Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 361296, 19 pages
doi:10.1155/2009/361296

Research Article

Theoretical Study of a Chain Sliding on a Fixed Support

Jérôme Bastien[1] and Claude-Henri Lamarque[2]

[1] *Centre de Recherche et d'Innovation sur le Sport, U.F.R.S.T.A.P.S., Université de Lyon, Université Claude Bernard-Lyon 1, 27-29, boulevard du 11 Novembre 1918, 69622 Villeurbanne Cedex, France*
[2] *Département Génie Civil et Bâtiment FRE CNRS 3237, École Nationale des Travaux Publics de l'Etat, Université de Lyon, Rue Maurice Audin, 69518 Vaulx-en-Velin Cedex, France*

Correspondence should be addressed to Jérôme Bastien, jerome.bastien@univ-lyon1.fr

Received 3 July 2009; Accepted 7 December 2009

Recommended by Jerzy Warminski

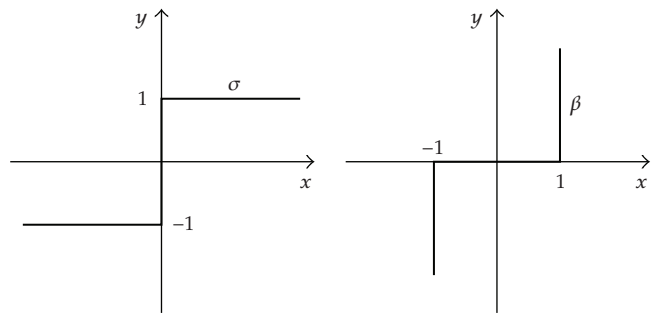
A chain sliding on a fixed support, made out of some elementary rheological models (dry friction element and linear spring) can be covered by the existence and uniqueness theory for maximal monotone operators. Several behavior from quasistatic to dynamical are investigated. Moreover, classical results of numerical analysis allow to use a numerical implicit Euler scheme.

1. Introduction

This paper is the next step of a series of previous works dealing with modelling of discrete mechanical systems with finite number of degrees of freedom involving assemblies of classical smooth constitutive elements (in the mechanical point of view they correspond to linear or non linear springs, dashpots) and nonsmooth ones mainly based on St-Venant Elements. Let us cite basic rheological models [1], with different applications and developpements [2–7]. Delay or stochastic frame have also been investigated in [8–10].

In this paper we examine a new model: it can be associated with motion of a discretized beam "sliding" on soil. We do not give more details on this discretization.

This paper is organized as follows in Section 2, the model is described. In Section 3, the general model is adapted to different dynamical, semi-dynamical or quasistatic cases. In Section 4, existence and uniqueness is addressed. In Section 5, numerical scheme is described and its convergence obtained.



(a) The graph σ (b) The graph β

Figure 1: Two useful multivalued maximal monotone graphs.

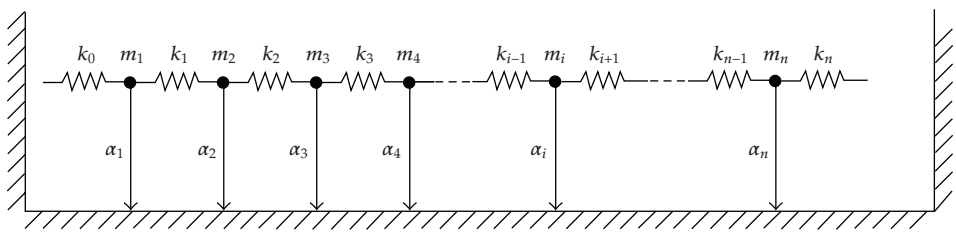


Figure 2: The studied model with mechanical components.

2. Description of the Model

We refer to previous works for description of some rheological models (see for example [1, 6]).

We consider the model of Figure 2. $(m_i)_{1\le i\le n}$ (with $m_i \ge 0$) correspond to masses, $(k_i)_{0\le i\le n}$ to stiffness, and $(\alpha_i)_{1\le i\le n}$ to St-Venant elements thresholds.

The reader is referred to Appendix A.

Let σ be the multivalued graph sign defined by (see Figure 1(a)).

$$\sigma(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0. \end{cases} \tag{2.1}$$

According to [11], this graph is maximal monotone. Therefore:

$$\forall x \in \mathbb{R}, \quad \sigma(x) = \partial|x|. \tag{2.2}$$

Let us assume (see Figure 3) the following

(i) This mechanical system is submitted to external forces $(F_i)_{0\le i\le n+1}$: F_0 is exerted on the spring with stiffness k_0; For $1 \le i \le n$, F_i is exerted on material point of mass m_i; F_{n+1} is exerted on the spring with stiffness k_{n+1}.

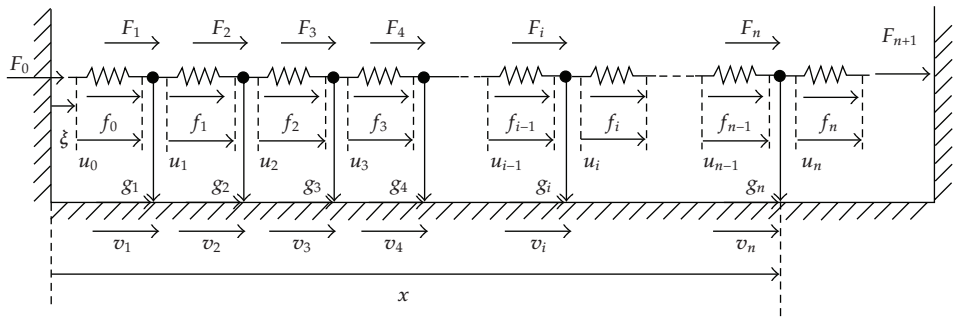


Figure 3: The studied model with external forces $(F_i)_{0\le i\le n}$, friction forces $(g_i)_{1\le i\le n}$, linear forces $(f_i)_{0\le i\le n}$, and with displacements $(u_i)_{0\le i\le n}$, $(v_i)_{1\le i\le n}$, x and ξ.

(ii) For $1 \le i \le n$, g_i is the friction force exerted by the support of the ith St-Venant.

(iii) For $0 \le i \le n$, f_i is elastic linear force exerted by the ith spring.

(iv) For $0 \le i \le n$, u_i is the displacement of the ith spring.

(v) For $1 \le i \le n$, v_i is the displacement of the ith St-Venant element.

(vi) ξ is the displacement of the spring with stiffness k_0.

(vii) x is the displacement of the material point of mass m_n.

These two last notations are justified by the study of particular cases in the next sections.

The different equations of the model are successively given by the fundamental Newton law:

$$\forall i \in \{1,\dots,n\}, \quad m_i\ddot{v}_i = F_i + f_{i-1} - f_i + g_i, \tag{2.3a}$$

by the constitutive laws of linear springs:

$$\forall i \in \{0,\dots,n\}, \quad f_i = -k_i u_i, \tag{2.3b}$$

by the constitutive of laws St-Venant elements:

$$\forall i \in \{1,\dots,n\}, \quad g_i \in -\alpha_i \sigma(\dot{v}_i), \tag{2.3c}$$

by the geometrical connexions:

$$\forall i \in \{0,\dots,n-1\}, \quad \xi + u_0 + u_1 + \cdots + u_i = v_{i+1}, \tag{2.3d}$$

$$\xi + u_0 + u_1 + \cdots + u_n = x, \tag{2.3e}$$

and finally by the boundary conditions:

$$F_0 = f_0, \tag{2.3f}$$

$$F_{n+1} = f_n. \tag{2.3g}$$

We can observe that (2.3d)–(2.3e) are equivalent to

$$\xi + u_0 = v_1, \tag{2.4a}$$

$$\forall i \in \{1, \ldots, n-1\}, \quad v_{i+1} - v_i = u_i, \tag{2.4b}$$

$$x - v_n = u_n. \tag{2.4c}$$

Now, we study systems (2.3a), (2.3b), (2.3c), (2.3f), (2.3g), and (2.4a)–(2.4c).

3. Transformations of Equations

Now, as in [1, 6], we transform system (2.3a)-(2.3b)-(2.3c)-(2.3f)-(2.3g)-(2.4b)-(2.4c) to rewrite it under the usual form (A.7) according to different kinds of problem and of boundary conditions.

Let us assume that the external forcing $F_1, \ldots, F_n$ are known.

3.1. Dynamical Case

We assume in this section that

$$\forall i \in \{1, \ldots, n\}, \quad m_i > 0. \tag{3.1}$$

Equations (2.3a)-(2.3b)-(2.3c)-(2.4a)–(2.4c) imply

$$m_1 \ddot{v}_1 + \alpha_1 \sigma(\dot{v}_1) - k_0 \xi + (k_0 + k_1) v_1 - k_1 v_2 \ni F_1, \tag{3.2a}$$

$$\forall i \in \{2, \ldots, n-1\}, \quad m_i \ddot{v}_i + \alpha_i \sigma(\dot{v}_i) - k_{i-1} v_{i-1} + (k_{i-1} + k_i) v_i - k_i v_{i+1} \ni F_i, \tag{3.2b}$$

$$m_n \ddot{v}_n + \alpha_n \sigma(\dot{v}_n) - k_{n-1} v_{n-1} + (k_{n-1} + k_n) v_n - k_n x \ni F_n. \tag{3.2c}$$

3.1.1. Clamped Mechanical System

We assume that our mechanical system is clamped at its two extremities so that we can write the boundary conditions:

$$\xi = 0, \qquad x = 0, \tag{3.3a}$$

$$\text{and the reactions } F_0 \text{ and } F_{n+1} \text{ are unknown.} \tag{3.3b}$$

We set, for all $q \in \mathbb{N}^*$,

$$K(q) = \begin{pmatrix} k_0 + k_1 & -k_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 & \cdots & \cdots & 0 \\ & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -k_{q-2} & k_{q-2} + k_{q-1} & -k_{q-1} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -k_{q-1} & k_{q-1} + k_q \end{pmatrix} \in \mathcal{M}_q(\mathbb{R}). \tag{3.4}$$

Thus, by setting

$$V = {}^t(v_1, \ldots, v_n) \in \mathbb{R}^n, \tag{3.5a}$$

$$F = {}^t(F_1, \ldots, F_n) \in \mathbb{R}^n, \tag{3.5b}$$

$$K = K(n), \tag{3.5c}$$

$$\mathcal{M} = \begin{pmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ 0 & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & m_n \end{pmatrix} \tag{3.5d}$$

and defining the maximal monotone operator $\mathcal{A}$ by

$$\mathcal{A}(v_1, \ldots, v_n) = \alpha_1 \sigma(v_1) \times \cdots \times \alpha_n \sigma(v_n), \tag{3.5e}$$

equations (3.2a)–(3.2c) imply the system of equations

$$M\ddot{V} + \mathcal{A}\dot{V} + KV \ni F. \tag{3.6}$$

Reactions F_0 and F_{n+1} can be determined thanks to (2.3f)-(2.3g) which give

$$F_0 = -k_0 v_1, \tag{3.7a}$$

$$F_{n+1} = k_n v_n. \tag{3.7b}$$

Set

$$p = 2n, \tag{3.8a}$$

$$M = \begin{pmatrix} I_n & 0 \\ 0 & \mathcal{M}^{-1} \end{pmatrix} \in \mathcal{M}_{2n}(\mathbb{R}), \tag{3.8b}$$

where I_n is the identity of $\in \mathcal{M}_n(\mathbb{R})$ and for $t \in \mathbb{R}$, $X = (V_1, V_2) \in \mathbb{R}^{2n}$, with $V_2 = (V_{2,1}, \ldots, V_{2,n})$,

$$\mathcal{G}(t, (V_1, V_2)) = \begin{pmatrix} V_2 \\ \mathcal{M}^{-1}F - \mathcal{M}^{-1}KV_1 \end{pmatrix}, \tag{3.8c}$$

$$\phi(V_1, V_2) = \sum_{i=1}^{n} \alpha_i |V_{2,i}|. \tag{3.8d}$$

Then, the system (3.6) is equivalent to (A.7) (see Appendix A).

Reciprocally, if (3.6) and (3.7a)-(3.7b) hold, we define x, ξ, $(u_i)_{0\le i\le n}$, $(f_i)_{0\le i\le n}$, and $(g_i)_{1\le i\le n}$ successivelly by

$$\begin{gathered} (2.3b), \\ (2.4a)–(2.4c), \\ (3.3a)\text{-}(3.3b), \\ \forall i \in \{1, \ldots, n\}, \quad g_i = m_i \ddot{v}_i - F_i - f_{i-1} + f_i. \end{gathered} \tag{3.9}$$

Then, we can deduce (2.3a), (2.3b), (2.3c), (2.3f), (2.3g), (2.4b), and (2.4c).

3.1.2. Clamped-Free Mechanical System

We assume that our mechanical system is clamped at its left extremity and free at its right extremity so that we can write the boundary condition:

$$\xi = 0, \tag{3.10a}$$

$$\text{reaction } F_0 \text{ is unknown}, \tag{3.10b}$$

$$\text{displacement } x \text{ is unknown}, \tag{3.10c}$$

$$\text{and external forcing } F_{n+1} \text{ is known}. \tag{3.10d}$$

As in Section 3.1.1, by setting

$$V = {}^t(v_1, \dots, v_n) \in \mathbb{R}^n, \tag{3.11a}$$

$$F = {}^t(F_1, \dots, F_{n-1}, F_n - F_{n+1}) \in \mathbb{R}^n, \tag{3.11b}$$

$$\widetilde{K} = \begin{pmatrix} k_0 + k_1 & -k_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 & \cdots & \cdots & 0 \\ & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -k_{n-2} & k_{n-2} + k_{n-1} & -k_{n-1} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & -k_{n-1} & k_{n-1} \end{pmatrix} \in \mathcal{M}_n(\mathbb{R}), \tag{3.11c}$$

$$\mathcal{M} \text{ and } \mathcal{A} \text{ as in (3.5d)-(3.5e)}, \tag{3.11d}$$

we can prove that equations (3.2a)–(3.2c) imply the system of equations

$$M\ddot{V} + \mathcal{A}\dot{V} + \widetilde{K}V \ni F. \tag{3.12}$$

Reactions F_0 and displacement x can be determined thanks to

$$F_0 = -k_0 v_1, \tag{3.13a}$$

$$x = -\frac{F_{n+1}}{k_n} + v_n. \tag{3.13b}$$

As in Section 3.1.1, let us set

$$p = 2n, \tag{3.14a}$$

$$M \text{ and } \phi \text{ defined by (3.8b)–(3.8d)}, \tag{3.14b}$$

and for $t \in \mathbb{R}$, $X = (V_1, V_2) \in \mathbb{R}^{2n}$, with $V_2 = (V_{2,1}, \dots, V_{2,n})$,

$$\mathcal{G}(t, (V_1, V_2)) = \begin{pmatrix} V_2 \\ \mathcal{M}^{-1}F - \mathcal{M}^{-1}\widetilde{K}V_1 \end{pmatrix}. \tag{3.14c}$$

Then, system (3.12) is equivalent to (A.7).

As in Section 3.1.1, reciprocally, if (3.12) and (3.13a)-(3.13b) hold, we define x, ξ, $(u_i)_{0\le i\le n}$, $(f_i)_{0\le i\le n}$, and $(g_i)_{1\le i\le n}$ successivelly by

$$\begin{gathered}(2.3b),\\ (2.4a)–(2.4c),\\ (3.3a)\text{-}(3.3b),\\ \text{Last equation of (3.9).}\end{gathered} \tag{3.15}$$

Then, we can deduce (2.3a), (2.3b), (2.3c), (2.3f), (2.3g), (2.4b), and (2.4c).

3.2. Semi-Dynamical Case

In this section, we assume that

$$\forall i \in \{1,\dots,n-1\}, \quad m_i = 0, \tag{3.16a}$$

$$m_n = m > 0. \tag{3.16b}$$

Equation (3.2b) implies

$$\forall i \in \{2,\dots,n-1\}, \quad \alpha_i\sigma(\dot{v}_i) + g_i \ni 0, \tag{3.17a}$$

with

$$\forall i \in \{2,\dots,n-1\}, \quad g_i = -F_i - k_{i-1}v_{i-1} + (k_{i-1}+k_i)v_i - k_i v_{i+1}. \tag{3.17b}$$

As in [6, 7], we introduce β, the inverse graph of σ (in the sens of [11], see Figure 1(b)):

$$\beta(x) = \begin{cases} \emptyset & \text{if } x \in]-\infty,-1[\bigcup]1,+\infty[,\\ \{0\} & \text{if } x \in]-1,1[,\\ \mathbb{R}_- & \text{if } x = -1,\\ \mathbb{R}_+ & \text{if } x = 1. \end{cases} \tag{3.18}$$

We have

$$\forall x \in \mathbb{R}, \quad \beta(x) = \partial\psi_{[-1,1]}(x), \tag{3.19}$$

where $\partial\psi_{[-1,1]}$ is the convex indicatrix function of the convex domain $[-1,1]$. Thus, (3.17a) is equivalent to

$$\forall i \in \{2,\dots,n-1\}, \quad \dot{v}_i + \partial\psi_{[-\alpha_i,\alpha_i]}(g_i) \ni 0. \tag{3.20}$$

Similarly, (3.2a) gives

$$\dot{v}_1 + \partial\psi_{[-\alpha_1,\alpha_1]}(g_1) \ni 0, \tag{3.21a}$$

with

$$g_1 = -F_1 - k_0\xi + (k_0 + k_1)v_1 - k_1 v_2. \tag{3.21b}$$

and (3.2c) gives

$$m\ddot{v}_n + \alpha_n\sigma(\dot{v}_n) - k_{n-1}v_{n-1} + (k_{n-1} + k_n)v_n - k_n x \ni F_n. \tag{3.22}$$

3.2.1. Clamped Mechanical System

We assume that our mechanical system is clamped at its two extremities so that we can write the boundary conditions (3.3a)-(3.3b). As in [6, 7], let us set

$$V = {}^t(v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1}, \tag{3.23a}$$

$$G = {}^t(g_1, \ldots, g_{n-1}) \in \mathbb{R}^{n-1}, \tag{3.23b}$$

$$F = {}^t(F_1, \ldots, F_{n-1}) \in \mathbb{R}^{n-1}, \tag{3.23c}$$

$$Z = {}^t(F_1, \ldots, F_{n-2}, F_{n-1} + k_{n-1}v_n) \in \mathbb{R}^{n-1}, \tag{3.23d}$$

$$C = [\alpha_1, \alpha_1] \times \cdots \times [-\alpha_{n-1}, \alpha_{n-1}] \subset \mathbb{R}^{n-1}, \tag{3.23e}$$

$$\widehat{K} = K(n-1) \in \mathcal{M}_{n-1}(\mathbb{R}), \tag{3.23f}$$

where $K(q)$ is defined by (3.4). Thus, according to (3.17b)–(3.21b), we have

$$G = \widehat{K}V - Z, \tag{3.24}$$

and from (3.17a)–(3.21a) we can write

$$\dot{V} + \partial\psi_C(G) \ni 0, \tag{3.25}$$

Under the assumption

$$k_{n-1} \geq 0 \quad \forall i \in \{1, \ldots, n-1\},\ k_i > 0, \tag{3.26}$$

the matrix $\widehat{K}$ is symmetric definite positive (see proof in Lemma B.1 of Appendix B), so that

$$V = \widehat{K}^{-1}(G + Z), \tag{3.27}$$

and (3.25) gives

$$\widehat{K}^{-1}(\dot{G}+\dot{Z})+\partial\psi_C(G)\ni 0, \tag{3.28}$$

which is equivalent to

$$\dot{G}+\widehat{K}\partial\psi_C(G)\ni -\dot{Z}. \tag{3.29}$$

For q integer and u vector of $\mathbb{R}^m$, we denote by

$$[u]_q \tag{3.30}$$

the qth component of u. Equation (3.22) gives

$$m\ddot{v}_n+\alpha_n\sigma(\dot{v}_n)-k_{n-1}v_{n-1}+(k_{n-1}+k_n)v_n\ni F_n, \tag{3.31}$$

which can be rewritten under the following form:

$$\ddot{v}_n+\frac{\alpha_n}{m}\sigma(\dot{v}_n)-\frac{k_{n-1}}{m}\left[\widehat{K}^{-1}(G+Z)\right]_{n-1}+\frac{k_{n-1}+k_n}{m}v_n\ni\frac{F_n}{m}. \tag{3.32}$$

Let u be the vector of $\mathbb{R}^{n-1}$ defined by

$$u={}^t(0,\ldots,0,1). \tag{3.33}$$

Note that

$$Z=F+k_{n-1}v_nu. \tag{3.34}$$

We set

$$p=n+1, \tag{3.35a}$$

$$M=\begin{pmatrix}\widehat{K} & 0\\ 0 & I_2\end{pmatrix}\in\mathcal{M}_{n+1}(\mathbb{R}), \tag{3.35b}$$

and for all $t\in\mathbb{R}$, $G\in\mathbb{R}^{n-1}$, $a,b\in\mathbb{R}$, $X={}^t(G,a,b)$

$$\mathcal{G}(t,X)=\begin{pmatrix}-\dot{F}-k_{n-1}bu\\ b\\ \dfrac{F_n}{m}+\dfrac{k_{n-1}}{m}\left[\widehat{K}^{-1}(G+F+k_{n-1}au)\right]_{n-1}-\dfrac{k_{n-1}+k_n}{m}a\end{pmatrix} \tag{3.35c}$$

$$\phi(X)=\psi_{[\alpha_1,\alpha_1]\times\cdots\times[-\alpha_{n-1},\alpha_{n-1}]\times\{0\}\times\{0\}}(X)+\frac{\alpha_n}{m}|b|. \tag{3.35d}$$

Then, system (3.29)–(3.32) is equivalent to (A.7).

Reactions F_0 and F_{n+1} can be determined thanks to

$$F_0 = -k_0\left[\widehat{K}^{-1}(G + F + k_{n-1}au)\right]_1, \tag{3.36a}$$

$$F_{n+1} = k_n a. \tag{3.36b}$$

Reciprocally, as in Section 3.1.1, if (3.29)–(3.32) hold, we can determine G and Z thanks to

$$\begin{gathered} G = {}^t([X]_1, \ldots, [X]_{n-1}), \\ Z = k_{n-1}[X]_n u + F. \end{gathered} \tag{3.37}$$

then we can calculate V thanks to (3.27). Successively, x, ξ, $(u_i)_{0\le i\le n}$, and $(f_i)_{0\le i\le n}$ are defined by

$$\begin{gathered} (2.3b), \\ (2.4a)\text{–}(2.4c), \\ (3.3a)\text{-}(3.3b). \end{gathered} \tag{3.38}$$

Then, we can deduce (2.3a), (2.3b), (2.3c), (2.3f), (2.3g), (2.4b), and (2.4c).

3.2.2. Clamped-Free Mechanical System

We assume that our mechanical system is clamped at its left extremity and free at its right extremity so that we can write boundary condition (3.10a)–(3.10d).

The calculus are similar to those of Section 3.2.1; Equation (3.29) holds and (3.31) is replaced by

$$m\ddot{v}_n + \alpha_n\sigma(\dot{v}_n) - k_{n-1}v_{n-1} + k_{n-1}v_n + F_{n+1} \ni F_n. \tag{3.39}$$

Using notations (3.23a)–(3.23f), we obtain the system (A.7), where we set

$$p = n + 1, \tag{3.40a}$$

$$M \text{ and } \phi \text{ are defined by (3.35b) and (3.35d)}, \tag{3.40b}$$

and for all $t \in \mathbb{R}$, $G \in \mathbb{R}^{n-1}$, $a, b \in \mathbb{R}$, $X = {}^t(G, a, b)$,

$$\mathcal{G}(t, X) = \begin{pmatrix} -\dot{F} - k_{n-1}bu \\ b \\ \dfrac{F_n - F_{n+1}}{m} + \dfrac{k_{n-1}}{m}\left[\widehat{K}^{-1}(G + F + k_{n-1}au)\right]_{n-1} - \dfrac{k_{n-1}}{m}a \end{pmatrix} \in \mathcal{M}_{n+1}(\mathbb{R}). \tag{3.40c}$$

The reaction F_0 and the displacement x can be determined thanks to (3.36a) and

$$x = -\frac{F_{n+1}}{k_n} + a. \tag{3.41}$$

3.3. Quasistatic Case

In this section, we assume that

$$\forall i \in \{1, \ldots, n\}, \quad m_i = 0. \tag{3.42}$$

As it has been previously noticed, (3.17a)-(3.17b) and (3.21a)-(3.21b) are not modified, and (3.22) gives

$$\dot{v}_n + \partial\psi_{[-\alpha_n,\alpha_n]}(g_n) \ni 0, \tag{3.43a}$$

with

$$g_n = F_n - k_{n-1}v_{n-1} + (k_{n-1} + k_n)v_n - k_n x. \tag{3.43b}$$

3.3.1. Clamped Mechanical System

We assume that our mechanical system is clamped at its two extremities so that we can write the boundary conditions (3.3a)-(3.3b).

As in Section 3.2.1, following [6, 7], we set

$$V = {}^t(v_1, \ldots, v_n) \in \mathbb{R}^n, \tag{3.44a}$$

$$G = {}^t(g_1, \ldots, g_n) \in \mathbb{R}^n, \tag{3.44b}$$

$$F = {}^t(F_1, \ldots, F_n) \in \mathbb{R}^n, \tag{3.44c}$$

$$C = [\alpha_1, \alpha_1] \times \cdots \times [-\alpha_n, \alpha_n] \subset \mathbb{R}^n, \tag{3.44d}$$

$$K = K(n) \in \mathcal{M}_n(\mathbb{R}), \tag{3.44e}$$

where $K(q)$ is defined by (3.4).

Thus, we have

$$\begin{gathered} G = KV - F, \\ \dot{V} + \partial\psi_C(G) \ni 0 \end{gathered} \tag{3.45}$$

Under assumption

$$k_n \geq 0, \quad \forall i \in \{1, \ldots, n-1\}, \ k_i > 0, \tag{3.46}$$

the matrix K is symmetric definite positive (see proof in Lemma B.1), so that

$$V = K^{-1}(G + F), \tag{3.47}$$

$$\dot{G} + K\partial\psi_C(G) \ni -\dot{F}. \tag{3.48}$$

We set

$$p = n, \tag{3.49a}$$

$$M = K \in \mathcal{M}_n(\mathbb{R}), \tag{3.49b}$$

and, for all $t \in \mathbb{R}$, for all $X \in \mathbb{R}^n$

$$\mathcal{G}(t, X) = -\dot{F}, \tag{3.49c}$$

$$\phi(X) = \psi_C(X). \tag{3.49d}$$

Then, the system (3.48) is equivalent to (A.7).

Reactions F_0 and F_{n+1} can be determined thanks to

$$F_0 = -k_0\left[K^{-1}(G + F)\right]_1, \tag{3.50a}$$

$$F_{n+1} = k_n\left[K^{-1}(G + F)\right]_n. \tag{3.50b}$$

3.3.2. Clamped-Free Mechanical System

We assume that our mechanical system is clamped at its left extremity so that we can write the boundary condition (3.10a) and (3.10b). Boundary conditions for its right extremity is given later.

The calculus is similar to those of [6, 7].

(i) First Case: Displacement-Force Model

We assume that the displacement x is known and that the force F_{n+1} is unknown.

We introduce V, G, C and matrix K defined by (3.44a), (3.44b), (3.44d), and (3.44e) and F defined by

$$F = {}^t\,(F_1, \ldots, F_{n-1}, F_n + k_n x) \in \mathbb{R}^n, \tag{3.51}$$

and we obtain, as in Section 3.3.1,

$$G = KV - F, \tag{3.52}$$

$$\dot{G} + K\partial\psi_C(G) \ni -\dot{F}. \tag{3.53}$$

By setting

$$p = n, \tag{3.54a}$$

$$M \text{ and } \phi \text{ are defined by (3.49b) and (3.49d)}, \tag{3.54b}$$

and, for all $t \in \mathbb{R}$, for all $X \in \mathbb{R}^n$,

$$\mathcal{G}(t, X) = -\dot{F}, \tag{3.54c}$$

we remark that system (3.53) is equivalent to (A.7).

Reactions F_0 and F_{n+1} can be determined thanks to

$$F_{n0} = -k_0 \left[K^{-1}(G + F) \right]_1, \tag{3.55a}$$

$$F_{n+1} = -k_n x + k_n \left[K^{-1}(G + F) \right]_n. \tag{3.55b}$$

(ii) Second Case: Force-Displacement Model

We assume that external forcing F_{n+1} are known and displacement x is unknown.

The calculus are similar to the previous case.

Equation (3.43b) is replaced by

$$g_n = F_{n+1} - F_n - k_{n-1} v_{n-1} + k_{n-1} v_n. \tag{3.56}$$

Following the same method, we introduce V, G, and C defined by (3.44a)-(3.44b)-(3.44d), and matrix $\widetilde{K}$ defined by (3.11c). Vector F is defined by

$$F = {}^t \left(F_1, \ldots, F_{n-1}, F_n - F_{n+1} \right). \tag{3.57}$$

So, (3.52) is replaced by

$$G = \widetilde{K} V - F, \tag{3.58}$$

and (3.48) is replaced by

$$\dot{G} + \widetilde{K} \partial \psi_C(G) \ni -\dot{F}. \tag{3.59}$$

Remark 3.1. As in [6], let us notice that matrix $\widetilde{K}$ defined by (3.11c) for force-displacement model corresponds to matrix $K(n)$ for displacement-force model defined by (3.4) with

$$k_n = 0. \tag{3.60}$$

According to previous remark, assumption

$$\forall i \in \{1, \ldots, n-1\}, \quad k_i > 0 \tag{3.61}$$

and Lemma B.1 ensure that matrix $\widetilde{K}$ is symmetric definite positive. Thus, like previously, the system is equivalent to

$$\dot{G} + \widetilde{K}\partial\psi_C(G) \ni -\dot{F}. \tag{3.62}$$

By giving p, ϕ defined by (3.49a)–(3.49d), $\mathcal{G}$ defined by for all $t \in \mathbb{R}$, for all $X \in \mathbb{R}^n$,

$$\mathcal{G}(t, X) = -\dot{F}, \tag{3.63a}$$

and M defined by

$$M = \widetilde{K} \in \mathcal{M}_n(\mathbb{R}), \tag{3.63b}$$

we remark that system (3.62) is equivalent to (A.7). Reactions F_0 and displacement x can be determined thanks to

$$F_0 = -k_0\left[\widetilde{K}^{-1}(G+F)\right]_1, \tag{3.64a}$$

$$x = -\frac{F_{n+1}}{k_n} + \left[\widetilde{K}^{-1}(G+F)\right]_n. \tag{3.64b}$$

4. Existence of Uniqueness Results

Thus, as proved in [1], all the systems of Section 3 can be written under the form (A.7) and, according to Proposition A.1 (see Appendix A), have a unique solution. For all systems, Table 1 provides the corresponding integer p, function ϕ, and matrix M. It is easy to prove that ϕ is convex proper and lower semi-continuous function on $\mathbb{R}^p$ and that M is symmetric positive definite.

5. Convergence of Numerical Scheme

All the models examined here can be written under the form (A.7). Based on [1, 12], general writing of the implicit Euler scheme corresponds to

$$\begin{gathered} \forall n \in \{0, \ldots, N-1\}, \quad \frac{X_{n+1} - X_n}{h} + M\partial\phi(X_n) \ni \mathcal{G}(t_n, X_n), \\ X_0 = \xi. \end{gathered} \tag{5.1}$$

with time step h, discretized time $t_n = hn$, and approximations $X_0, \ldots, X_N$ of the exact solution provided by the numerical scheme. Previous studies [12] ensure that this numerical

Table 1: The dimension of the system, the convex function and the symmetric positive definite matrix used for the above described mechanical models.

System	p	function ϕ	matrix M
(3.6)	$2n$	$\phi(V_1, V_2) = \sum_{i=1}^{n} \alpha_i \lvert V_{2,i}\rvert$	$\begin{pmatrix} I_n & 0 \\ 0 & \mathcal{M}^{-1} \end{pmatrix}$
(3.12)	$2n$	$\phi(V_1, V_2) = \sum_{i=1}^{n} \alpha_i \lvert V_{2,i}\rvert$	$\begin{pmatrix} I_n & 0 \\ 0 & \mathcal{M}^{-1} \end{pmatrix}$
(3.29)–(3.32)	$n+1$	$\phi(t,(G,a,b)) = \psi_{[-\alpha_1,\alpha_1]\times\cdots\times[-\alpha_{n-1},\alpha_{n-1}]\times\{0\}\times\{0\}}(G,a,b) + \frac{\alpha_n}{m}\lvert b\rvert$	$\begin{pmatrix} \hat{K} & 0 \\ 0 & I_2 \end{pmatrix}$
(3.29)–(3.39)	$n+1$	$\phi(t,(G,a,b)) = \psi_{[-\alpha_1,\alpha_1]\times\cdots\times[-\alpha_{n-1},\alpha_{n-1}]\times\{0\}\times\{0\}}(G,a,b) + \frac{\alpha_n}{m}\lvert b\rvert$	$\begin{pmatrix} \hat{K} & 0 \\ 0 & I_2 \end{pmatrix}$
(3.48)	n	$\phi(X) = \psi_{[-\alpha_1,\alpha_1]\times\cdots\times[-\alpha_n,\alpha_n]}(X)$	K
(3.53)	n	$\phi(X) = \psi_{[-\alpha_1,\alpha_1]\times\cdots\times[-\alpha_n,\alpha_n]}(X)$	K
(3.62)	n	$\phi(X) = \psi_{[-\alpha_1,\alpha_1]\times\cdots\times[-\alpha_n,\alpha_n]}(X)$	$\tilde{K}$

scheme is convergent with order 1/2 (systems (3.6), (3.12), (3.29)–(3.32), and (3.29)–(3.39)) or 1 (systems (3.48), (3.53), and (3.62)).

In practice for computation of solutions, three cases can be distinguished, based on further expression of X_{n+1}:

$$X_{n+1} = \left[I + hM\partial\phi\right]^{-1}(X_n + h\mathcal{G}(t_n, X_n)), \tag{5.2}$$

where I is the identity and $[I + hM\partial\phi]^{-1}$ is the inverse of the graph $I + hM\partial\phi$ (see [11]). According to [11], $[I + hM\partial\phi]^{-1}$ is a monovalued operator, providing a unique solution $X_{n+1} \in \mathbb{R}^p$. In the first case, effective computations of X_{n+1} associated with diagonal matrix M is explicit: this situation corresponds to systems (3.6) and (3.12). In the second case, ϕ is defined as the indicatrix function of a closed convex set: this situation corresponds to systems (3.48), (3.53), and (3.62). Effective computation of X_{n1} is given by the projection of a given vector on a closed convex set (see [6]). In the third case (for systems (3.29)–(3.32) and (3.29)–(3.39)), ϕ is involving indicatrix function of a closed convex set and a norm function. In such case, computation of X_{n+1} leads to the following problem: according to [11], X_{n+1} is the solution of minimization problem: considering $\|\cdot\|_M$ the norm define by the inner product given by (A.4)

$$\|x\|_M = \sqrt{{}^t x\, M^{-1} x}, \tag{5.3a}$$

$$Z_n = X_n + h\mathcal{G}(t_n, X_n) \tag{5.3b}$$

solve

$$\min_{x\in D(\partial\phi)} \phi(x) + \frac{1}{2h}\|x - Z_n\|_M^2 \tag{5.3c}$$

and such problem can be solved in practice following efficient algorithms [13].

6. Conclusion

In this paper, a mechanical system involving finite degrees of freedom and nonsmooth terms have been investigated from the mechanical point of view. Dynamical, semi-dynamical, and quasistatic modeling have been established. The main results are theoretical ones:

(i) all the problems are well posed;

(ii) it has been explained how a numerical approximation of solutions can be effectively computed.

All the mechanical systems have been considered in a deterministic frame. Theoretical results and corresponding effective computations could be extended to the stochastic frame.

Appendices

A. A Few Theoretical Reminders about the Class of Maximal Monotone Differential Equations Used

The reader is referred to [11]. Let $\langle , \rangle$ be scalar product on $\mathbb{R}^p$. If ϕ is a convex proper and lower semi-continuous function from $\mathbb{R}^p$ to $]-\infty, +\infty]$, we can define its subdifferential $\partial\phi$ by

$$\begin{gathered} y \in \partial\phi(x) \Longleftrightarrow \forall h \in \mathbb{R}^p, \quad \phi(x+h) - \phi(x) \geq \langle y, h \rangle, \\ D(\partial\phi) = \{x : \partial\phi(x) \neq \emptyset\}. \end{gathered} \tag{A.1}$$

Moreover, $\partial\phi$ is a maximal monotone graph in $\mathbb{R}^p \times \mathbb{R}^p$.

If C is a closed convex nonempty subset of $\mathbb{R}^p$, we denote by ψ_C the indicatrix of C defined by

$$\forall x \in C, \quad \psi_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases} \tag{A.2}$$

In this particular case, $\partial\psi_C$, which is the subdifferential of ψ_C, is given by

$$\forall (x, y) \in C \times \mathbb{R}^p, \quad y \in \partial\psi_C(x) \Longleftrightarrow \forall z \in C, \quad \langle y, x - z \rangle \geq 0, \tag{A.3a}$$

$$\forall x \notin C, \quad \partial\psi_C(x) = \emptyset. \tag{A.3b}$$

The domain of the maximal monotone operator $\partial\psi_C$ is equal to C.

We observe that if $\mathbb{R}^p$ is equipped with its canonical scalar product $\langle , \rangle$, and with another scalar product,

$$\langle x, y \rangle_M = {}^t x M^{-1} y, \tag{A.4}$$

where M is symmetric positive definite, then we can relate the subdifferential $\partial\phi$ of ϕ relatively to the canonical scalar product $\langle , \rangle$ and the subdifferential $\partial_M\phi$ relatively to $\langle , \rangle_M$ by

$$\partial_M\phi(x) = M\partial\phi(x). \tag{A.5}$$

We give now the general mathematical formulation of our problem. We assume that T is strictly positive and that $\mathcal{G}$ is a function from $[0,T]\times\mathbb{R}^p$ to $\mathbb{R}^p$ which is Lipschitz continuous with respect to its second argument, that is, there exists $\omega \geq 0$ such that

$$\forall t \in [0,T], \quad \forall X_1, X_2 \in \mathbb{R}^p, \quad \|\mathcal{G}(t,X_1) - \mathcal{G}(t,X_2)\| \leq \omega\|X_1 - X_2\|. \tag{A.6a}$$

Moreover, we assume that

$$\forall Y \in \mathbb{R}^p, \quad \mathcal{G}(\cdot, Y) \in L^\infty(0,T;\mathbb{R}^p). \tag{A.6b}$$

Proposition A.1. *If the matrix M is symmetric positive definite and ϕ is convex proper and lower semicontinuous on $\mathbb{R}^p$, under assumptions* (A.6a)-(A.6b)*, for all $\xi \in D(\partial\phi)$, there exists a unique function X in $W^{1,1}(0,T;\mathbb{R}^p)$ such that*

$$\begin{gathered} \dot{X}(t) + M\partial\phi(X(t)) \ni \mathcal{G}(t,X(t)) \quad \textit{a.e. on }]0,T[, \\ X(0) = \xi, \end{gathered} \tag{A.7}$$

where the differential inclusion can be written as an inequality: for almost every t in $]0,T[$,

$$\forall h \in \mathbb{R}^p, \quad \phi(X(t)+h) - \phi(X(t)) \geq \langle G(t,X(t)) - \dot{X}(t), h\rangle_M. \tag{A.8}$$

Proof of this result can be found in [1, Proposition 3.1], based on [11, Proposition 3.13, page 107] and (A.5).

B. $K(q)$ Defined by (3.4) Is Symmetric Definite Positive

Lemma B.1. *Under assumption*

$$k_q \geq 0, \qquad \forall i \in \{1,\ldots,q-1\}, \quad k_i > 0, \tag{B.1}$$

matrix $K(q)$ defined by (3.4) *is symmetric definite positive.*

Proof. We have, for all $X = (x_1, \ldots, x_q) \in \mathbb{R}^q$,

$$\begin{aligned} {}^t XK(q)X &= \sum_{i=1}^{q}(k_{i-1} + k_i)x_i^2 - 2\sum_{i=1}^{q} k_i x_i x_{i+1}, \\ &= k_0 x_1^2 + \sum_{i=2}^{q} k_{i-1} x_i^2 + \sum_{i=1}^{q-1} k_i x_i^2 + k_{q-1} x_{q-1}^2 - 2\sum_{i=1}^{q-1}(k_i k_{i+1}) x_{i+1}, \\ &= k_0 x_1^2 + \sum_{i=1}^{q-1} k_i (x_{i+1} - x_i)^2 + k_q x_q^2. \end{aligned} \tag{B.2}$$

Under assumption (B.1), ${}^t XK(q)X = 0$ then implies $X = 0$. □

References

[1] J. Bastien, M. Schatzman, and C.-H. Lamarque, "Study of some rheological models with a finite number of degrees of freedom," *European Journal of Mechanics A*, vol. 19, no. 2, pp. 277–307, 2000.

[2] J. Bastien, M. Schatzman, and C.-H. Lamarque, "Study of an elastoplastic model with an infinite number of internal degrees of freedom," *European Journal of Mechanics A*, vol. 21, no. 2, pp. 199–222, 2002.

[3] J. Bastien and C.-H. Lamarque, "Maximal monotone model with history term," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, no. 5–7, pp. e199–e207, 2005.

[4] J. Bastien and C.-H. Lamarque, "Non smooth dynamics of mechanical systems with history term," *Nonlinear Dynamics*, vol. 47, no. 1–3, pp. 115–128, 2007.

[5] J. Bastien, G. Michon, L. Manin, and R. Dufour, "An analysis of the modified Dahl and Masing models: application to a belt tensioner," *Journal of Sound and Vibration*, vol. 302, no. 4-5, pp. 841–864, 2007.

[6] J. Bastien and C.-H. Lamarque, "Persoz's gephyroidal model described by a maximal monotone differential inclusion," *Archive of Applied Mechanics*, vol. 78, no. 5, pp. 393–407, 2008.

[7] J. Bastien and C.-H. Lamarque, "A finite dimensional mechanical systme with a cascade of non smooth constitutive terms," in *Proceedings of European Conference on Nonlinear Oscillations (ENOC '08)*, Saint-Petersbourg, Russia, June-July 2008.

[8] C.-H. Lamarque, J. Bastien, and M. Holland, "Study of a maximal monotone model with a delay term," *SIAM Journal on Numerical Analysis*, vol. 41, no. 4, pp. 1286–1300, 2003.

[9] C.-H. Lamarque, J. Bastien, and M. Holland, "Maximal monotone model with delay term of convolution," *Mathematical Problems in Engineering*, vol. 2005, no. 4, pp. 437–453, 2005.

[10] C.-H. Lamarque, F. Bernardin, and J. Bastien, "Study of a rheological model with a friction term and a cubic term: deterministic and stochastic cases," *European Journal of Mechanics A*, vol. 24, no. 4, pp. 572–592, 2005.

[11] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematics Studies, no. 5. Notas de Matemática (50), North-Holland, Amsterdam, The Netherlands, 1973.

[12] J. Bastien and M. Schatzman, "Numerical precision for differential inclusions with uniqueness," *M2AN Mathematical Modelling and Numerical Analysis*, vol. 36, no. 3, pp. 427–460, 2002.

[13] V. Acary and B. Brogliato, *Numerical Methods for Nonsmooth Dynamical Systems. Applications in Mechanics and Electronics*, Springer, Berlin, Germany, 2008.