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Research Article

The Well-Posedness of the Dirichlet Problem in the Cylindric Domain for the Multidimensional Wave Equation

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In the theory of hyperbolic PDEs, the boundary-value problems with conditions on the entire boundary of the domain serve typically as the examples of the ill-posedness. The paper shows the unique solvability of the Dirichlet problem in the cylindric domain for the multidimensional wave equation. We also establish the criterion for the unique solvability of the equation.

One of the fundamental problems of mathematical physics—the analysis of the behavior of the vibrating string—has been shown to be ill-posed when the boundary-value conditions are defined on the entire boundary ([1]). Furthermore, this problem (known as Dirichlet problem) has been shown to be ill-posed not only for the wave equation but for hyperbolic PDEs more generally (see [2, 3]). Some progress was done in [4] which showed that for some rectangles the solution of this problem existed under sufficient differentiability conditions. Further analyses of this problem reverted to functional analysis methods (see, e.g., [5]), which has the serious shortcoming of making the applications of such results in physics and engineering highly difficult. Moreover, most studies have concentrated so far on the 2D wave equation.

This paper studies the Dirichlet problem, using the classical methods, in the cylindric domain for the multidimensional wave equation. We show that the problem is well-posed. We also establish the criterion for the unique solvability of the problem.

Let Ω_α be the cylindric domain of the Euclidean space E_{m+1} of points $(x_1, \dots, x_m, t)$, bounded by the cylinder $\Gamma = \{(x,t) : |x| = 1\}$, the planes $t = \alpha > 0$ and $t = 0$, where $|x|$ is the length of the vector $x = (x_1, \dots, x_m)$.

Let us denote, respectively, with Γ_α, S_α, and S_0 the parts of these surfaces that form the boundary $\partial\Omega_\alpha$ of the domain Ω_α.

We study, in the domain Ω_α, the multidimensional wave equation

$$\Delta_x u - u_{tt} = 0, \tag{1}$$

where Δ_x is the Laplace operator on the variables $x_1, \dots, x_m$, $m \geq 2$.

Hereafter, it is useful to move from the Cartesian coordinates $x_1, \dots, x_m, t$ to the spherical ones $r, \theta_1, \dots, \theta_m$, $t, r \geq 0$, $0 \leq \theta_1 < 2\pi$, $0 \leq \theta_i \leq \pi$, $i = 2, 3, \dots, m-1$.

Problem 1 (Dirichlet). Find the solution of (1) in the domain Ω_α, in the class $C(\overline{\Omega}_\alpha) \cap C^2(\Omega_\alpha)$, that satisfies the following boundary-value conditions:

$$u|_{S_\alpha} = \varphi(r,\theta), \qquad u|_{\Gamma_\alpha} = \psi(t,\theta), \qquad u|_{S_0} = \tau(r,\theta). \tag{2}$$

Let $\{Y^k_{n,m}(\theta)\}$ be a system of linearly independent spherical functions of order n, $1 \leq k \leq k_n$, $(m-2)!n!k_n = (n+m-3)!(2n+m-2)$, and let $W^l_2(S_0)$, $l = 0, 1, \dots$ be Sobolev spaces.

The following lemmata hold ([6]).

Lemma 1. *Let $f(r,\theta) \in W^l_2(S_0)$. If $l \geq m-1$, then the series*

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} f^k_n(r) Y^k_{n,m}(\theta), \tag{3}$$

as well as the series obtained through its differentiation of order $p \leq l - m + 1$, converge absolutely and uniformly.

Lemma 2. *For $f(r,\theta) \in W^l_2(S_0)$, it is necessary and sufficient that the coefficients of the series* (3) *satisfy the inequalities*

$$\left| f^1_0(r) \right| \leq c_1, \quad \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} n^{2l} \left| f^k_n(r) \right|^2 \leq c_2, \quad c_1, c_2 = \text{const}. \tag{4}$$

Let's denote as $\overline{\varphi}^k_n(r)$, $\psi^k_n(t)$, and $\overline{\tau}^k_n(r)$ the coefficients of the series (3), respectively, of the functions $\varphi(r,\theta)$, $\psi(t,\theta)$, and $\tau(r,\theta)$.

Theorem 3. *If $\varphi(r,\theta) \in W^l_2(S_\alpha)$, $\psi(t,\theta) \in W^l_2(\Gamma_\alpha)$, $\tau(r,\theta) \in W^l_2(S_0)$, $l > 3m/2$, and*

$$\sin \mu_s \alpha \neq 0, \quad s = 1, 2, \dots, \tag{5}$$

then Problem 1 is uniquely solvable, where μ_s are the positive nulls of the Bessel function of first type $J_{n+(m-2)/2}(z)$.

Theorem 4. *The solution of Problem 1 is unique if and only if condition* (5) *is satisfied.*

Proof of Theorem 3. In the spherical coordinates, (1) takes the form

$$u_{rr} + \frac{m-1}{r}u_r - \frac{1}{r^2}\delta u - u_{tt} = 0,$$
$$\delta \equiv -\sum_{j=1}^{m-1}\frac{1}{g_j \sin^{m-j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(\sin^{m-j-1}\theta_j\frac{\partial}{\partial\theta_j}\right), \quad g_1 = 1,\ g_j = \left(\sin\theta_1\cdots\sin\theta_{j-1}\right)^2,\ j > 1. \tag{6}$$

It is known (see [6]) that the spectrum of the operator δ consists of eigenvalues $\lambda_n = n(n+m-2),\ n = 0,1,\ldots,$ to each of which correspond k_n orthonormalized eigenfunctions $Y_{n,m}^k(\theta)$.

Given that solution of the problem that we are looking for belongs to the class $C(\overline{\Omega}_\alpha)\cap C^2(\Omega_\alpha)$, we can look for it in the form of the series

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\overline{u}_n^k(r,t)Y_{n,m}^k(\theta), \tag{7}$$

where $\overline{u}_n^k(r,t)$ are the functions to be determined.

Substituting (7) into (6) and using the orthogonality of the spherical functions $Y_{n,m}^k(\theta)$ ([6]), we get

$$\overline{u}_{nrr}^k + \frac{m-1}{r}\overline{u}_{nr}^k - \overline{u}_{ntt}^k - \frac{\lambda_n}{r^2}\overline{u}_n^k = 0, \quad k = \overline{1,k_n},\ n = 0,1,\ldots, \tag{8}$$

and given this, the boundary-value conditions (2), taking into account Lemma 1, will take the form

$$\overline{u}_n^k(r,0) = \overline{\tau}_n^k(r), \qquad \overline{u}_n^k(r,\alpha) = \overline{\varphi}_n^k(r), \qquad \overline{u}_n^k(1,t) = \psi_n^k(t), \quad k = \overline{1,k_n},\ n = 0,1,\ldots. \tag{9}$$

In (8) and (9), making the substitution of variables

$$\overline{\vartheta}_n^k(r,t) = \overline{u}_n^k(r,t) - \psi_n^k(t), \tag{10}$$

we get

$$\overline{\vartheta}_{nrr}^k + \frac{m-1}{r}\overline{\vartheta}_{nr}^k - \overline{\vartheta}_{ntt}^k - \frac{\lambda_n}{r^2}\overline{\vartheta}_n^k = \overline{f}_n^k(r,t),$$
$$\overline{\vartheta}_n^k(r,0) = \overline{\tau}_n^k(r), \quad \overline{\vartheta}_n^k(r,\alpha) = \varphi_n^k(r), \quad \overline{\vartheta}_n^k(1,t) = 0, \quad k = \overline{1,k_n},\ n = 0,1,\ldots, \tag{11}$$
$$\overline{f}_n^k(r,t) = \psi_{ntt}^k + \frac{\lambda_n}{r^2}\psi_n^k, \qquad \tau_n^k(r) = \overline{\tau}_n^k(r) - \psi_n^k(0), \qquad \varphi_n^k(r) = \overline{\varphi}_n^k(r) - \psi_n^k(\alpha).$$

Making the substitution of the variable $\overline{\vartheta}_n^k(r,t) = r^{(1-m)/2}\vartheta_n^k(r,t)$, we can reduce the problem (11) to the following problem

$$\begin{gathered} L\vartheta_n^k \equiv \vartheta_{nrr}^k - \vartheta_{ntt}^k + \frac{\overline{\lambda}_n}{r^2}\vartheta_n^k = f_n^k(r,t), \\ \vartheta_n^k(r,0) = \tilde{\tau}_n^k(r), \qquad \vartheta_n^k(r,\alpha) = \tilde{\varphi}_n^k(r), \qquad \vartheta_n^k(1,t) = 0, \\ \overline{\lambda}_n = \frac{(m-1)(3-m) - 4\lambda_n}{4}, \qquad f_n^k(r,t) = r^{(1-m)/2}\overline{f}_n^k(r,t), \\ \tilde{\tau}_n^k(r) = r^{(1-m)/2}\tau_n^k(r), \qquad \tilde{\varphi}_n^k(r) = r^{(1-m)/2}\varphi_n^k(r). \end{gathered} \tag{12}$$

We look for the solution of the problem (12) in the form $\vartheta_n^k(r,t) = \vartheta_{1n}^k(r,t) + \vartheta_{2n}^k(r,t)$, where $\vartheta_{1n}^k(r,t)$ is the solution of the problem

$$\begin{gathered} L\vartheta_{1n}^k = f_n^k(r,t), \\ \vartheta_{1n}^k(r,0) = 0, \qquad \vartheta_{1n}^k(r,\alpha) = 0, \qquad \vartheta_{1n}^k(1,t) = 0 \end{gathered} \tag{13}$$

whereas $\vartheta_{2n}^k(r,t)$ is the solution of the problem

$$\begin{gathered} L\vartheta_{2n}^k = 0, \\ \vartheta_{2n}^k(r,0) = \tilde{\tau}_n^k(r), \qquad \vartheta_{2n}^k(r,\alpha) = \tilde{\varphi}_n^k(r), \qquad \vartheta_{2n}^k(1,t) = 0. \end{gathered} \tag{14}$$

We analyze the solutions of the above problems, analogously to [7], in the form

$$\vartheta_n^k(r,t) = \sum_{s=1}^{\infty} R_s(r)T_s(t); \tag{15}$$

moreover, let

$$f_n^k(r,t) = \sum_{s=1}^{\infty} a_s(t)R_s(r), \qquad \tilde{\tau}_n^k(r) = \sum_{s=1}^{\infty} b_s R_s(r), \qquad \tilde{\varphi}_n^k(r) = \sum_{s=1}^{\infty} d_s R_s(r). \tag{16}$$

Substituting (15) into (13) and taking into account (16), we get

$$R_{srr} + \frac{\overline{\lambda}_n}{r^2}R_s + \mu R_s = 0, \quad 0 < r < 1, \tag{17}$$

$$R_s(1) = 0, \qquad |R_s(0)| < \infty, \tag{18}$$

$$T_{stt} + \mu T_s = -a_s(t), \quad 0 < t < \alpha, \tag{19}$$

$$T_s(0) = T_s(\alpha) = 0. \tag{20}$$

The bounded solution of the problems (17) and (18) is (see [8])

$$R_s(r) = \sqrt{r} J_\upsilon(\mu_s r), \tag{21}$$

where $\upsilon = n + (m-2)/2,\ \mu = \mu_s^2$.

The general solution of (19) can be represented in the form (see [8])

$$T_s(t) = c_{1s}\cos\mu_s t + c_{2s}\sin\mu_s t + \frac{\cos\mu_s t}{\mu_s}\int_0^t a_s(\xi)\sin\mu_s\xi\, d\xi - \frac{\sin\mu_s t}{\mu_s}\int_0^t a_s(\xi)\cos\mu_s\xi\, d\xi, \tag{22}$$

where c_{1s} and c_{2s} are arbitrary constants; satisfying the condition (20), we will get

$$\begin{gathered} c_{1s} = 0, \\ c_{2s}\mu_s \sin\mu\alpha = -\cos\mu_s\alpha\int_0^\alpha a_s(\xi)\sin\mu_s\xi\, d\xi - \sin\mu_s\alpha\int_0^\alpha a_s(\xi)\cos\mu_s\xi\, d\xi. \end{gathered} \tag{23}$$

Substituting (21) into (16), we get

$$\begin{gathered} r^{-1/2} f_n^k(r,t) = \sum_{s=1}^{\infty} a_s(t) J_\upsilon(\mu_s r), \qquad r^{-1/2}\tilde{\tau}_n^k(r) = \sum_{s=1}^{\infty} b_s J_\upsilon(\mu_s r), \\ r^{-1/2}\tilde{\varphi}_n^k(r) = \sum_{s=1}^{\infty} d_s J_\upsilon(\mu_s r), \quad 0 < r < 1. \end{gathered} \tag{24}$$

Series (24) are the decompositions into the Fourier-Bessel series (see [9]), if

$$a_s(t) = \frac{2}{\left[J_{\upsilon+1}(\mu_s)\right]^2}\int_0^1 \sqrt{\xi} f_n^k(\xi,t) J_\upsilon(\mu_s\xi)\, d\xi, \tag{25}$$

$$b_s = \frac{2}{\left[J_{\upsilon+1}(\mu_s)\right]^2}\int_0^1 \sqrt{\xi}\tilde{\tau}_n^k(\xi) J_\upsilon(\mu_s\xi)\, d\xi, \qquad d_s = \frac{2}{\left[J_{\upsilon+1}(\mu_s)\right]^2}\int_0^1 \sqrt{\xi}\tilde{\varphi}_n^k(\xi) J_\upsilon(\mu_s\xi)\, d\xi, \tag{26}$$

$\mu_s, s = 1, 2, \ldots$ are positive nulls of the Bessel functions, set in the increasing order.

From (21)–(23) we get the solution of the problem (13):

$$\begin{aligned} \vartheta_{1n}^k(r,t) = \sum_{s=1}^{\infty}\frac{\sqrt{r}}{\mu_s}\Bigg\{ &\left[\int_0^\alpha a_s(\xi)\cos\mu_s\xi d\xi - \cot\mu_s\alpha\int_0^\alpha a_s(\xi)\sin\mu_s d\xi\right]\sin\mu_s t \\ &+ \cos\mu_s t\int_0^t a_s(\xi)\sin\mu_s\xi d\xi - \sin\mu_s t\int_0^t a_s(\xi)\cos\mu_s d\xi \Bigg\} J_\upsilon(\mu_s r), \end{aligned} \tag{27}$$

where $a_s(t)$ is determined from (25).

Next, substituting (15) into (14) and taking into account (16), we will get

$$T_{stt} + \mu_s^2 T_s = 0, \quad 0 < t < \alpha, \tag{28}$$

$$T_s(0) = b_s, \qquad T_s(\alpha) = d_s. \tag{29}$$

The general solution of (28) will become

$$T_s(t) = c'_{1s} \cos \mu_s t + c'_{2s} \sin \mu_s t; \tag{30}$$

satisfying the condition (29), we will get

$$\begin{aligned} c'_{1s} &= b_s, \\ c'_{2s} &= \frac{d_s}{\sin \mu_s \alpha} - b_s \cot \mu_s \alpha. \end{aligned} \tag{31}$$

From (21), (30), and (31) we find the solution of the problem (14):

$$\vartheta_{2n}^k(r,t) = \sum_{s=1}^{\infty} \sqrt{r} \left[b_s \cos \mu_s t - \left(\frac{d_s}{\sin \mu_s \alpha} - b_s \cot \mu_s \alpha \right) \sin \mu_s t \right] J_\upsilon(\mu_s r), \tag{32}$$

where b_s and d_s are found from (26).

Thus, the unique solution of Problem 1 is the function

$$u(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} \left\{ \varphi_n^k(t) + r^{(1-m)/2} \left[\vartheta_{1n}^k(r,t) + \vartheta_{2n}^k(r,t) \right] \right\} Y_{n,m}^k(\theta), \quad t > 0, \tag{33}$$

where $\vartheta_{1n}^k(r,t)$ and $\vartheta_{2n}^k(r,t)$ are determined from (27) and (32).

Taking into account the formula (see [9]) $J'_\upsilon(z) = J_{\upsilon-1}(z) + J_{\upsilon+1}(z)$, the estimates (see [6, 9])

$$\begin{aligned} |J_\upsilon(z)| &\le \frac{1}{\Gamma(1+\upsilon)} \left(\frac{z}{2} \right)^\upsilon, \qquad k_n \le c_1 n^{m-2}, \\ \left| \frac{\partial^q}{\partial \theta_j^q} Y_{n,m}^k(\theta) \right| &\le c_2 n^{m/2-1+q}, \qquad j = \overline{1, m-1}, \ q = 0, 1, \ldots, \end{aligned} \tag{34}$$

where $\Gamma(z)$ is the gamma-function, the lemmata, and the bounds on the given functions $\varphi(r,\theta), \psi(t,\theta)$, and $\tau(r,\theta)$, we can show that the obtained solution (33) belongs to the class $C(\overline{\Omega}_\alpha) \cap C^2(\Omega_\alpha)$.

Theorem 3 is proven. □

Proof of Theorem 4. If condition (5) is satisfied, then from Theorem 3, it follows that the solution of Problem 1 is unique.

Now, suppose condition (5) does not hold, at least for one $s = 1$.

Then, if we look for the solution of the homogeneous problem, corresponding to Problem 1, in the form (7), then we get to the problem

$$L\vartheta_n^k = 0,$$
$$\vartheta_n^k(r,0) = 0, \qquad \vartheta_n^k(r,\alpha) = 0, \qquad \vartheta_n^k(1,t) = 0, \quad k = \overline{1,k_n}, \; n = 0,1,\ldots, \tag{35}$$

the solution of which is the function

$$\vartheta_n^k(r,t) = \sqrt{r}\sin\mu_l t J_{n+(m-2)/2}(\mu_l r). \tag{36}$$

Therefore, the nontrivial solution of homogeneous Problem 1 is written as

$$u(r,\theta,t) = \sum_{n=2}^{\infty}\sum_{k=1}^{k_n} n^{-l} r^{(2-m)/2}\sin\mu_l t J_{n+(m-2)/2}(\mu_l r) Y_{n,m}^k(\theta). \tag{37}$$

From estimates (34) it follows that $u \in C(\overline{\Omega}_\alpha) \cap C^2(\Omega_\alpha)$, if $l > 3m/2$. □

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