtain

$$\left(-\eta^2 - ia\eta^3\right)u(t) = (b + ic\eta)Au(t) + e^{i\eta t}y. \tag{3.3}$$

Taking $t = 0$ we conclude that $(l(\eta) - A)$ is bijective and

$$u(t) = \frac{1}{b + ic\eta}R(l(\eta), A)e^{i\eta t}y. \tag{3.4}$$

Define $e_\eta(t) = e^{i\eta t}$ and $(e_\eta \otimes y)(t) = e_\eta(t)y$. We have the identity $\|e_\eta \otimes x\|_\alpha = K_\alpha|\eta|^\alpha\|x\|$ where $K_\alpha = 2\sup_{t>0} t^{-\alpha}\sin(t/2)$ (see [6, section 3]). Hence

$$\begin{aligned} K_\alpha|\eta|^\alpha\left\|\frac{\eta^3}{b + ic\eta}R(l(\eta), A)y\right\| &= \left\|e_\eta \otimes \frac{\eta^3}{b + ic\eta}R(l(\eta), A)y\right\|_\alpha = \|u'''\|_\alpha \\ &\le \|u\|_{\alpha+3} = \|Lf\|_{\alpha+3} \le \|L\|\|f\|_\alpha \\ &\le \|L\|(\|f\|_\alpha + \|f(0)\|) = \|L\|\left(\|e_\eta \otimes y\|_\alpha + \|y\|\right) \\ &\le \|L\|(K_\alpha|\eta|^\alpha + 1)\|y\|. \end{aligned} \tag{3.5}$$

Therefore, for $\varepsilon > 0$ we have

$$\sup_{|\eta|>\varepsilon}\left\|\frac{\eta^3}{b + ic\eta}R(l(\eta), A)y\right\| \le \|L\|\sup_{|\eta|>\varepsilon}\left(1 + \frac{1}{K_\alpha|\eta|^\alpha}\right)\|y\| < \infty. \tag{3.6}$$

On the other hand, since $\{1/(b + ic\eta)\}_{\eta\in\mathbb{R}}$ is bounded and $\eta \to \eta^3 R(l(\eta), A)$ is continuous at $\eta = 0$, we obtain (ii) and the proof is complete. □

In what follows, we denote by id^k the function: $s \to (is)^k$ for all $s \in \mathbb{R}$, and $k \in \mathbb{N}$. As before, we also use the notation

$$\begin{aligned} l(s) &:= -s^2\frac{1 + ias}{b + ics}, \\ M(s) &:= \frac{1}{b + ics}R(l(s), A), \quad \forall\, s \in \mathbb{R}. \end{aligned} \tag{3.7}$$

Lemma 3.3. *Assume that*

$$\sup_{s\in\mathbb{R}}\left\|s^3 M(s)\right\| < \infty, \tag{3.8}$$

then $id^2 \cdot M$ and $id^3 \cdot M$ are $\dot{C}^\alpha$- multipliers in $\mathcal{B}(X)$. Moreover M and $id \cdot M$ are $\dot{C}^\alpha$-multipliers in $\mathcal{B}(X, D(A))$.

Proof. Define $\kappa(s) := 1/(b+ias)$. We first observe that the functions $\theta(s) := \kappa'(s)/\kappa(s)$ and $\vartheta(s) := l'(s)/l(s)$ have the property that $s\theta(s), s\vartheta(s), s^2\theta'(s)$ and $s^2\vartheta'(s)$ are bounded on $\mathbb{R}$. We next claim that M is a $\dot{C}^\alpha$-multiplier. In fact, note that by hypothesis $\sup_{|s|>\varepsilon}\|M(s)\| < \infty$ for each $\varepsilon > 0$, and the function $s \to M(s)$ is continuous at $t = 0$ since $b > 0$. Hence $M(s)$ is bounded. Moreover, defining $\xi(s) := l(s)/\kappa(s) = -s^2 - ias^3$ we have

$$M'(s) = \theta(s)M(s) - \vartheta(s)\xi(s)[M(s)]^2, \tag{3.9}$$

where $s\xi(s)$ is of order s^4 and then $Q(s) := \xi(s)[M(s)]^2$ is bounded by (3.8). It follows that $sM'(s)$ is bounded. Next, we have the identity

$$s^2M''(s) = s^2\theta'(s)M(s) + s\theta(s)sM'(s) - s^2\vartheta'(s)Q(s) - s^2\vartheta(s)Q'(s). \tag{3.10}$$

where the first three terms on the right hand side are bounded. For the last term, we have

$$s^2\vartheta(s)Q'(s) = [s\vartheta(s)]^2Q(s) - s\vartheta(s)s\theta(s)Q(s) + 2s\vartheta(s)\xi(s)M(s)sM'(s). \tag{3.11}$$

It is clear that the first two terms on the right hand side are bounded. We observe that the last term also is bounded. In fact, note that by hypothesis $\sup_{|s|>\varepsilon}\|\xi(s)M(s)\| < \infty$ for each $\varepsilon > 0$ and the function $s \to \xi(s)M(s)$ is continuous at $s = 0$. Hence $\xi(s)M(s)$ is bounded. This completes the proof of the Lemma. □

Lemma 3.4. *Let $0 < \alpha < 1$, $k \in \mathbb{N}$ and $u, v \in C^\alpha(\mathbb{R}, X)$. The following assertions are equivalent:*

(i) *$u \in C^{\alpha+k}(\mathbb{R}, X)$ and $u^{(k)} - v$ is constant.*

(ii) *$\int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds = \int_{\mathbb{R}} u(s)(\mathcal{F}\, id^k \cdot \varphi)(s)ds$ for all $\varphi \in \mathfrak{D}(\mathbb{R}\setminus\{0\})$.*

Proof. (i) $\Rightarrow$ (ii). Let $\Phi \in \mathfrak{D}(\mathbb{R}\{0\})$. Then $\int_{\mathbb{R}} v(s)(\mathcal{F}\varphi)(s)ds = \int_{\mathbb{R}} u^{(k)}(s)(\mathcal{F}\varphi)(s)ds = (-1)^k\int_{\mathbb{R}} u(s)(\mathcal{F}\varphi)^{(k)}(s)ds = \int_{\mathbb{R}} u(s)\mathcal{F}(id^k\cdot\varphi)(s)ds$.
(ii) $\Rightarrow$ (i). Let $\Phi \in \mathfrak{D}(\mathbb{R}\setminus\{0\})$ and $\psi(s) = \varphi(s)/s^k$. Then $\psi \in \mathfrak{D}(\mathbb{R}\setminus\{0\})$ and $\mathcal{F}\varphi = (\mathcal{F}\psi)^{(k)}$. Let $w(t) = \int_0^t (t-s)^{k-1}v(s)ds$. Then integration by parts and assumption give $\int_{\mathbb{R}} w(s)(\mathcal{F}\varphi)(s)ds = \int_{\mathbb{R}} u(s)(\mathcal{F}\varphi)(s)ds$. It follows from [10, Theorems 4.8.2 and 4.8.1] that $w - u$ is a polynomial. Since $\|w(t)\| \le c(1+|t|^{\alpha+k})$ it follows that $u(t) = w(t) + y_0 + ty_1 + t^2y_2 + \cdots + t^k y_k = \int_0^t (t-s)^{k-1} v(s)ds + y_0 + ty_1 + t^2 y_2 + \cdots + t^k y_k$ for some vectors $y_0, y_1, \ldots, y_k \in X$. Thus $u^{(k)} = v + x$ for some vector $x \in X$. □

The following theorem, which is one of the main results in this paper, shows that the converse of Proposition 3.2 is valid.

Theorem 3.5. *Let A be a closed linear operator defined on a Banach space X. Then the following assertions are equivalent:*

(i) *Equation* (3.1) *is C^α-well posed;*

(ii) *$l(\eta) := -\eta^2((1+ia\eta)/(b+ic\eta)) \in \rho(A)$ for all $\eta \in \mathbb{R}$ and*

$$\sup_{\eta\in\mathbb{R}}\left\|\frac{\eta^3}{b+ic\eta}R(l(\eta),A)\right\| < \infty. \tag{3.12}$$

Proof. The implication $(i) \Rightarrow (ii)$ follows by Proposition 3.2. We now prove the converse implication.

Let $f \in C^{\alpha}(\mathbb{R}, X)$. By Lemma 3.3 there exists $u_1, u_2 \in C^{\alpha}(\mathbb{R}, [D(A)])$ and $u_3, u_4 \in C^{\alpha}(\mathbb{R}, X)$ such that

$$\int_{\mathbb{R}} u_1(s)(\mathcal{F}\phi_1)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_1 \cdot M)(s)f(s)ds, \tag{3.13}$$

$$\int_{\mathbb{R}} u_2(s)(\mathcal{F}\phi_2)(s)ds = \int_{\mathbb{R}} \mathcal{F}(\phi_2 \cdot id \cdot M)(s)f(s)ds, \tag{3.14}$$

$$\int_{\mathbb{R}} u_3(s)(\mathcal{F}\phi_3)(s)ds = \int_{\mathbb{R}} \mathcal{F}\Big(\phi_3 \cdot id^2 \cdot M\Big)(s)f(s)ds, \tag{3.15}$$

$$\int_{\mathbb{R}} u_4(s)(\mathcal{F}\phi_4)(s)ds = \int_{\mathbb{R}} \mathcal{F}\Big(\phi_4 \cdot id^3 \cdot M\Big)(s)f(s)ds \tag{3.16}$$

for all $\Phi_i \in C_0^{\infty}(\mathbb{R} \setminus \{0\})$ $(i = 1, 2, 3, 4)$. Choosing $\Phi_1 = id \cdot \Phi_2$ in (3.13), it follows from Lemma 3.4 that $u_1 \in C^{1+\alpha}(\mathbb{R}, X)$ and

$$u_1' = u_2 + y_1, \tag{3.17}$$

for some $y_1 \in X$. Now we can choose $\phi_2 = id \cdot \phi_3$ in (3.14), it follows that $u_1 \in C^{\alpha+2}(\mathbb{R}, X)$ and

$$u_1'' = u_3 + y_2, \tag{3.18}$$

for some $y_2 \in X$. In a similar way, we can see that $u_1 \in C^{\alpha+3}(\mathbb{R}, X)$ and

$$u_1''' = u_4 + y_3, \tag{3.19}$$

for some $y_3 \in X$. From the definition of $M(s)$ we obtain $(b + ics)(l(s) - A)M(s) = I$. Taking into account the definition of $l(s)$ we get $[-s^2(1 + ics) - (b + ics)A]M(s) = I$. Then we deduce the identity

$$(is)^2 M(s) + a(is)^3 M(s) = bAM(s) + icsAM(s) + I. \tag{3.20}$$

We multiply the above identity by ϕ, take Fourier transforms and then integrate over $\mathbb{R}$ after taking the values at $f(s)$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \mathcal{F}\Big(\phi \cdot id^2 \cdot M\Big)(s)f(s)ds + a\int_{\mathbb{R}} \mathcal{F}\Big(\phi \cdot id^3 \cdot M\Big)(s)f(s)ds \\ &\quad = b\int_{\mathbb{R}} A\mathcal{F}(\phi \cdot M)(s)f(s)ds + c\int_{\mathbb{R}} A\mathcal{F}(\phi \cdot id \cdot M)(s)f(s)ds + \int_{\mathbb{R}} \mathcal{F}(\phi)(s)f(s)ds \end{aligned} \tag{3.21}$$

for all $\phi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Using (3.17), (3.18) and (3.19) in the above identity we conclude that

$$\begin{aligned}&\int_{\mathbb{R}} u_1''(s)(\mathcal{F}\phi)(s)ds + a\int_{\mathbb{R}} u_1'''(s)(\mathcal{F}\phi)(s)ds \\ &\quad = b\int_{\mathbb{R}} Au_1(s)(\mathcal{F}\phi)(s)ds + c\int_{\mathbb{R}} Au_1'(s)(\mathcal{F}\phi)(s)ds + \int_{\mathbb{R}} \mathcal{F}(\phi)(s)f(s)ds\end{aligned} \tag{3.22}$$

for all $\phi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. By Lemma 3.4 there exists $z \in X$ such that

$$u_1''(t) + au_1'''(t) = bAu_1(t) + cAu_1'(t) + f(t) + z, \qquad t \in \mathbb{R}. \tag{3.23}$$

We define

$$u(t) = u_1(t) + \frac{1}{b}A^{-1}(z). \tag{3.24}$$

Then, we can show that u solves (3.1) and that $u \in C^{\alpha+3}(\mathbb{R}, X) \cap C^{\alpha+1}(\mathbb{R}, [D(A)]) \cap C^{\alpha}(\mathbb{R}, [D(A)])$.

In order to prove uniqueness, suppose that

$$u''(t) + au'''(t) = bAu(t) + cAu'(t), \qquad t \in \mathbb{R}. \tag{3.25}$$

where $u \in C^{\alpha+3}(\mathbb{R}, X) \cap C^{\alpha+1}(\mathbb{R}, [D(A)]) \cap C^{\alpha}(\mathbb{R}, [D(A)])$. Let $\sigma > 0$. We define $L_\sigma u$ by

$$(L_\sigma u)(\rho) = \hat{u}(\sigma + i\rho) - \hat{u}(-\sigma + i\rho), \qquad \rho \in \mathbb{R} \tag{3.26}$$

where the hat indicates the Carleman transform (see e.g. [11]). By [12, Proposition A.2(i)], we have that

$$\int_{\mathbb{R}} u(\rho)(\mathcal{F}\phi)(\rho)d\rho = \lim_{\sigma \downarrow 0}\int_{\mathbb{R}} (L_\sigma u)(\rho)\phi(\rho)d\rho \tag{3.27}$$

for all $\phi \in \mathcal{S}(\mathbb{R})$, the Schwartz space of smooth rapidly decreasing functions on $\mathbb{R}$. We will prove that the right term in (3.27) is zero, from which $u \equiv 0$ proving the theorem. In fact, by [12, Proposition A.2(iii)] we have

$$\begin{aligned}(b + c(\sigma + i\rho))(l_\sigma(\rho) - A)(L_\sigma u)(\rho) &= 2\sigma cA\hat{u}(-\sigma + i\rho) \\ &\quad - \left[2\sigma(\sigma + i\rho) + 2\sigma a(\sigma + i\rho)^2\right]\hat{u}(-\sigma + i\rho) \\ &\quad - [2\sigma + 2\sigma a(\sigma + i\rho)]\hat{u}'(-\sigma + i\rho) - 2\sigma a\hat{u}''\prime(-\sigma + i\rho) \\ &: = H_{a,c}(\sigma, \rho),\end{aligned} \tag{3.28}$$

where

$$l_\sigma(\rho) = (\sigma + i\rho)^2 \frac{1 + a(\sigma + i\rho)}{b + c(\sigma + i\rho)}. \tag{3.29}$$

Observe that $l_0(\rho) = l(\rho) \in \rho(A)$ for all $\rho \in \mathbb{R}$. Therefore we have

$$(l_\sigma(\rho) - l(\rho))(l(\rho) - A)^{-1}(L_\sigma u)(\rho) + (L_\sigma u)(\rho) = \frac{1}{b + c(\sigma + i\rho)}(l(\rho) - A)^{-1} H_{a,c}(\sigma, \rho). \tag{3.30}$$

Let $\phi \in C_0^\infty(\mathbb{R})$. Multiplying by ϕ and integrating over $\mathbb{R}$ the above identity we obtain

$$\int_{\mathbb{R}} (L_\sigma u)(\rho)\phi(\rho) d\rho = \int_{\mathbb{R}} N_\sigma(\rho) H_{a,c}(\sigma, \rho) d\rho - \int_{\mathbb{R}} M_\sigma(\rho)(L_\sigma u)(\rho) d\rho \tag{3.31}$$

where

$$\begin{aligned} N_\sigma(\rho) &= \frac{1}{b + c(\sigma + i\rho)} \phi(\rho)(l(\rho) - A)^{-1} \\ M_\sigma(\rho) &= \phi(\rho)(l_\sigma(\rho) - l(\rho))(l(\rho) - A)^{-1}. \end{aligned} \tag{3.32}$$

We note that in [12, Lemma A.4],

$$\lim_{\sigma \downarrow 0} \int_{\mathbb{R}} M_\sigma(\rho) L_\sigma(u)(\rho) d\rho = 0. \tag{3.33}$$

It remains to prove that

$$\lim_{\sigma \downarrow 0} \int_{\mathbb{R}} N_\sigma(\rho) H_{a,c}(\sigma, \rho) d\rho = 0. \tag{3.34}$$

In fact, since $(L_\sigma u)(\rho) = \int_{\mathbb{R}} e^{-\sigma|t|} e^{-i\rho t} u(t) dt$, we have

$$\begin{aligned} \int_{\mathbb{R}} M_\sigma(\rho) L_\sigma(u)(\rho) d\rho &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\rho t} M_\sigma(\rho) d\rho e^{-\sigma|t|} u(t) dt \\ &= \int_{\mathbb{R}} (\mathcal{F} M_\sigma)(t) e^{-\sigma|t|} u(t) dt. \end{aligned} \tag{3.35}$$

Then

$$\left\|\int_{\mathbb{R}} M_\sigma(\rho) L_\sigma(u)(\rho)\,d\rho\right\| \leq \int_{\mathbb{R}} \|(\mathcal{F} M_\sigma)(t)\| \|u(t)\| dt. \\ \leq 2C\left(\|M_\sigma\|_{L^1} + \left\|M''_\sigma\right\|_{L^1}\right). \tag{3.36}$$

It is easy to check that $\|M_\sigma\|_{L^1} + \|M''_\sigma\|_{L^1} \to 0$ as $\sigma \to 0$, proving (3.34).

We write

$$H_{a,c}(\sigma,\rho) = I_1(\sigma,\rho) + I_2(\sigma,\rho) + I_3(\sigma,\rho) + I_4(\sigma,\rho). \tag{3.37}$$

We first prove that

$$\lim_{\sigma\downarrow 0}\int_{\mathbb{R}} N_\sigma(\rho) I_1(\sigma,\rho)\,d\rho = 0. \tag{3.38}$$

In fact, we apply Fubini's theorem to obtain

$$\begin{aligned} \int_{\mathbb{R}} N_\sigma(\rho) I_1(\sigma,\rho)\,d\rho &= 2\sigma c\int_{\mathbb{R}} N_\sigma(\rho)\widehat{Au}(-\sigma+i\rho)\,d\rho \\ &= -2\sigma c\int_{-\infty}^{0}\left[\int_{\mathbb{R}} e^{-i\rho t} N_\sigma(\rho)\,d\rho\right] e^{\sigma t} Au(t)\,dt \\ &= -2\sigma c\int_{-\infty}^{0} (\mathcal{F} N_\sigma)(t) e^{\sigma t} Au(t)\,dt \end{aligned} \tag{3.39}$$

It follows from [[12], Lemma A.3] that

$$\left\|\int_{-\infty}^{0} (\mathcal{F} N_\sigma)(t) e^{\sigma t} Au(t)\,dt\right\| \leq 2C\left(\|N_\sigma\|_{L^1} + \left\|N''_\sigma\right\|_{L^1}\right), \tag{3.40}$$

where C is a positive constant. Taking into account (3.39) and (3.40) we deduce (3.38).

We next prove that

$$\lim_{\sigma\downarrow 0}\int_{\mathbb{R}} N_\sigma(\rho) I_2(\sigma,\rho)\,d\rho = 0. \tag{3.41}$$

In fact, define $N^a_\sigma(\rho) = [1 + a(\sigma + i\rho)](\sigma + i\rho)N_\sigma(\rho)$. Then

$$\begin{aligned} \int_{\mathbb{R}} N_\sigma(\rho) I_2(\sigma,\rho)\,d\rho &= -2\sigma\int_{\mathbb{R}}\int_{-\infty}^{0} e^{(\sigma-i\rho)s} N^a_\sigma(\rho) u(s)\,ds\,d\rho \\ &= -2\sigma\int_{-\infty}^{0} (\mathcal{F} N^a_\sigma)(s) e^{as} u(s)\,ds. \end{aligned} \tag{3.42}$$

By [12, Lemma $A.3$], we have for $0 \le \sigma \le \varepsilon$,

$$\left\| \int_{-\infty}^{0} (\mathcal{F} N_\sigma^a)(s) e^{as} u(s) ds \right\| \le 2C \sup_{0\le\sigma\le\varepsilon} \left(\|N_\sigma^a\|_{L^1} + \left\| (N_\sigma^a)'' \right\|_{L^1} \right), \tag{3.43}$$

where $C > 0$. Therefore, we deduce (3.41). Proceeding in the same way we obtain

$$\lim_{\sigma\downarrow 0} \int_{\mathbb{R}} N_\sigma(\rho) I_j(\sigma,\rho) d\rho = 0, \qquad j = 3, 4. \tag{3.44}$$

This completes the proof of the assertion (3.34). □

Corollary 3.6. *The solution u of problem* (3.1) *given by Theorem 3.5 satisfies the following maximal regularity property: $u, u' \in C^\alpha(\mathbb{R}; [D(A)])$ and $Au, Au', u'', u''' \in C^\alpha(\mathbb{R}; X)$. Moreover, there exists a constant $C > 0$ independent of $f \in C^\alpha(\mathbb{R}; X)$ such that*

$$\|u\|_\alpha + \|u'\|_\alpha + \|u''\|_\alpha + \|u'''\|_\alpha + \|Au\|_\alpha + \|Au'\|_\alpha \le C\|f\|_\alpha. \tag{3.45}$$

The following consequence of Theorem 3.5 is remarkable in the study of C^α well posedness for flexible structural systems. We recall that $l(\eta) := -\eta^2((1 + ia\eta)/(b + ic\eta))$.

Corollary 3.7. *If A is the generator of a bounded analytic semigroup, then* (3.1) *is C^α-well posed.*

Proof. Since A generates a bounded analytic semigroup, we have that $\{\tau : Re\,\tau > 0\} \subseteq \rho(A)$ and there is a constant $M > 0$ such that $\|\tau(\tau - A)^{-1}\| \le M$ for $Re\,\tau > 0$. Note that

$$\frac{\eta^3}{b + ic\eta} R(l(\eta); A) = \frac{-\eta}{1 + i\alpha\eta} l(\eta) R(l(\eta); A) \tag{3.46}$$

and that

$$Re(l(\eta)) = \frac{b + ac\eta^2}{b^2 + c^2\eta^2} > 0 \quad \text{for each } \eta \in \mathbb{R}. \tag{3.47}$$

We conclude that $l(\eta) \in \rho(A)$ and

$$\sup_{\eta\in\mathbb{R}} \left\| \frac{\eta^3}{b + ic\eta} R(l(\eta), A) \right\| < \infty. \tag{3.48}$$

The conclusion follows by Theorem 3.5. □

For example, if A is a normal operator on a Hilbert space H satisfying

$$\sigma(A) \subset \{ z \in \mathbb{C} : \arg(-z) < \delta \} \tag{3.49}$$

for some $\delta \in [0, \pi/2)$, then A generates a bounded analytic semigroup. In particular, the semigroup generated by a self-adjoint operator that is bounded above is analytic of angle $\pi/2$. Another important class of generators of analytic semigroups is provided by squares of group generators.

Example 3.8. Since the Laplacian Δ is the generator a bounded analytic semigroup (the diffusion semigroup) in $X = L^p(\mathbb{R}^N)$ $(1 \leq p < \infty)$, we obtain that for each $f \in C^\alpha(\mathbb{R}, L^p(\mathbb{R}^N))$ the problem

$$u_{tt}(t,x) + \lambda u_{ttt}(t,x) = c^2\big(\Delta u(t,x) + \mu \Delta u_t(t,x)\big) + f(t,x) \tag{3.50}$$

has a unique solution $u \in C^{\alpha+3}(\mathbb{R}, L^p(\mathbb{R}^N)) \cap C^{\alpha+1}(\mathbb{R}, W^{2,p}(\mathbb{R}^N)) \cap C^\alpha(\mathbb{R}, W^{2,p}(\mathbb{R}^N))$.

Since it is also well known that the Dirichlet Laplacian Δ generates a bounded analytic semigroup on $L^2(\Omega)$, where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^3$, we obtain the following consequence for our initial problem.

Corollary 3.9. *If Ω is a bounded domain with boundary of class C^2 in $\mathbb{R}^3$ then for each $f \in C^\alpha(\mathbb{R}, L^2(\Omega))$, the problem* (3.50) *is C^α-well posed, that is, has a unique solution $u \in C^{\alpha+3}(\mathbb{R}, L^2(\Omega)) \cap C^{\alpha+1}(\mathbb{R}, H^2(\Omega) \cap H_0^1(\Omega)) \cap C^\alpha(\mathbb{R}, H^2(\Omega) \cap H_0^1(\Omega))$.*

We note that the same assertion remain true for all $p \in [1, \infty)$.

Acknowledgment

The first author is partially supported by CNPQ/Brazil. The second author is partially financed by Laboratorio de Análisis Estocástico, Proyecto Anillo ACT-13.

References

[1] S. K. Bose and G. C. Gorain, "Exact controllability and boundary stabilization of torsional vibrations of an internally damped flexible space structure," *Journal of Optimization Theory and Applications*, vol. 99, no. 2, pp. 423–442, 1998.

[2] A. Bátkai and S. Piazzera, *Semigroups for Delay Equations*, vol. 10 of *Research Notes in Mathematics*, A. K. Peters, Wellesley, Mass, USA, 2005.

[3] S. K. Bose and G. C. Gorain, "Exact controllability and boundary stabilization of flexural vibrations of an internally damped flexible space structure," *Applied Mathematics and Computation*, vol. 126, no. 2-3, pp. 341–360, 2002.

[4] S. K. Bose and G. C. Gorain, "Stability of the boundary stabilised internally damped wave equation $y'' + \lambda y''' = c^2(\Delta y + \mu \Delta y')$ in a bounded domain in $\mathbb{R}^n$," *Indian Journal of Mathematics*, vol. 40, no. 1, pp. 1–15, 1998.

[5] T.-J. Xiao and J. Liang, *The Cauchy Problem for Higher-Order Abstract Differential Equations*, vol. 1701 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1998.

[6] W. Arendt, C. Batty, and S. Bu, "Fourier multipliers for Hölder continuous functions and maximal regularity," *Studia Mathematica*, vol. 160, no. 1, pp. 23–51, 2004.

[7] V. Keyantuo and C. Lizama, "Periodic solutions of second order differential equations in Banach spaces," *Mathematische Zeitschrift*, vol. 253, no. 3, pp. 489–514, 2006.

[8] D. Mugnolo, "A variational approach to strongly damped wave equations," in *Functional Analysis and Evolution Equations: The Günter Lumer Volume*, pp. 503–514, Birkhäuser, Basel, Switzerland, 2008.

[9] R. Chill and S. Srivastava, "L^p-maximal regularity for second order Cauchy problems," *Mathematische Zeitschrift*, vol. 251, no. 4, pp. 751–781, 2005.

[10] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, vol. 96 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 2001.

[11] J. Prüss, *Evolutionary Integral Equations and Applications*, vol. 87 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 1993.

[12] V. Keyantuo and C. Lizama, "Hölder continuous solutions for integro-differential equations and maximal regularity," *Journal of Differential Equations*, vol. 230, no. 2, pp. 634–660, 2006.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization