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Research Article

Shannon Wavelets for the Solution of Integrodifferential Equations

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Shannon wavelets are used to define a method for the solution of integrodifferential equations. This method is based on (1) the Galerking method, (2) the Shannon wavelet representation, (3) the decorrelation of the generalized Shannon sampling theorem, and (4) the definition of connection coefficients. The Shannon sampling theorem is considered in a more general approach suitable for analysing functions ranging in multifrequency bands. This generalization coincides with the Shannon wavelet reconstruction of $L_2(\mathbb{R})$ functions. Shannon wavelets are C^{∞}-functions and their any order derivatives can be analytically defined by some kind of a finite hypergeometric series (connection coefficients).

1. Introduction

In recent years wavelets have been successfully applied to the wavelet representation of integro-differential operators, thus giving rise to the so-called wavelet solutions of PDE and integral equations. While wavelet solutions of PDEs can be easily find in a large specific literature, the wavelet representation of integro-differential operators cannot be considered completely achieved and only few papers discuss in depth this question with particular regards to methods for the integral equations. Some of them refer to the Haar wavelets [1–3] to the harmonic wavelets [4–9] and to the spline-Shannon wavelets [10–13]. These methods are mainly based on the Petrov-Galerkin method with a suitable choice of the collocation points [14]. Alternatively to the collocation method, there has been also proposed, for the solution of PDEs, the evaluation of the differential operators on the wavelet basis, thus defining the so-called connection coefficients [6, 15–21].

Wavelets [22] are localized functions which are a useful tool in many different applications: signal analysis, data compression, operator analysis, PDE solving (see, e.g., [15, 23] and references therein), vibration analysis, and solid mechanics [23]. Very often wavelets have been used only as any other kind of orthogonal functions, without taking into consideration their fundamental properties. The main feature of wavelets is, in fact, their possibility to split objects into different scale components [22, 23] according to the multiscale resolution analysis. For the $L_2(\mathbb{R})$ functions, that is, functions with decay to infinity, wavelets give the best approximation. When the function is localized in space, that is, the bottom length of the function is within a short interval (function with a compact support), such as pulses, any other reconstruction, but wavelets, leads towards undesirable problems such as the Gibbs phenomenon when the approximation is made in the Fourier basis. Wavelets are the most expedient basis for the analysis of impulse functions (pulses) [24, 25].

Among the many families of wavelets, Shannon wavelets [17] offer some more specific advantages, which are often missing in the others. In fact, Shannon wavelets

(1) are analytically defined;

(2) are infinitely differentiable;

(3) are sharply bounded in the frequency domain, thus allowing a decomposition of frequencies in narrow bands;

(4) enjoy a generalization of the Shannon sampling theorem, which extend to all range of frequencies [17]

(5) give rise to the connection coefficients which can be analytically defined [15–17] for any order derivatives, while for the other wavelet families they were computed only numerically and only for the lower order derivatives [18, 19, 21].

The (Shannon wavelet) connection coefficients are obtained in [17] as a finite series (for any order derivatives). In Latto's method [18, 20, 21], instead, these coefficients were obtained only (for the Daubechies wavelets) by using the inclusion axiom but in approximated form and only for the first two-order derivatives. The knowledge of the derivatives of the basis enables us to approximate a function and its derivatives and it is an expedient tool for the projection of differential operators in the numerical computation of the solution of both partial and ordinary differential equations [6, 15, 23, 26].

The wavelet reconstruction by using Shannon wavelets is also a fundamental step in the analysis of functions-operators. In fact, due to their definition Shannon wavelets are box functions in the frequency domain, thus allowing a sharp decorrelation of frequencies, which is an important feature in many physical-engineering applications. In fact, the reconstruction by Shannon wavelets ranges in multifrequency bands. Comparing with the Shannon sampling theorem where the frequency band is only one, the reconstruction by Shannon wavelets can be done for functions ranging in all frequency bands (see, e.g., [17]). The Shannon sampling theorem [27], which plays a fundamental role in signal analysis and applications, will be generalized, so that under suitable hypotheses a few set of values (samples) and a preliminary chosen Shannon wavelet basis enable us to completely represent, by the wavelet coefficients, the continuous signal and its frequencies.

The Shannon wavelet solution of an integrodifferential equation (with functions localized in space and slow decay in frequency) will be computed by using the Petrov-Galerkin method and the connection coefficients. The wavelet coefficients enable to represent the solution in the frequency domain singling out the contribution to different frequencies.

This paper is organized as follows. Section 2 deals with some preliminary remarks and properties of Shannon wavelets also in frequency domain; the reconstruction of a function is given in Section 3 together with the generalization of the Shannon sampling theorem;

the error of the wavelet approximation is computed. The wavelet reconstruction of the derivatives of the basis and the connection coefficients are given in Section 4. Section 5 deals with the Shannon wavelet solution of an integrodifferential equation and an example is given at last in Section 6.

2. Shannon Wavelets

Shannon wavelets theory (see, e.g., [16, 17, 28, 29]) is based on the scaling function $\varphi(x)$ (also known as sinc function)

$$\varphi(x) = \operatorname{sinc} x \overset{\text{def}}{=} \frac{\sin \pi x}{\pi x} = \frac{e^{\pi i x} - e^{-\pi i x}}{2\pi i x}, \tag{2.1}$$

and the corresponding wavelet [16, 17, 28, 29]

$$\begin{aligned} \psi(x) &= \frac{\sin \pi(x - 1/2) - \sin 2\pi(x - 1/2)}{\pi(x - 1/2)} \\ &= \frac{e^{-2i\pi x}\left(-i + e^{i\pi x} + e^{3i\pi x} + ie^{4i\pi x}\right)}{(\pi - 2\pi x)}. \end{aligned} \tag{2.2}$$

From these functions a multiscale analysis [22] can be derived. The dilated and translated instances, depending on the scaling parameter n and space shift k, are

$$\begin{aligned} \varphi_k^n(x) &= 2^{n/2}\varphi(2^n x - k) = 2^{n/2}\frac{\sin \pi(2^n x - k)}{\pi(2^n x - k)} \\ &= 2^{n/2}\frac{e^{\pi i(2^n x - k)} - e^{-\pi i(2^n x - k)}}{2\pi i(2^n x - k)}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \psi_k^n(x) &= 2^{n/2}\ \frac{\sin \pi(2^n x - k - 1/2) - \sin 2\pi(2^n x - k - 1/2)}{\pi(2^n x - k - 1/2)} \\ &= \frac{2^{n/2}}{2\pi(2^n x - k + 1/2)}\sum_{s=1}^{2} i^{1+s} e^{s\pi i(2^n x - k)} - i^{1-s} e^{-s\pi i(2^n x - k)} \end{aligned} \tag{2.4}$$

respectively.

2.1. Properties of the Shannon Scaling and Wavelet Functions

By a direct computation it can be easily seen that

$$\varphi_k^0(h) = \delta_{kh}, \quad (h, k \in \mathbb{Z}), \tag{2.5}$$

with δ_{kh} Kroneker symbol, so that

$$\varphi_k^0(x) = 0, \quad x = h \neq k \quad (h, k \in \mathbb{Z}), \tag{2.6}$$

$$\psi_k^n(x) = 0, \quad x = 2^{-n}\left(k + \frac{1}{2} \pm \frac{1}{3}\right), \quad (n \in \mathbb{N}, k \in \mathbb{Z}). \tag{2.7}$$

It is also

$$\lim_{x \to 2^{-n}(h+1/2)} \psi_k^n(x) = -2^{n/2}\delta_{hk}. \tag{2.8}$$

Thus, according to (2.5), (2.8), for each fixed scale n, we can choose a set of points x:

$$x \in \{h\} \cup \left\{2^{-n}\left(h + \frac{1}{2} \pm \frac{1}{3}\right)\right\}, \quad (n \in \mathbb{N}, h \in \mathbb{Z}), \tag{2.9}$$

where either the scaling functions or the wavelet vanishes, but it is important to notice that when the scaling function is zero, the wavelet is not and viceversa. As we shall see later, this property will simplify the numerical methods based on collocation point.

Since they belong to $L_2(\mathbb{R})$, both families of scaling and wavelet functions have a (slow) decay to zero; in fact, according to their definition (2.3), (2.4)

$$\lim_{x \to \pm\infty} \varphi_k^n(x) = 0, \qquad \lim_{x \to \pm\infty} \psi_k^n(x) = 0, \tag{2.10}$$

it can be also easily checked that for a fixed x_0

$$\begin{aligned} &\varphi_{k+1}^n(x_0) < \varphi_k^n(x_0), \quad \frac{\varphi_{k+1}^n(x_0)}{\varphi_k^n(x_0)} = \frac{2^n x - k}{2^n x - k + 1} < 1, \\ &\frac{\psi_{k+1}^n(x_0)}{\psi_k^n(x_0)} = \frac{2^{n+1}x - 2k - 1}{2^{n+1}x - 2k - 3} \times \frac{2\sin(\pi(2^n x - k)) - 1}{2\sin(\pi(2^n x - k)) + 1}. \end{aligned} \tag{2.11}$$

Since

$$\begin{gathered} \lim_{x \to \infty} \frac{2^{n+1}x - 2k - 1}{2^{n+1}x - 2k - 3} = 1, \\ 2\sin(\pi(2^n x - k)) - 1 < 2\sin(\pi(2^n x - k)) + 1, \end{gathered} \tag{2.12}$$

it is

$$\lim_{x \to \infty} \frac{\psi_{k+1}^n(x)}{\psi_k^n(x)} < 1. \tag{2.13}$$

Analogously we have

$$\frac{\psi_k^{n+1}(x_0)}{\psi_k^n(x_0)} = \frac{\sqrt{2}(2^{n+1}x - 2k - 1)}{2^{n+2}x - 2k - 1} \times \frac{\cos(\pi(2^{n+1}x - k)) - \sin(2\pi(2^{n+1}x - k))}{\cos(\pi(2^n x - k)) - \sin(2\pi(2^n x - k))},$$
$$\lim_{x \to 2^{-n}(k+1/2)} \frac{\psi_{k+1}^{n+1}(x)}{\psi_k^n(x)} = \frac{2\sqrt{2}(\cos k\pi - \sin 2k\pi)}{(2k-1)\pi} = \frac{(-1)^k 2\sqrt{2}}{(2k-1)\pi}, \quad \left|\frac{(-1)^k 2\sqrt{2}}{(2k-1)\pi}\right| < 1. \tag{2.14}$$

The maximum and minimum values of these functions can be easily computed. The maximum value of the scaling function $\varphi_k^0(x)$ can be found in correspondence of $x = k$

$$\max\left[\varphi_k^0(x_M)\right] = 1, \quad x_M = k. \tag{2.15}$$

The min value of $\varphi_k^0(x)$ can be computed only numerically and it is

$$\min\left[\varphi_k^0(x)\right] \cong \varphi_k^0(x_m) = \frac{\sin\sqrt{2}\pi}{\sqrt{2}\pi}, \quad x_m = k - 1 \pm \sqrt{2}. \tag{2.16}$$

The minimum of the wavelet $\psi_k^n(x)$ can be found in correspondence of the middle point of the zeroes (2.7) so that

$$\min\left[\psi_k^n(x_m)\right] = -2^{n/2}, \quad x_m = 2^{-n-1}(2k+1), \tag{2.17}$$

and the max values of $\psi_k^n(x)$ are

$$\max\left[\psi_k^n(x_M)\right] = 2^{n/2}\frac{3\sqrt{3}}{\pi}, \quad x_M = \begin{cases} -2^{-n}\left(k + \dfrac{1}{6}\right), \\ \dfrac{2^{-n-1}}{3}(18k+7). \end{cases} \tag{2.18}$$

2.2. Shannon Wavelets Theory in the Fourier Domain

Let

$$\widehat{f}(\omega) = \widehat{f(x)} \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \tag{2.19}$$

be the Fourier transform of the function $f(x) \in L_2(\mathbb{R})$, and

$$f(x) = 2\pi\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}\,d\omega \tag{2.20}$$

its inverse transform.

The Fourier transform of (2.1), (2.2) gives us

$$\widehat{\varphi}(\omega) = \frac{1}{2\pi}\chi(\omega + 3\pi) = \begin{cases} \dfrac{1}{2\pi}, & -\pi \leq \omega < \pi \\ 0, & \text{elsewhere,} \end{cases} \tag{2.21}$$

and [17]

$$\widehat{\psi}(\omega) = \frac{1}{2\pi}e^{-i\omega}\left[\chi(2\omega) + \chi(-2\omega)\right] \tag{2.22}$$

with

$$\chi(\omega) = \begin{cases} 1, & 2\pi \leq \omega < 4\pi, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.23}$$

Analogously for the dilated and translated instances of scaling/wavelet function, in the frequency domain, it is

$$\begin{aligned} \widehat{\varphi}_k^n(\omega) &= \frac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi\left(\frac{\omega}{2^n} + 3\pi\right), \\ \widehat{\psi}_k^n(\omega) &= -\frac{2^{-n/2}}{2\pi}e^{-i\omega(k+1/2)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right) + \chi\left(\frac{-\omega}{2^{n-1}}\right)\right]. \end{aligned} \tag{2.24}$$

It can be seen that

$$\chi(\omega + 3\pi)\left[\chi\left(\frac{\omega}{2^{n-1}}\right) + \chi\left(\frac{-\omega}{2^{n-1}}\right)\right] = 0 \tag{2.25}$$

so that by using the function $\widehat{\varphi}_k^0(\omega)$ and $\widehat{\psi}_k^n(\omega)$ there is a decorrelation into different non-overlapping frequency bands.

For each $f(x) \in L_2(\mathbb{R})$ and $g(x) \in L_2(\mathbb{R})$, the inner product is defined as

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x, \tag{2.26}$$

which, according to the Parseval equality, can be expressed also as

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x = 2\pi\int_{-\infty}^{\infty} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}\mathrm{d}\omega = 2\pi\left\langle \widehat{f}, \widehat{g} \right\rangle, \tag{2.27}$$

where the bar stands for the complex conjugate.

With respect to the inner product (2.26). The following can be shown. [16, 17]

Theorem 2.1. *Shannon wavelets are orthonormal functions, in the sense that*

$$\langle \psi_k^n(x), \psi_h^m(x) \rangle = \delta^{nm}\delta_{hk}, \tag{2.28}$$

With δ^{nm}, δ_{hk} being the Kroenecker symbols.

For the proof see [17]. Moreover we have [16, 17].

Theorem 2.2. *The translated instances of the Shannon scaling functions $\varphi_k^n(x)$, at the level $n = 0$, are orthogonal, in the sense that*

$$\left\langle \varphi_k^0(x), \varphi_h^0(x) \right\rangle = \delta_{kh}, \tag{2.29}$$

being $\varphi_k^0(x) \stackrel{\text{def}}{=} \varphi(x-k)$.

See the proof in [17].

The scalar product of the (Shannon) scaling functions with respect to the corresponding wavelets is characterized by the following [16, 17].

Theorem 2.3. *The translated instances of the Shannon scaling functions $\varphi_k^n(x)$, at the level $n = 0$, are orthogonal to the Shannon wavelets, in the sense that*

$$\left\langle \varphi_k^0(x), \psi_h^m(x) \right\rangle = 0, \quad m \geq 0, \tag{2.30}$$

being $\varphi_k^0(x) \stackrel{\text{def}}{=} \varphi(x-k)$.

Proof is in [17].

3. Reconstruction of a Function by Shannon Wavelets

Let $f(x) \in L_2(\mathbb{R})$ be a function such that for any value of the parameters $n, k \in \mathbb{Z}$, it is

$$\left| \int_{-\infty}^{\infty} f(\mathrm{x})\varphi_k^0(x)\mathrm{d}x \right| \leq A_k < \infty, \qquad \left| \int_{-\infty}^{\infty} f(x)\psi_k^n(x)\mathrm{d}x \right| \leq B_k^n < \infty, \tag{3.1}$$

and $\mathcal{B} \subset L_2(\mathbb{R})$ the Paley-Wiener space, that is, the space of band limited functions, that is,

$$\operatorname{supp} \widehat{f} \subset [-b, b], \quad b < \infty. \tag{3.2}$$

According to the sampling theorem (see, e.g., [27] and references therein) we have the following.

Theorem 3.1 (Shannon). *If* $f(x) \in L_2(\mathbb{R})$ *and* $\operatorname{supp} \widehat{f} \subset [-\pi, \pi]$, *the series*

$$f(x) = \sum_{k=-\infty}^{\infty} \alpha_k \varphi_k^0(x) \tag{3.3}$$

uniformly converges to $f(x)$, *and*

$$\alpha_k = f(k). \tag{3.4}$$

Proof (see also [17]). In order to compute the values of the coefficients we have to evaluate the series in correspondence of the integer:

$$f(h) = \sum_{k=-\infty}^{\infty} \alpha_k \varphi_k^0(h) \overset{(2.5)}{=} \sum_{k=-\infty}^{\infty} \alpha_k \delta_{kh} = \alpha_h, \tag{3.5}$$

having taken into account (2.5).

The convergence follows from the hypotheses on $f(x)$. In particular, the importance of the band limited frequency can be easily seen by applying the Fourier transform to (3.3):

$$\begin{aligned}
\widehat{f}(\omega) &= \sum_{k=-\infty}^{\infty} f(k) \widehat{\varphi}_k^0(x) \\
&\overset{(2.24)}{=} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} f(k) e^{-i\omega k} \chi(\omega + 3\pi) \\
&= \frac{1}{2\pi} \chi(\omega + 3\pi) \sum_{k=-\infty}^{\infty} f(k) e^{-i\omega k}
\end{aligned} \tag{3.6}$$

so that

$$\widehat{f}(\omega) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\sum_{k=-\infty}^{\infty} f(k) e^{-i\omega k}, & \omega \in [-\pi, \pi] \\ 0, & \omega \notin [-\pi, \pi]. \end{cases} \tag{3.7}$$

In other words, if the function is band limited (i.e., with compact support in the frequency domain), it can be completely reconstructed by a discrete Fourier series. The Fourier coefficients are the values of the function $f(x)$ sampled at the integers. □

As a generalization of the Paley-Wiener space, and in order to generalize the Shannon theorem to unbounded intervals, we define the space $\mathcal{B}_{\psi} \supseteq \mathcal{B}$ of functions $f(x)$ such that the integrals

$$\alpha_k \overset{\text{def}}{=} \left\langle f(x), \varphi_k^0(x) \right\rangle \overset{(2.27)}{=} \int_{-\infty}^{\infty} f(x)\overline{\varphi_k^0(x)}\mathrm{d}x,$$
$$\beta_k^n \overset{\text{def}}{=} \langle f(x), \psi_k^n(x) \rangle \overset{(2.27)}{=} \int_{-\infty}^{\infty} f(x)\overline{\psi_k^n(x)}\mathrm{d}x \tag{3.8}$$

exist and are finite. According to (2.26), (2.27), it is in the Fourier domain that

$$\begin{aligned}
\alpha_k &\overset{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\varphi_k^0(x)\mathrm{d}x \overset{(14)}{=} 2\pi\langle \widehat{f(x)}, \widehat{\varphi_k^0(x)} \rangle = 2\pi \int_{-\infty}^{\infty} \widehat{f}(\omega)\overline{\widehat{\varphi}_k^0(\omega)}\mathrm{d}\omega \\
&\overset{(2.24)}{=} 2\pi \int_{-\infty}^{\infty} \widehat{f}(\omega)\frac{1}{2\pi}e^{i\omega k}\chi(\omega + 3\pi)\mathrm{d}\omega \overset{(2.23)}{=} \int_{-\pi}^{\pi} \widehat{f}(\omega)e^{i\omega k}\mathrm{d}\omega, \\
\beta_k^n &\overset{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\psi_k^n(x)\mathrm{d}x \overset{(2.27)}{=} 2\pi\langle \widehat{f(x)}, \widehat{\psi_k^n(x)} \rangle \\
&\overset{(2.24)}{=} -2\pi \int_{-\infty}^{\infty} \widehat{f}(\omega)\frac{2^{-n/2}}{2\pi}e^{i\omega(k+1/2)/2^n}\left[\chi\left(\frac{\omega}{2^{n-1}}\right) + \chi\left(\frac{-\omega}{2^{n-1}}\right)\right]\mathrm{d}\omega \\
&\overset{(2.23)}{=} -2^{-n/2}\left[\int_{2^n\pi}^{2^{n+1}\pi} \widehat{f}(\omega)e^{i\omega(k+1/2)/2^n}\mathrm{d}\omega + \int_{-2^{n+1}\pi}^{-2^n\pi} \widehat{f}(\omega)e^{i\omega(k+1/2)/2^n}\mathrm{d}\omega\right],
\end{aligned} \tag{3.9}$$

so that

$$\alpha_k = \int_{-\pi}^{\pi} \widehat{f}(\omega)e^{i\omega k}\mathrm{d}\omega$$
$$\beta_k^n = -2^{-n/2}\left[\int_{2^n\pi}^{2^{n+1}\pi} \widehat{f}(\omega)e^{i\omega(k+1/2)/2^n}\mathrm{d}\omega + \int_{-2^{n+1}\pi}^{-2^n\pi} \widehat{f}(\omega)e^{i\omega(k+1/2)/2^n}\mathrm{d}\omega\right]. \tag{3.10}$$

For the unbounded interval, let us prove the following.

Theorem 3.2 (Shannon generalized theorem). *If $f(x) \in B_{\psi} \subset L_2(\mathbb{R})$ and* supp $\widehat{f} \subseteq \mathbb{R}$, *the series*

$$f(x) = \sum_{h=-\infty}^{\infty} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x) \tag{3.11}$$

converges to $f(x)$, with α_h and β_k^n given by (3.8) *and* (3.10). *In particular, when* supp $\widehat{f} \subseteq [-2^{N+1}\pi, 2^{N+1}\pi]$, *it is*

$$f(x) = \sum_{h=-\infty}^{\infty} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{N} \sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x). \tag{3.12}$$

Proof. The representation (3.11) follows from the orthogonality of the scaling and Shannon wavelets (Theorems 2.1, 2.2, and 2.3). The coefficients, which exist and are finite, are given by (3.8). The convergence of the series is a consequence of the wavelet axioms. □

It should be noticed that

$$\operatorname{supp} \widehat{f} = [-\pi, \pi] \bigcup_{n=0,\ldots,\infty} \left[-2^{n+1}\pi, -2^{n}\pi\right] \cup \left[2^{n}\pi, 2^{n+1}\pi\right], \tag{3.13}$$

so that for a band limited frequency signal, that is, for a signal whose frequency belongs to the band $[-\pi, \pi]$, this theorem reduces to the Shannon sampling theorem. More in general, the representation (3.11) takes into account more frequencies ranging in different bands. In this case we have some nontrivial contributions to the series coefficients from all bands, ranging from $[-2^N\pi, 2^N\pi]$:

$$\operatorname{supp} \widehat{f} = [-\pi, \pi] \bigcup_{n=0,\ldots,N} \left[-2^{n+1}\pi, -2^{n}\pi\right] \cup \left[2^{n}\pi, 2^{n+1}\pi\right]. \tag{3.14}$$

In the frequency domain, (3.11) gives

$$\begin{aligned}
\widehat{f}(\omega) &= \sum_{h=-\infty}^{\infty} \alpha_h \widehat{\varphi}_h^0(\omega) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \beta_k^n \widehat{\varphi}_k^n(\omega) \\
\widehat{f}(\omega) &\overset{(2.24)}{=} \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \alpha_h e^{-i\omega h} \chi(\omega + 3\pi) \\
&\quad - \frac{1}{2\pi} \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-n/2} \beta_k^n e^{-i\omega(k+1/2)/2^n} \left[\chi\left(\frac{\omega}{2^{n-1}}\right) + \chi\left(\frac{-\omega}{2^{n-1}}\right)\right].
\end{aligned} \tag{3.15}$$

That is,

$$\begin{aligned}
\widehat{f}(\omega) = \frac{1}{2\pi} \chi(\omega + 3\pi) \sum_{h=-\infty}^{\infty} \alpha_h e^{-i\omega h} \\
- \frac{1}{2\pi} \chi\left(\frac{\omega}{2^{n-1}}\right) \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-n/2} \beta_k^n e^{-i\,\omega(k+1/2)/2^n} \\
- \frac{1}{2\pi} \chi\left(\frac{-\omega}{2^{n-1}}\right) \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-n/2} \beta_k^n e^{-i\,\omega(k+1/2)/2^n}.
\end{aligned} \tag{3.16}$$

Moreover, taking into account (2.5), (2.7), we can write (3.11) as

$$f(x) = \sum_{h=-\infty}^{\infty} f(h) \varphi_h^0(x) - \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-n/2} f_n\left(2^{-n}\left(k + \frac{1}{2}\right)\right) \psi_k^n(x) \tag{3.17}$$

with

$$f_n(x) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} \langle f(x), \psi_k^n(x) \rangle \psi_k^n(x). \tag{3.18}$$

3.1. Error of the Shannon Wavelet Approximation

Let us fix an upper bound for the series of (3.11) in a such way that we can only have the approximation

$$f(x) \cong \sum_{h=-K}^{K} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{N} \sum_{k=-S}^{S} \beta_k^n \psi_k^n(x). \tag{3.19}$$

This approximation can be estimated by the following

Theorem 3.3 (Error of the Shannon wavelet approximation). *The error of the approximation* (3.19) *is given by*

$$\begin{aligned} &\left| f(x) - \sum_{h=-K}^{K} \alpha_h \ \varphi_h^0(x) + \sum_{n=0}^{N} \sum_{k=-S}^{S} \beta_k^n \psi_k^n(x) \right| \\ &\quad \le \left| f(-K-1) + f(K+1) - \frac{3\sqrt{3}}{\pi} \left[f\left(2^{-N-1}\left(-S-\frac{1}{2}\right)\right) + f\left(2^{-N-1}\left(S+\frac{3}{2}\right)\right) \right] \right|. \end{aligned} \tag{3.20}$$

Proof. The error of the approximation (3.19) is defined as

$$\begin{aligned} &f(x) - \sum_{h=-K}^{K} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{N} \sum_{k=-S}^{S} \beta_k^n \psi_k^n(x) \\ &\quad = \sum_{h=-\infty}^{-K-1} \alpha_h \ \varphi_h^0(x) + \sum_{h=K+1}^{\infty} \alpha_h \varphi_h^0(x) + \sum_{n=N+1}^{\infty} \left[\sum_{k=-\infty}^{-S-1} \beta_k^n \psi_k^n(x) + \sum_{k=S+1}^{\infty} \beta_k^n \psi_k^n(x) \right]. \end{aligned} \tag{3.21}$$

Concerning the first part of the r.h.s, it is

$$\begin{aligned} \sum_{h=-\infty}^{-K-1} \alpha_h \ \varphi_h^0(x) + \sum_{h=K+1}^{\infty} \alpha_h \varphi_h^0(x) &\le \max_{x \in \mathbb{R}} \left[\sum_{h=-\infty}^{-K-1} \alpha_h \varphi_h^0(x) + \sum_{h=K+1}^{\infty} \alpha_h \varphi_h^0(x) \right] \\ &= \sum_{h=-\infty}^{-K-1} \alpha_h \ \varphi_h^0(h) + \sum_{h=K+1}^{\infty} \alpha_h \ \varphi_h^0(h) \\ &\stackrel{(2.5)}{=} \sum_{h=-\infty}^{-K-1} \alpha_h + \sum_{h=K+1}^{\infty} \alpha_h \stackrel{(3.3)}{=} \sum_{h=-\infty}^{-K-1} f(h) + \sum_{h=K+1}^{\infty} f(h), \end{aligned} \tag{3.22}$$

and since $f(x) \in L_2(\mathbb{R})$ is a decreasing function,

$$\sum_{h=-\infty}^{-K-1} \alpha_h \varphi_h^0(x) + \sum_{h=K+1}^{\infty} \alpha_h \varphi_h^0(x) \leq f(-K-1) + f(K+1). \tag{3.23}$$

Analogously, it is

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \left[\sum_{k=-\infty}^{-S-1} \beta_k^n \psi_k^n(x) + \sum_{k=S+1}^{\infty} \beta_k^n \psi_k^n(x) \right] &\leq \max_{x \in \mathbb{R}} \sum_{n=N+1}^{\infty} \left[\sum_{k=-\infty}^{-S-1} \beta_k^n \psi_k^n(x) + \sum_{k=S+1}^{\infty} \beta_k^n \psi_k^n(x) \right] \\
&\overset{(2.18)}{=} \sum_{n=N+1}^{\infty} \left[\sum_{k=-\infty}^{-S-1} \beta_k^n \psi_k^n \left(\frac{2^{-n-1}(18k+7)}{3} \right) + \sum_{k=S+1}^{\infty} \beta_k^n \psi_k^n \left(\frac{2^{-n-1}(18k+7)}{3} \right) \right] \\
&= \sum_{n=N+1}^{\infty} \left[\sum_{k=-\infty}^{-S-1} \beta_k^n 2^{n/2} \frac{3\sqrt{3}}{\pi} + \sum_{k=S+1}^{\infty} \beta_k^n 2^{n/2} \frac{3\sqrt{3}}{\pi} \right] = \frac{3\sqrt{3}}{\pi} \sum_{n=N+1}^{\infty} 2^{n/2} \left[\sum_{k=-\infty}^{-S-1} \beta_k^n + \sum_{k=S+1}^{\infty} \beta_k^n \right] \\
&\overset{(3.17)}{=} -\frac{3\sqrt{3}}{\pi} \sum_{n=N+1}^{\infty} 2^{n/2} \left[\sum_{k=-\infty}^{-S-1} 2^{-n/2} f\left(2^{-n} \left(k + \frac{1}{2} \right) \right) + \sum_{k=S+1}^{\infty} 2^{-n/2} f\left(2^{-n} \left(k + \frac{1}{2} \right) \right) \right],
\end{aligned} \tag{3.24}$$

so that

$$\sum_{n=N+1}^{\infty} \left[\sum_{k=-\infty}^{-S-1} \beta_k^n \psi_k^n(x) + \sum_{k=S+1}^{\infty} \beta_k^n \psi_k^n(x) \right] \leq -\frac{3\sqrt{3}}{\pi} \left[f\left(2^{-N-1} \left(-S - \frac{1}{2} \right) \right) + f\left(2^{-N-1} \left(S + \frac{3}{2} \right) \right) \right] \tag{3.25}$$

from where (3.20) follows. □

4. Reconstruction of the Derivatives

Let $f(x) \in L_2(\mathbb{R})$ and let $f(x)$ be a differentiable function $f(x) \in C^p$ with p sufficiently high. The reconstruction of a function $f(x)$ given by (3.11) enables us to compute also its derivatives in terms of the wavelet decomposition:

$$\frac{d^\ell}{dx^\ell} f(x) = \sum_{h=-\infty}^{\infty} \alpha_h \frac{d^\ell}{dx^\ell} \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \frac{d^\ell}{dx^\ell} \psi_k^n(x), \tag{4.1}$$

so that, according to (3.11), the derivatives of $f(x)$ are known when the derivatives

$$\frac{d^\ell}{dx^\ell} \varphi_h^0(x), \qquad \frac{d^\ell}{dx^\ell} \psi_k^n(x) \tag{4.2}$$

are given.

Indeed, in order to represent differential operators in wavelet bases, we have to compute the wavelet decomposition of the derivatives:

$$\begin{aligned} \frac{d^\ell}{dx^\ell}\varphi_h^0(x) &= \sum_{k=-\infty}^{\infty} \lambda_{hk}^{(\ell)}\ \varphi_k^0(x), \\ \frac{d^\ell}{dx^\ell}\psi_h^m(x) &= \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} \gamma^{(\ell)}{}_{hk}^{mn}\psi_k^n(x), \end{aligned} \tag{4.3}$$

being

$$\lambda_{kh}^{(\ell)} \overset{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell}\varphi_k^0(x), \varphi_h^0(x) \right\rangle, \qquad \gamma^{(\ell)}{}_{kh}^{nm} \overset{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell}\psi_k^n(x), \psi_h^m(x) \right\rangle \tag{4.4}$$

the connection coefficients [15–21, 26, 29] (or refinable integrals).

Their computation can be easily performed in the Fourier domain, thanks to the equality (2.27). In fact, in the Fourier domain the ℓ-order derivative of the (scaling) wavelet functions is

$$\widehat{\frac{d^\ell}{dx^\ell}\varphi_k^n(x)} = (i\omega)^\ell \widehat{\varphi}_k^n(\omega), \qquad \widehat{\frac{d^\ell}{dx^\ell}\psi_k^n(x)} = (i\omega)^\ell \widehat{\psi}_k^n(\omega), \tag{4.5}$$

and according to (2.24),

$$\begin{aligned} \widehat{\frac{d^\ell}{dx^\ell}\varphi_k^n(x)} &= (i\omega)^\ell \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n} \chi\left(\frac{\omega}{2^n} + 3\pi\right), \\ \widehat{\frac{d^\ell}{dx^\ell}\psi_k^n(x)} &= -(i\omega)^\ell \frac{2^{-n/2}}{2\pi} e^{-i\omega(k+1/2)/2^n} \left[\chi\left(\frac{\omega}{2^{n-1}}\right) + \chi\left(-\frac{\omega}{2^{n-1}}\right)\right]. \end{aligned} \tag{4.6}$$

Taking into account (2.27), we can easily compute the connection coefficients in the frequency domain

$$\lambda_{kh}^{(\ell)} = 2\pi \left\langle \widehat{\frac{d^\ell}{dx^\ell}\varphi_k^0(x)}, \widehat{\varphi_h^0(x)} \right\rangle, \qquad \gamma^{(\ell)}{}_{kh}^{nm} = 2\pi \left\langle \widehat{\frac{d^\ell}{dx^\ell}\psi_k^n(x)}, \widehat{\psi_h^m(x)} \right\rangle \tag{4.7}$$

with the derivatives given by (4.6).

If we define

$$\mu(m) = \operatorname{sign}(m) = \begin{cases} 1, & m > 0, \\ -1, & m < 0, \\ 0, & m = 0, \end{cases} \tag{4.8}$$

the following has been shown [16, 17].

Theorem 4.1. *The any order connection coefficients* $(4.4)_1$ *of the Shannon scaling functions* $\varphi_k^0(x)$ *are*

$$\lambda_{kh}^{(\ell)} = \begin{cases} (-1)^{k-h} \dfrac{i^\ell}{2\pi} \displaystyle\sum_{s=1}^{\ell} \dfrac{\ell! \pi^s}{s![i(k-h)]^{\ell-s+1}} \left[(-1)^s - 1\right], k \neq h, \\ \dfrac{i^\ell \pi^{\ell+1}}{2\pi(\ell+1)} \left[1 + (-1)^\ell\right], k = h, \end{cases} \tag{4.9}$$

or, shortly,

$$\begin{aligned} \lambda_{kh}^{(\ell)} &= \frac{i^\ell \pi^\ell}{2(\ell+1)} \left[1 + (-1)^\ell\right] \left(1 - \left|\mu(k-h)\right|\right) \\ &\quad + (-1)^{k-h} \left|\mu(k-h)\right| \frac{i^\ell}{2\pi} \sum_{s=1}^{\ell} \frac{\ell! \pi^s}{s![i(k-h)]^{\ell-s+1}} \left[(-1)^s - 1\right]. \end{aligned} \tag{4.10}$$

For the proof see [17].

Analogously for the connection coefficients $(4.4)_2$ we have the following.

Theorem 4.2. *The any order connection coefficients* $(4.7)_2$ *of the Shannon scaling wavelets* $\psi_k^n(x)$ *are*

$$\begin{aligned} \gamma^{(\ell)}{}_{kh}^{nm} &= \delta^{nm} \Bigg\{ i^\ell \left(1 - \left|\mu(h-k)\right|\right) \frac{\pi^\ell 2^{n\ell-1}}{\ell+1} \left(2^{\ell+1} - 1\right) \left(1 + (-1)^\ell\right) \\ &\quad + \mu(h-k) \sum_{s=1}^{\ell+1} (-1)^{[1+\mu(h-k)](2\ell-s+1)/2} \frac{\ell! i^{\ell-s} \pi^{\ell-s}}{(\ell-s+1)! |h-k|^s} (-1)^{-s-2(h+k)} 2^{n\ell-s-1} \\ &\quad \times \left\{ 2^{\ell+1} \left[(-1)^{4h+s} + (-1)^{4k+\ell}\right] - 2^s \left[(-1)^{3k+h+\ell} + (-1)^{3h+k+s}\right] \right\} \Bigg\}, \end{aligned} \tag{4.11}$$

respectively, for $\ell \geq 1$, *and* $\gamma^{(0)}{}_{kh}^{nm} = \delta_{kh} \delta^{nm}$.

For the proof see [17].

Theorem 4.3. *The connection coefficients are recursively given by the matrix at the lowest scale level:*

$$\gamma^{(\ell)}{}_{kh}^{nn} = 2^{\ell(n-1)} \gamma^{(\ell)}{}_{kh}^{11}. \tag{4.12}$$

Moreover it is

$$\gamma^{(2\ell+1)}{}_{kh}^{nn} = -\gamma^{(2\ell+1)}{}_{hk}^{nn}, \qquad \gamma^{(2\ell)}{}_{kh}^{nn} = \gamma^{(2\ell)}{}_{hk}^{nn}. \tag{4.13}$$

If we consider a dyadic discretisation of the x-axis such that

$$x_k = 2^{-n} \left(k + \frac{1}{2}\right), \quad k \in \mathbb{Z} \tag{4.14}$$

according to (2.8), the $(4.3)_2$ at dyadic points $x_k = 2^{-n}(k+1/2)$ becomes

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\varphi_k^n(x)\right]_{x=x_k} = -2^{n/2}\sum_{h=-\infty}^{\infty}\gamma'^{nn}_{kh}. \tag{4.15}$$

For instance, in $x_1 = 2^{-1}(1+1/2)$

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\varphi_1^1(x)\right]_{x=x_1=3/4} = -2^{1/2}\sum_{h=-\infty}^{\infty}\gamma_{1h}^{11} \cong -2^{1/2}\sum_{h=-2}^{2}\gamma_{1h}^{11} = -2^{1/2}\left(\frac{1}{6}+\frac{1}{4}\right) = -\frac{5\sqrt{2}}{12}. \tag{4.16}$$

Analogously it is

$$\varphi_k^n\left(2^{-n}\left(k+\frac{1}{2}\right)\right) = \frac{2^{1+n/2}}{\pi}, \quad k\in\mathbb{Z}, \tag{4.17}$$

from where, in $x_k = (k+1/2)$, it is

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\varphi_k^0(x)\right]_{x=x_k} = \frac{2}{\pi}\sum_{h=-\infty}^{\infty}\lambda_{kh}. \tag{4.18}$$

5. Wavelet Solution of the Integrodifferential Equation

Let us consider the following linear integrodifferential equation:

$$A\frac{\mathrm{d}u}{\mathrm{d}x} = B\int_{-\infty}^{\infty}k(x,y)u(y)\mathrm{d}y + u(x) + q(x) \quad (A,B\in\mathbb{R}), \tag{5.1}$$

which includes as special cases the integral equation $(A = 0, B \neq 0)$ and the differential equation $(A\neq 0, B=0)$. When $A = B = 0$, there is the trivial solution $u(x) = -q(x)$.

It is assumed that the kernel is in the form:

$$k(x,y) = f(x)g(y), \tag{5.2}$$

and the given functions $f(x)\in L_2(\mathbb{R})$, $g(x)\in L_2(\mathbb{R})$, $q(x)\in L_2(\mathbb{R})$, so that, according to (3.11)

$$\begin{aligned} f(x) &= \sum_{h=-\infty}^{\infty} f_h\varphi_h^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} f_k^n\psi_k^n(x),\\ g(x) &= \sum_{h=-\infty}^{\infty} g_h\varphi_h^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} g_k^n\psi_k^n(x),\\ q(x) &= \sum_{h=-\infty}^{\infty} q_h\varphi_h^0(x) + \sum_{n=0}^{\infty}\sum_{k=-\infty}^{\infty} q_k^n\psi_k^n(x), \end{aligned} \tag{5.3}$$

with the wavelet coefficients $f_h, f_k^n, g_h, g_k^n, q_h, q_k^n$ given by (3.8).

The analytical solution of (5.1) can be obtained as follows.

Theorem 5.1. *The solution of* (5.1)*, in the degenerate case* (5.2)*, in the Fourier domain is*

$$\hat{u}(\omega) = \frac{2\pi\ B\langle \hat{g}(\omega), \hat{q}(\omega)/(Ai\omega - 1)\rangle}{(1-2\pi B)\left\langle \hat{g}(\omega), \hat{f}(\omega)/(Ai\omega - 1)\right\rangle}\ \frac{\hat{f}(\omega)}{Ai\omega - 1} + \frac{\hat{q}(\omega)}{Ai\omega - 1}. \tag{5.4}$$

Proof. The Fourier transform of (5.1), with kernel as (5.2), is

$$\begin{aligned} A\widehat{\frac{\mathrm{d}u}{\mathrm{d}x}} &= B\widehat{f(x)}\int_{-\infty}^{\infty} g(y)u(y)\mathrm{d}y + \widehat{u(x)} + \widehat{q(x)}, \\ Ai\omega\, \hat{u}(\omega) &= 2\pi B\hat{f}(\omega)\langle \hat{g}(\omega), \hat{u}(\omega)\rangle + \hat{u}(\omega) + \hat{q}(\omega), \\ \hat{u}(\omega) &= 2\pi B\frac{\hat{f}(\omega)}{(Ai\omega - 1)}\langle \hat{g}(\omega), \hat{u}(\omega)\rangle + \frac{\hat{q}(\omega)}{(Ai\omega - 1)}, \end{aligned} \tag{5.5}$$

that is,

$$\hat{u}(\omega) = 2\pi B\frac{\hat{f}(\omega)}{(Ai\omega - 1)}\langle \hat{g}(\omega), \hat{u}(\omega)\rangle + \frac{\hat{q}(\omega)}{(Ai\omega - 1)}. \tag{5.6}$$

By the inner product with $\hat{g}(\omega)$ there follows

$$\langle \hat{g}(\omega), \hat{u}(\omega)\rangle = 2\pi B\left\langle \hat{g}(\omega), \frac{\hat{f}(\omega)}{(Ai\omega - 1)}\right\rangle\langle \hat{g}(\omega), \hat{u}(\omega)\rangle + \left\langle \hat{g}(\omega), \frac{\hat{q}(\omega)}{(Ai\omega - 1)}\right\rangle, \tag{5.7}$$

so that

$$\langle \hat{g}(\omega), \hat{u}(\omega)\rangle = \frac{\langle \hat{g}(\omega), \hat{q}(\omega)/(Ai\omega - 1)\rangle}{(1-2\pi B)\left\langle \hat{g}(\omega), \hat{f}(\omega)/(Ai\omega - 1)\right\rangle}. \tag{5.8}$$

If we put this equation into (5.6), we get (5.4). □

Although the existence of solution is proven, the computation of the Fourier transform could not be easily performed. Therefore the numerical computation is searched in the wavelet approximation.

The wavelet solution of (5.1) can be obtained as follows: it is assumed that the unknown function and its derivative can be written as

$$
\begin{aligned}
u(x) &= \sum_{h=-\infty}^{\infty} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x), \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \sum_{h=-\infty}^{\infty} \alpha_h \frac{\mathrm{d}}{\mathrm{d}x} \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \beta_k^n \psi_k^n(x) \\
&\overset{(4.3)}{=} \sum_{h=-\infty}^{\infty} \alpha_h \sum_{s=-\infty}^{\infty} \lambda'_{hs} \varphi_s^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \sum_{m=0}^{\infty} \sum_{s=-\infty}^{\infty} \gamma'^{nm}_{sk} \psi_s^m(x),
\end{aligned}
\tag{5.9}
$$

and the integral can be written as

$$
\int_{-\infty}^{\infty} g(y) u(y) \mathrm{d}y = \langle g, u \rangle = \sum_{h=-\infty}^{\infty} \alpha_h g_h + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n g_k^n. \tag{5.10}
$$

There follows the system

$$
\begin{aligned}
&\sum_{h=-\infty}^{\infty} \alpha_h \sum_{s=-\infty}^{\infty} \lambda'_{hs} \varphi_s^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \sum_{m=0}^{\infty} \sum_{s=-\infty}^{\infty} \gamma'^{nm}_{sk} \psi_s^m(x) \\
&\quad = \sum_{h=-\infty}^{\infty} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x) \\
&\qquad + \left[\sum_{h=-\infty}^{\infty} \alpha_h g_h + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n g_k^n \right] \left[\sum_{h=-\infty}^{\infty} f_h \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} f_k^n \psi_k^n(x) \right] \\
&\qquad + \sum_{h=-\infty}^{\infty} q_h \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} q_k^n \psi_k^n(x),
\end{aligned}
\tag{5.11}
$$

and, according to the definition of the connection coefficients,

$$
\begin{aligned}
&\sum_{h=-\infty}^{\infty} \alpha_h \sum_{s=-\infty}^{\infty} \lambda'_{hs} \varphi_s^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \beta_k^n \gamma'^{nn}_{sk} \psi_s^n(x) \\
&\quad = \sum_{h=-\infty}^{\infty} \alpha_h \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \psi_k^n(x) \\
&\qquad + \left[\sum_{h=-\infty}^{\infty} \alpha_h g_h + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n g_k^n \right] \left[\sum_{h=-\infty}^{\infty} f_h \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} f_k^n \psi_k^n(x) \right] \\
&\qquad + \sum_{h=-\infty}^{\infty} q_h \varphi_h^0(x) + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} q_k^n \psi_k^n(x).
\end{aligned}
\tag{5.12}
$$

By the inner product and taking into account the orthogonality conditions (Theorems 2.1, 2.2, and 2.3) it is

$$\sum_{h=-\infty}^{\infty} \alpha_h \lambda'_{hk} = \alpha_k + \left[\sum_{h=-\infty}^{\infty} \alpha_h g_h + \sum_{n=0}^{\infty} \sum_{h=-\infty}^{\infty} \beta_h^n g_h^n \right] f_k + q_k, \tag{5.13}$$

or

$$\sum_{h=-\infty}^{\infty} \left(\lambda'_{hk} - \delta_{hk} - g_h f_k \right) \alpha_h = \left[\sum_{n=0}^{\infty} \sum_{h=-\infty}^{\infty} \beta_h^n g_h^n \right] f_k + q_k, \quad (k \in \mathbb{Z}). \tag{5.14}$$

Analogously, it is

$$\sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n \gamma'^{nj}_{kr} = \beta_r^j + \left[\sum_{h=-\infty}^{\infty} \alpha_h g_h + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_k^n g_k^n \right] f_r^j + q_r^j \tag{5.15}$$

or, according to (4.11), and rearranging the indices

$$\sum_{h=-\infty}^{\infty} \beta_h^n \left(\gamma'^{nn}_{hk} - \delta_{hk} \right) - f_k^n \sum_{m=0}^{\infty} \sum_{h=-\infty}^{\infty} \beta_h^m g_h^m = f_k^n \sum_{h=-\infty}^{\infty} \alpha_h g_h + q_k^n. \tag{5.16}$$

Thus the solution of (5.1) is $(5.9)_1$ with the wavelet coefficients given by the algebraic system

$$\begin{gathered} \sum_{h=-\infty}^{\infty} \left(\lambda'_{hk} - \delta_{hk} - g_h f_k \right) \alpha_h = \left[\sum_{n=0}^{\infty} \sum_{h=-\infty}^{\infty} \beta_h^n g_h^n \right] f_k + q_k \quad (k \in \mathbb{Z}), \\ \sum_{h=-\infty}^{\infty} \beta_h^n \left(\gamma'^{nn}_{hk} - \delta_{hk} \right) - f_k^n \sum_{m=0}^{\infty} \sum_{h=-\infty}^{\infty} \beta_h^m g_h^m = f_k^n \sum_{h=-\infty}^{\infty} \alpha_h g_h + q_k^n \quad (n \in \mathbb{N}, k \in \mathbb{Z}) \end{gathered} \tag{5.17}$$

and up to a fixed scale of approximation N, S:

$$\begin{gathered} \sum_{h=-S}^{S} \left(\lambda'_{hk} - \delta_{hk} - g_h f_k \right) \alpha_h = \left[\sum_{n=0}^{N} \sum_{h=-S}^{S} \beta_h^n g_h^n \right] f_k + q_k \quad (k \in \mathbb{Z}), \\ \sum_{h=-S}^{S} \beta_h^n \left(\gamma'^{nn}_{hk} - \delta_{hk} \right) - f_k^n \sum_{m=0}^{N} \sum_{h=-S}^{S} \beta_h^m g_h^m = f_k^n \sum_{h=-N}^{N} \alpha_h g_h + q_k^n \quad (n \in \mathbb{N}, k \in \mathbb{Z}). \end{gathered} \tag{5.18}$$

6. Example

Let us consider the following equation:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \int_{-\infty}^{\infty} e^{-x^2-|y|}u(y)\mathrm{d}y - \frac{x}{|x|}u(x) - e^{-x^2} \tag{6.1}$$

with the condition

$$u(0) = 1. \tag{6.2}$$

The analytical solution, as can be directly checked, is

$$u(x) = e^{-|x|}. \tag{6.3}$$

Since

$$f(x) = e^{-x^2}, \qquad g(x) = e^{-|x|}, \qquad q(x) = -e^{-x^2} \tag{6.4}$$

belong to $L_2(\mathbb{R})$, let us find the wavelet approximation by assuming that also $u(x)$ belongs to $L_2(\mathbb{R})$, so that they can be represented according to (5.3), (5.9).

At the level of approximation $N = 0, S = 0$, from (5.3) we have

$$\begin{gathered} f(x) = e^{-x^2} \cong 0.97\varphi_0^0(x), \qquad g(x) = e^{-|x|} \cong 0.80\varphi_0^0(x) + 0.04\psi_0^0(x), \\ q(x) = -e^{-x^2} \cong -0.97\varphi_0^0(x), \end{gathered} \tag{6.5}$$

so that

$$f_0 = 0.97, \qquad f_0^0 = 0, \qquad g_0 = 0.80, \qquad g_0^0 = 0.04, \qquad q_0 = -0.97, \qquad q_0^0 = 0. \tag{6.6}$$

System (5.18) becomes

$$\begin{gathered} \left(\lambda'_{00} - \delta_{00} - g_0 f_0\right)\alpha_0 = \beta_0^0 g_0^0 f_0 + q_0, \\ \beta_0^0\left(\gamma'^{00}_{00} - \delta_{00}\right) - f_0^0\beta_0^0 g_0^0 = f_0^0\alpha_0 g_0 + q_0^0, \end{gathered} \tag{6.7}$$

and, since $\lambda'_{00} = 0$ and $\gamma'^{00}_{00} = 0$, according to (6.6) we have

$$\begin{gathered} -1 - 0.80 \times 0.97\alpha_0 = -0.97, \\ -\beta_0^0 = 0, \end{gathered} \tag{6.8}$$

whose solution is

$$\alpha_0 = 0.548, \qquad \beta_0^0 = 0, \tag{6.9}$$

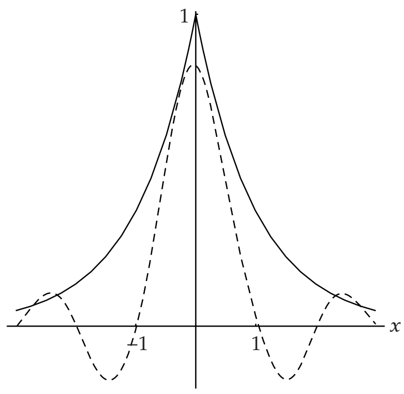

Figure 1: Wavelet approximations (shaded) of the analytical solution (plain) of (6.1) obtained by solving (5.17).

so that

$$u(x) \cong 0.548\varphi_0^0(x). \tag{6.10}$$

As expected, the approximation is very row (Figure 1(a)); in fact in order to get a satisfactory approximation we have to solve system (5.18) at least at the levels $N = 0, S = 5$ as shown in Figure 1(b).

7. Conclusion

In this paper the theory of Shannon wavelets combined with the connection coefficients methods and the Petrov-Galerkin method has been used to find the wavelet approximation of integrodifferential equations. Among the main advantages there is the decorrelation of frequencies, in the sense that the differential operator is splitted into its different frequency bands.

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