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Research Article

Soliton and Periodic Wave Solutions to the Osmosis $K(2,2)$ Equation

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Two types of traveling wave solutions to the osmosis $K(2, 2)$ equation $u_t + (u^2)_x - (u^2)_{xxx} = 0$ are investigated. They are characterized by two parameters. The expresssions for the soliton and periodic wave solutions are obtained.

1. Introduction

In 1993, Rosenau and Hyman [1] introduced a genuinely nonlinear dispersive equation, a special type of KdV equation, of the form

$$u_t + a(u^n)_x + (u^n)_{xxx} = 0, \quad n > 1, \tag{1.1}$$

where a is a constant and both the convection term $(u^n)_x$ and the dispersion effect term $(u^n)_{xxx}$ are nonlinear. These equations arise in the process of understanding the role of nonlinear dispersion in the formation of structures like liquid drops. Rosenau and Hyman derived solutions called compactons to (1.1) and showed that while compactons are the essence of the focusing branch where $a > 0$, spikes, peaks, and cusps are the hallmark of the defocusing branch where $a < 0$ which also supports the motion of kinks. Further, the negative branch, where $a < 0$, was found to give rise to solitary patterns having cusps or infinite slopes. The focusing branch and the defocusing branch represent two different models, each leading to a different physical structure. Many powerful methods were applied to construct the exact

solutions to (1.1), such as Adomain method [2], homotopy perturbation method [3], Exp-function method [4], variational iteration method [5], and variational method [6, 7]. In [8], Wazwaz studied a generalized forms of (1.1), that is $mK(n,n)$ equations and defined by

$$u^{n-1}u_t + a(u^n)_x + b(u^n)_{xxx} = 0, \quad n > 1, \tag{1.2}$$

where a, b are constants. He showed how to construct compact and noncompact solutions to (1.2) and discussed it in higher-dimensional spaces in [9]. Chen et al. [10] showed how to construct the general solutions and some special exact solutions to (1.2) in higher-dimensional spatial domains. He et al. [11] considered the bifurcation behavior of traveling wave solutions to (1.2). Under different parametric conditions, smooth and nonsmooth periodic wave solutions, solitary wave solutions, and kink and antikink wave solutions were obtained. Yan [12] further extended (1.2) to be a more general form

$$u^{m-1}u_t + a(u^n)_x + b\left(u^k\right)_{xxx} = 0, \quad nk \neq 1. \tag{1.3}$$

And using some direct ansatze, some abundant new compacton solutions, solitary wave solutions and periodic wave solutions to (1.3) were obtained. By using some transformations, Yan [13] obtained some Jacobi elliptic function solutions to (1.3). Biswas [14] obtained 1-soliton solution of equation with the generalized evolution term

$$\left(u^l\right)_t + a(u^m)u_x + b(u^n)_{xxx} = 0, \tag{1.4}$$

where a, b are constants, while l, m, and n are positive integers. Zhu et al. [15] applied the decomposition method and symbolic computation system to develop some new exact solitary wave solutions to the $K(2,2,1)$ equation

$$u_t + \left(u^2\right)_x - \left(u^2\right)_{xxx} + u_{xxxxx} = 0, \tag{1.5}$$

and the $K(3,3,1)$ equation

$$u_t + \left(u^3\right)_x - \left(u^3\right)_{xxx} + u_{xxxxx} = 0. \tag{1.6}$$

Recently, Xu and Tian [16] introduced the osmosis $K(2,2)$ equation

$$u_t + \left(u^2\right)_x - \left(u^2\right)_{xxx} = 0, \tag{1.7}$$

where the positive convection term $(u^2)_x$ means the convection moves along the motion direction, and the negative dispersive term $(u^2)_{xxx}$ denotes the contracting dispersion. They obtained the peaked solitary wave solution and the periodic cusp wave solution to (1.7). In [17], the authors obtained the smooth soliton solutions to (1.7). In this paper, following Vakhnenko and Parkes's strategy [18, 19] we continue to investigate the traveling wave

solutions to (1.7) and obtain soliton and periodic wave solutions. Our work in this paper covers and extends the results in [16, 17] and may help people to know deeply the described physical process and possible applications of the osmosis $K(2,2)$ equation.

The remainder of this paper is organized as follows. In Section 2, for completeness and readability, we repeat [19, Appendix A], which discusses the solutions to a first-order ordinary differential equaion. In Section 3, we show that, for traveling wave solutions, (1.7) may be reduced to a first-order ordinary differential equation involving two arbitrary integration constants a and b. We show that there are four distinct periodic solutions corresponding to four different ranges of values of a and restricted ranges of values of b. A short conclusion is given in Section 4.

2. Solutions to a First-Order Ordinary Differential Equaion

This section is due to Vakhnenko and Parkes (see [19, Appendix A]). For completeness and readability, we state it in the following.

Consider solutions to the following ordinary differential equation:

$$(\varphi\varphi_{\xi})^2 = \varepsilon^2 f(\varphi), \tag{2.1}$$

where

$$f(\varphi) = (\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi_3 - \varphi)(\varphi_4 - \varphi), \tag{2.2}$$

and $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are chosen to be real constants with $\varphi_1 \le \varphi_2 \le \varphi \le \varphi_3 \le \varphi_4$.

Following [20] we introduce ζ defined by

$$\frac{d\xi}{d\zeta} = \frac{\varphi}{\varepsilon}, \tag{2.3}$$

so that (2.1) becomes

$$(\varphi_{\zeta})^2 = f(\varphi). \tag{2.4}$$

Equation (2.4) has two possible forms of solution. The first form is found using result 254.00 in [21]. Its parametric form is

$$\begin{gathered} \varphi = \frac{\varphi_2 - \varphi_1 n \operatorname{sn}^2(w \mid m)}{1 - n \operatorname{sn}^2(w \mid m)}, \\ \xi = \frac{1}{\varepsilon p}\left(w\varphi_1 + (\varphi_2 - \varphi_1)\prod(n; w \mid m)\right), \end{gathered} \tag{2.5}$$

with w as the parameter, where

$$m = \frac{(\varphi_3 - \varphi_2)(\varphi_4 - \varphi_1)}{(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_1)}, \quad p = \frac{1}{2}\sqrt{(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_1)}, \quad w = p\zeta, \tag{2.6}$$

$$n = \frac{\varphi_3 - \varphi_2}{\varphi_3 - \varphi_1}. \tag{2.7}$$

In (2.5) $\mathrm{sn}(w \mid m)$ is a Jacobian elliptic function, where the notation is as used in [22, Chapter 16]. $\Pi(n; w \mid m)$ is the elliptic integral of the third kind and the notation is as used in [22, Section 17.2.15].

The solution to (2.1) is given in parametric form by (2.5) with w as the parameter. With respect to w, φ in (2.5) is periodic with period $2K(m)$, where $K(m)$ is the complete elliptic integral of the first kind. It follows from (2.5) that the wavelength λ of the solution to (2.1) is

$$\lambda = \frac{2}{\varepsilon p}\left|\varphi_1 K(m) + (\varphi_2 - \varphi_1)\prod(n \mid m)\right|, \tag{2.8}$$

where $\Pi(n \mid m)$ is the complete elliptic integral of the third kind.

When $\varphi_3 = \varphi_4$, $m = 1$, (2.5) becomes

$$\begin{gathered} \varphi = \frac{\varphi_2 - \varphi_1 n \tanh^2 w}{1 - n\tanh^2 w}, \\ \xi = \frac{1}{\varepsilon}\left(\frac{w\varphi_3}{p} - 2\tanh^{-1}\left(\sqrt{n}\tanh w\right)\right). \end{gathered} \tag{2.9}$$

The second form of the solution to (2.4) is found using result 255.00 in [21]. Its parametric form is

$$\begin{gathered} \varphi = \frac{\varphi_3 - \varphi_4 n \mathrm{sn}^2(w \mid m)}{1 - n\mathrm{sn}^2(w \mid m)}, \\ \xi = \frac{1}{\varepsilon p}\left(w\varphi_4 - (\varphi_4 - \varphi_3)\prod(n; w \mid m)\right), \end{gathered} \tag{2.10}$$

where m, p, w are as in (2.6), and

$$n = \frac{\varphi_3 - \varphi_2}{\varphi_4 - \varphi_2}. \tag{2.11}$$

The solution to (2.1) is given in parametric form by (2.10) with w as the parameter. The wavelength λ of the solution to (2.1) is

$$\lambda = \frac{2}{\varepsilon p}\left|\varphi_4 K(m) - (\varphi_4 - \varphi_3)\prod(n \mid m)\right|. \tag{2.12}$$

When $\varphi_1 = \varphi_2$, $m = 1$, (2.10) becomes

$$\varphi = \frac{\varphi_3 - \varphi_4 n \tanh^2 w}{1 - n \tanh^2 w},$$
$$\xi = \frac{1}{\varepsilon}\left(\frac{w\varphi_2}{p} + 2\tanh^{-1}\left(\sqrt{n}\tanh w\right)\right). \tag{2.13}$$

3. Solitary and Periodic Wave Solutions to (1.7)

Equation (1.7) can also be written in the form

$$u_t + 2uu_x - 6u_x u_{xx} - 2uu_{xxx} = 0. \tag{3.1}$$

Let $u = \varphi(\xi) + c$ with $\xi = x - ct$ be a traveling wave solution to (3.1), then it follows that

$$-c\varphi_\xi + 2\varphi\varphi_\xi - 6\varphi_\xi\varphi_{\xi\xi} - 2\varphi\varphi_{\xi\xi\xi} = 0, \tag{3.2}$$

where φ_ξ is the derivative of function φ with respect to ξ.

Integrating (3.2) twice with respect to ξ yields

$$(\varphi\varphi_\xi)^2 = \frac{1}{4}\left(\varphi^4 - \frac{4c}{3}\varphi^3 + a\varphi^2 + b\right), \tag{3.3}$$

where a and b are two arbitrary integration constants.

Equation (3.3) is in the form of (2.1) with $\varepsilon = 1/2$ and $f(\varphi) = (\varphi^4 - (4c/3)\varphi^3 + a\varphi^2 + b)$. For convenience we define $g(\varphi)$ and $h(\varphi)$ by

$$f(\varphi) = \varphi^2 g(\varphi) + b, \quad \text{where } g(\varphi) = \varphi^2 - \frac{4c}{3}\varphi + a,$$
$$f'(\varphi) = 2\varphi h(\varphi), \quad \text{where } h(\varphi) = 2\varphi^2 - 2c\varphi + a, \tag{3.4}$$

and define φ_L, φ_R, b_L, and b_R by

$$\varphi_L = \frac{1}{2}\left(c - \sqrt{c^2 - 2a}\right), \qquad \varphi_R = \frac{1}{2}\left(c + \sqrt{c^2 - 2a}\right),$$
$$b_L = -\varphi_L^2 g(\varphi_L) = \frac{a^2}{4} - \frac{1}{2}c^2 a + \frac{c^4}{6} - \frac{1}{6}\left(c^3 - 2ac\right)\sqrt{c^2 - 2a},$$
$$b_R = -\varphi_L^2 g(\varphi_L) = \frac{a^2}{4} - \frac{1}{2}c^2 a + \frac{c^4}{6} + \frac{1}{6}\left(c^3 - 2ac\right)\sqrt{c^2 - 2a}. \tag{3.5}$$

Obviously, φ_L, φ_R are the roots of $h(\varphi) = 0$.

Without loss of generality, we suppose the wave speed $c > 0$. In the following, suppose that $a < c^2/2$ and $a \neq 0$ for each value $c > 0$, such that $f(\varphi)$ has three distinct stationary points:

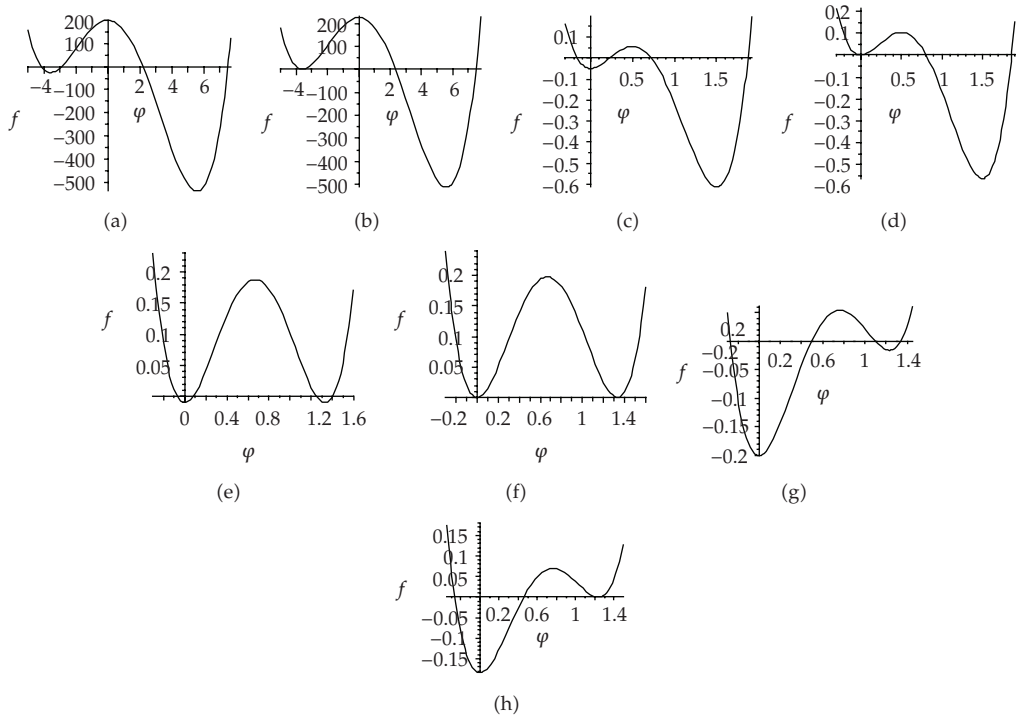

Figure 1: The curve of $f(\varphi)$ for the wave speed $c = 2$. (a) $a = -40$, $b = 200$; (b) $a = -40$, $b = 226.0424$; (c) $a = 1.5$, $b = -0.05$; (d) $a = 1.5$, $b = 0$; (e) $a = 16/9$, $b = -0.1$; (f) $a = 16/9$, $b = 0$; (g) $a = 17/9$, $b = -0.24$; (h) $a = 17/9$, $b = -0.1842$.

φ_L, φ_R, 0 and comprise two minimums separated by a maximum. Under this assumption, (1.7) has periodic and solitary wave solutions that have different analytical forms depending on the values of a and b as follows.

(1) $a < 0$

In this case $\varphi_L < 0 < \varphi_R$ and $f(\varphi_L) > f(\varphi_R)$. For each value $a < 0$ and $0 < b < b_L$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(a)), there are periodic loop-like solutions to (3.3) given by (2.10) so that $0 < m < 1$, and with wavelength given by (2.12). See Figure 2(a) for an example.

The case $a < 0$ and $b = b_L$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(b)) corresponds to the limit $\varphi_1 = \varphi_2 = \varphi_L$ so that $m = 1$, and then the solution is a loop-like solitary wave given by (2.13) with $\varphi_2 \le \varphi < \varphi_R$ and

$$\begin{aligned} \varphi_3 &= \frac{1}{2}\sqrt{c^2 - 2a} + \frac{c}{6} - \frac{1}{3}\sqrt{c^2 + 3c\sqrt{4 - 2a}}, \\ \varphi_4 &= \frac{1}{2}\sqrt{c^2 - 2a} + \frac{c}{6} + \frac{1}{3}\sqrt{c^2 + 3c\sqrt{4 - 2a}}. \end{aligned} \tag{3.6}$$

See Figure 3(a) for an example.

(2) $0 < a < 4c^2/9$

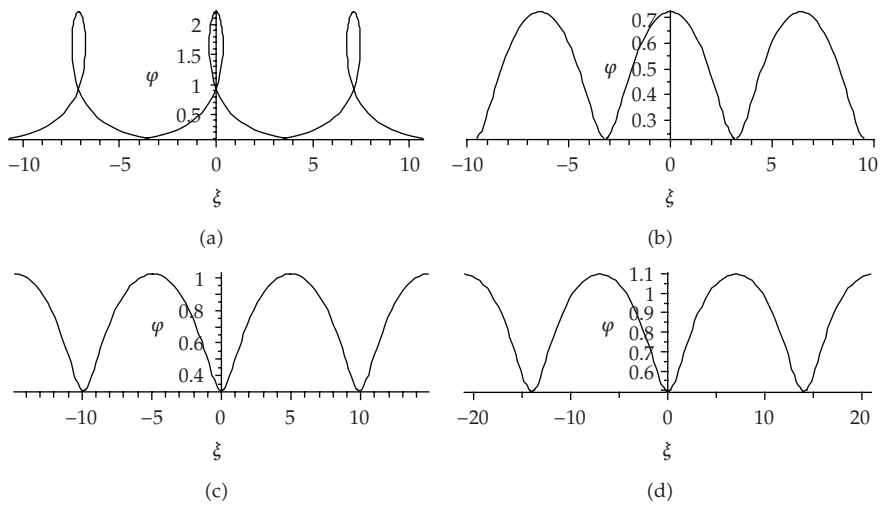

Figure 2: Periodic solutions to (3.3) with $0 < m < 1$ and the wave speed $c = 2$. (a) $a = -40$, $b = 200$ so $m = 0.8978$; (b) $a = 1.5$, $b = -0.05$ so $m = 0.6893$; (c) $a = 16/9$, $b = -0.1$ so $m = 0.8254$; (d) $a = 17/9$, $b = -0.24$ so $m = 0.8412$.

In this case $0 < \varphi_L < \varphi_R$ and $f(\varphi_R) < f(0)$. For each value $0 < a < 4c^2/9$ and $b_L < b < 0$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(c)), there are periodic valley-like solutions to (3.3) given by (2.10) so that $0 < m < 1$, and with wavelength given by (2.12). See Figure 2(b) for an example.

The case $0 < a < 4c^2/9$ and $b = 0$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(d)) corresponds to the limit $\varphi_1 = \varphi_2 = 0$ so that $m = 1$, and then the solution can be given by (2.13) with φ_3 and φ_4 given by the roots of $g(\varphi) = 0$, namely,

$$\varphi_3 = \frac{2c}{3} - \sqrt{\frac{4c^2}{9} - a}, \qquad \varphi_4 = \frac{2c}{3} + \sqrt{\frac{4c^2}{9} - a}. \tag{3.7}$$

In this case we obtain a weak solution, namely, the periodic downward-cusp wave

$$\varphi = \varphi(\xi - 2j\xi_m), \quad (2j-1)\xi_m < \xi < (2j+1)\xi_m, \quad j = 0, \pm 1, \pm 2, \ldots, \tag{3.8}$$

where

$$\begin{gathered} \varphi(\xi) = \left(\varphi_3 - \varphi_4 \tanh^2\left(\frac{\xi}{4}\right)\right)\cosh^2\left(\frac{\xi}{4}\right), \\ \xi_m = 4\tanh^{-1}\sqrt{\frac{\varphi_3}{\varphi_4}}. \end{gathered} \tag{3.9}$$

See Figure 3(b) for an example.

(3) $a = 4c^2/9$

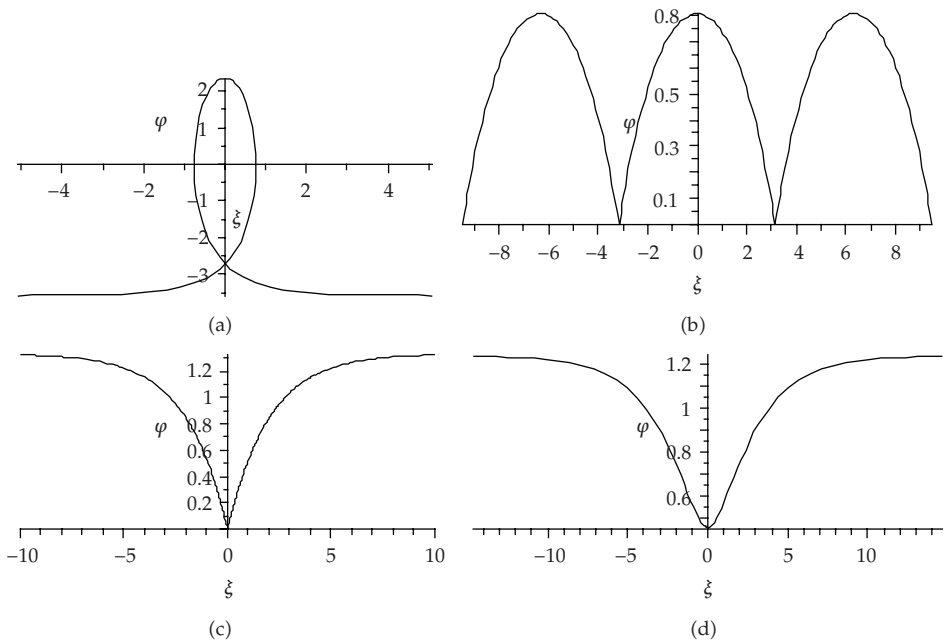

Figure 3: Solutions to (3.3) with $m = 1$ and the wave speed $c = 2$. (a) $a = -40$, $b = 226.0424$; (b) $a = 1.5$, $b = 0$; (c) $a = 16/9$, $b = 0$; (d) $a = 17/9$, $b = -0.1842$.

In this case $0 < \varphi_L < \varphi_R$ and $f(\varphi_R) = f(0)$. For $a = 4c^2/9$ and each value $b_L < b < 0$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(e)), there are periodic valley-like solutions to (3.3) given by (2.5) so that $0 < m < 1$, and with wavelength given by (2.8). See Figure 2(c) for an example.

The case $a = 4c^2/9$ and $b = 0$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(f)) corresponds to the limit $\varphi_3 = \varphi_4 = \varphi_R = 2c/3$ and $\varphi_1 = \varphi_2 = 0$ so that $m = 1$. In this case neither (2.9) nor (2.13) is appropriate. Instead we consider (3.3) with $f(\varphi) = (1/4)\varphi^2(\varphi - (2c/3))^2$ and note that the bound solution has $0 < \varphi < 2c/3$. On integrating (3.3) and setting $\varphi = 0$ at $\xi = 0$ we obtain a weak solution

$$\varphi = -\frac{2c}{3}\exp\left(-\frac{1}{2}|\xi|\right) + \frac{2c}{3}, \tag{3.10}$$

that is, a single valley-like peaked solution with amplitude $2c/3$. See Figure 3(c) for an example.

(4) $4c^2/9 < a < c^2/2$

In this case $0 < \varphi_L < \varphi_R$ and $f(\varphi_R) > f(0)$. For each value $4c^2/9 < a < c^2/2$ and $b_L < b < b_R$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(g)), there are periodic valley-like solutions to (3.3) given by (2.5) so that $0 < m < 1$, and with wavelength given by (2.8). See Figure 2(d) for an example.

The case $4c^2/9 < a < c^2/2$ and $b = b_R$ (a corresponding curve of $f(\varphi)$ is shown in Figure 1(h)) corresponds to the limit $\varphi_3 = \varphi_4 = \varphi_R$ so that $m = 1$, and then the solution is a velley-like solitary wave given by (2.10) with $\varphi_L < \varphi \leq \varphi_3$ and

$$\begin{aligned} \varphi_1 &= \frac{c}{6} - \frac{1}{2}\sqrt{c^2 - 2a} - \frac{1}{3}\sqrt{c^2 - 3c\sqrt{c^2 - 2a}}, \\ \varphi_2 &= \frac{c}{6} - \frac{1}{2}\sqrt{c^2 - 2a} + \frac{1}{3}\sqrt{c^2 - 3c\sqrt{c^2 - 2a}}. \end{aligned} \tag{3.11}$$

See Figure 3(d) for an example.

4. Conclusion

In this paper, we have found expressions for two types of traveling wave solutions to the osmosis $K(2,2)$ equation, that is, the soliton and periodic wave solutions. These solutions depend, in effect, on two parameters a and m. For $m = 1$, there are loop-like ($a < 0$), peakon ($a = 4c^2/9$), and smooth ($4c^2/9 < a < c^2/2$) soliton solutions. For $m = 1, 0 < a < 4c^2/9$ or $0 < m < 1$, $a < c^2/2$, and $a \neq 0$, there are periodic wave solutions.

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