Research Article

The Boundary Crossing Theorem and the Maximal Stability Interval

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The boundary crossing theorem and the zero exclusion principle are very useful tools in the study of the stability of family of polynomials. Although both of these theorem seem intuitively obvious, they can be used for proving important results. In this paper, we give generalizations of these two theorems and we apply such generalizations for finding the maximal stability interval.

1. Introduction

Consider the linear system

$$\dot{x} = Ax, \tag{1.1}$$

where A is an $n \times n$ matrix with constant coefficients and $x \in \mathbb{R}^n$. It is known that if the characteristic polynomial, $p_A(t)$ is a Hurwitz polynomial, that is, all of its roots have negative real part, then the origin is an asymptoticly stable equilibrium. Great amount of information has been published about these polynomials since Maxwell proposed the problem of finding conditions for verifying if a given polynomial has all of its roots with negative real part [1]. At first the researchers focused in the problem proposed for Maxwell and the Routh-Hurwitz criterion, the Hermite-Biehler theorem and other criteria were obtained. For reading the proofs of these results the books [2–4] can be consulted. After the scientists began to study other related problems, for instance, the problem of giving conditions for a given family

of polynomials consist of Hurwitz polynomials alone. This problem has its motivation in the applications because when a physical phenomenon is modeled, families of polynomials must be considered due the presence of uncertainties. Maybe the most famous results about families of Hurwitz polynomials are the Kharitonov's theorem [5] and the Edge theorem [6] which consider interval polynomials and polytopes of polynomials, respectively. However, other families have been studied also, for example, the cones and rays of polynomails (see [7, 8]) or the segments of polynomails ([9–11]). Besides in the study of Hurwitz polynomials, topological approaches have been used recently (see [12, 13]). Good references about Hurwitz polynomilas which were reported during 1987–1991 can be found in [14]. The books [2, 15, 16] are very recommendable works for consulting questions about families of stable polynomilas. Many proofs of results about the stability of families of polynomials are based in the boundary crossing theorem [17] and in the zero exclusion principle (see [2] for a proof). In this way the importance of these two results has been appreciated. Now, in this paper we give generalizations as the boundary crossing theorem as the zero exclusion principle and we apply these generalizations for giving an alternative method to calculate the maximal interval for robust stability, which was studied by Białas. We will explain the differences between both methods.

2. Main Results

We have divided this section in two parts: in the first subsection we present generalizations of the mentioned theorems and in the second subsection we apply such generalizations for calculating the maximal stability interval for robust stability.

2.1. Generalizations of the Boundary Crossing Theorem and Zero Exclusion Principle

First, we give the known boundary crossing theorem. Consider a family of polynomials $P(\lambda, t)$ satisfying the following assumption.

Assumption 2.1. $P(\lambda, t)$ *is a family of polynomials of*

   (1) fixed degree n,

   (2) coefficients are continuous respect λ in a fixed interval $I = [a, b]$.

   Also let us to consider the complex plane $\mathbb{C}$ and let $\mathcal{S} \subset \mathbb{C}$ be any given open set and denote the boundary of $\mathcal{S}$ by $\partial \mathcal{S}$ and the complement as $\mathcal{U} = \mathbb{C} - \mathcal{S}$. The following results are presented in [2, page 34].

Theorem 2.2 (boundary crossing theorem). *Under Assumption 2.1, if $P(a, t)$ has all its roots in $\mathcal{S}$ whereas $P(b, t)$ has at least one root in $\mathcal{U}$. Then there exists at least one ρ in $(a, b]$ such that*

   (i) $P(\rho, t)$ *has all its roots in $\mathcal{S} \cup \partial \mathcal{S}$,*

   (ii) $P(\rho, t)$ *has at least one root in $\partial \mathcal{S}$.*

Now suppose that $\delta(t, p)$ denotes a polynomial whose coefficients depend continuously on the parameter vector $p \in \mathbb{R}^l$ which varies in a set $\Omega \subset \mathbb{R}^l$ and thus generates the family of polynomials

$$\Delta(t) := \{\delta(t, p) \mid p \in \Omega\}. \tag{2.1}$$

Theorem 2.3 (zero exclusion principle). *Assume that the polynomial family* (2.1) *has constant degree, contains at least one stable polynomial, and Ω is pathwise connected. Then the entire family is stable if and only if*

$$0 \notin \Delta(t^*), \quad \forall\, t^* \in \partial\mathcal{S}. \tag{2.2}$$

Now we present our generalizations. In them we will consider $\mathcal{S} = \mathbb{C}^-$.

Theorem 2.4 (generalization 1 of Theorem 2.2). *Under Assumption 2.1, suppose that $P(a, t)$ has n_1 roots in $\mathbb{C}^-$ and $n - n_1$ roots in $\mathbb{C}^+$, and $P(b, t)$ has at most $n_1 - 1$ roots in $\mathbb{C}^-$ and at least $n - n_1 + 1$ roots in $\mathbb{C}^+$. Then there exists at least one ρ in $(a, b]$ such that*

    (i) $P(\rho, t)$ *has at least n_1 roots in $\mathbb{C}^- \cup i\mathbb{R}$,*

    (ii) $P(\rho, t)$ *has at least one root in $i\mathbb{R}$.*

Theorem 2.5 (generalization 2 of Theorem 2.2). *Under Assumption 2.1, suppose that $P(a, t)$ has n_1 roots in $\mathbb{C}^-$ and $n - n_1$ roots in $\mathbb{C}^+$, and $P(b, t)$ has m_1 roots in $\mathbb{C}^-$ and $n - m_1$ roots in $\mathbb{C}^+$. If $n_1 \neq m_1$, Then there exists at least one ρ in $(a, b]$ such that*

    (i) $P(\rho, t)$ *has at least n_1 roots in $\mathbb{C}^- \cup i\mathbb{R}$,*

    (ii) $P(\rho, t)$ *has at least $n - n_1$ roots in $\mathbb{C}^+ \cup i\mathbb{R}$,*

    (iii) $P(\rho, t)$ *has at least one root in $i\mathbb{R}$.*

Theorem 2.6 (generalization of Theorem 2.3). *Consider the polynomial family $P(\lambda, t)$ with constant degree, where $\lambda \in \Omega$ and $\Omega \subset \mathbb{R}^l$ is a pathwise connected set. Suppose there exists an element of the family with n_1 roots in $\mathbb{C}^-$ and $n - n_1$ roots in $\mathbb{C}^+$. Then the entire family still having n_1 roots in $\mathbb{C}^-$ and $n - n_1$ roots in $\mathbb{C}^+$ if and only if*

$$P(\lambda, i\omega) \neq 0 \quad \forall \lambda \in \Omega, \, \forall \omega \in \mathbb{R}. \tag{2.3}$$

2.2. Application: An Alternative Method for Calculating the Maximal Stability Interval

We begin this subsection with three important definitions that can be seen in pages 50 to 51 of [16].

Definition 2.7. Consider the uncertain polynomial $P(t, k) = p_0(t) + kp_1(t)$ with $p_0(t)$ assumed Hurwitz stable and the uncertainty bounding set $K = [k^-, k^+]$ with $k^- \leq 0$ and $k^+ \geq 0$. We define the subfamilies

$$
\begin{aligned}
\mathcal{P}(k^+) &= \{p(\cdot, k) \mid 0 \leq k \leq k^+\}, \\
\mathcal{P}(k^-) &= \{p(\cdot, k) \mid k^- \leq k \leq 0\}.
\end{aligned}
\tag{2.4}
$$

Definition 2.8 (maximal stability interval). Associated with the subfamily $\mathcal{P}(k^+)$ is the right-sided robustness margin

$$
k_{\max}^+ = \sup\{k^+ : \mathcal{P}(k^+) \text{ is robustly stable}\},
\tag{2.5}
$$

and asociated with the subfamily $\mathcal{P}(k^-)$ is the left-sided robustness margin

$$
k_{\min}^- = \inf\{k^- : \mathcal{P}(k^-) \text{ is robustly stable}\}.
\tag{2.6}
$$

Subsequently we call $K_{\max} = (k_{\min}^-, k_{\max}^+)$ the maximal interval for robust stability.

Definition 2.9. Given an $n \times n$ matrix M, we define $\lambda_{\max}^+(M)$ to be the maximum positive real eigenvalue of M. When M does not have any positive real eigenvalue, we take $\lambda_{\max}^+(M) = 0^+$. Similarly, we define $\lambda_{\min}^-(M)$ to be the minimum negative real eigenvalue of M. When M does no have any negative real eigenvalue, we take $\lambda_{\min}^-(M) = 0^-$.

Białas [10] proved the following theorem.

Theorem 2.10 (eigenvalue criterion). *Consider the uncertain polynomial $P(t, k) = p_0(t) + kp_1(t)$ with $P(t, 0) = p_0(t)$ Hurwitz stable and having positive coefficients and $\deg p_0(t) > \deg p_1(t)$. Then the maximal interval for robust stability is described by*

$$
\begin{aligned}
k_{\max}^+ &= \frac{1}{\lambda_{\max}^+(-H^{-1}(p_0)H(p_1))}, \\
k_{\min}^- &= \frac{1}{\lambda_{\min}^-(-H^{-1}(p_0)H(p_1))},
\end{aligned}
\tag{2.7}
$$

where $H(p_0)$ and $H(p_1)$ are the corresponding matrices of Hurwitz of $p_0(t)$ and $p_1(t)$, respectively, and for purpose of conformability of matrix multiplication, $H(p_1)$ is an $n \times n$ matrix obtained by treating $p_1(t)$ as an n-degree polynomial.

Now we will explain our approach for calculating the maximal stability interval. Let $p_0(t)$ be an n-degree polynomial and $p_1(t)$ a polynomial such that $n > \deg p_1(t)$. Consider the family of polynomials $p_c(t) = p_0(t) + kp_1(t)$. By evaluating $p_0(-t)$ and $p_c(t)$ in $i\omega$ we get

$$
\begin{aligned}
p_0(-i\omega) &= P\left(\omega^2\right) - i\omega Q\left(\omega^2\right), \\
p_c(i\omega) &= P\left(\omega^2\right) + kp\left(\omega^2\right) + i\omega\left[Q\left(\omega^2\right) + kq\left(\omega^2\right)\right],
\end{aligned}
\tag{2.8}
$$

where p, q, P, Q are polynomials. Then

$$
\begin{aligned}
p_c(i\omega)p_0(-i\omega) = {} & P^2\left(\omega^2\right) + \omega^2 Q^2\left(\omega^2\right) \\
& + k\left[p\left(\omega^2\right)P\left(\omega^2\right) + \omega^2 q\left(\omega^2\right)Q\left(\omega^2\right)\right] \\
& + ik\omega\left[q\left(\omega^2\right)P\left(\omega^2\right) - p\left(\omega^2\right)Q\left(\omega^2\right)\right].
\end{aligned}
\tag{2.9}
$$

Define the polynomials

$$
\begin{aligned}
F(\omega) &= p\left(\omega^2\right)P\left(\omega^2\right) + \omega^2 q\left(\omega^2\right)Q\left(\omega^2\right), \\
G(\omega) &= P^2\left(\omega^2\right) + \omega^2 Q^2\left(\omega^2\right), \\
H(\omega) &= q\left(\omega^2\right)P\left(\omega^2\right) - p\left(\omega^2\right)Q\left(\omega^2\right).
\end{aligned}
\tag{2.10}
$$

Therefore, we can rewrite $p_c(i\omega)p_0(-i\omega)$ as $p_c(i\omega)p_0(-i\omega) = G(\omega) + kF(\omega) + ik\omega H(\omega)$.

Definition 2.11. For an arbitrary polynomial $f(t)$ we define the set of its roots as

$$
R(f) = \{\zeta \in \mathbb{C} \mid f(\zeta) = 0\}.
\tag{2.11}
$$

Let $R(f)_{\mathbb{R}^+}$ denote the set of positive real elements of $R(f)$. It is clear that $R(f)_{\mathbb{R}^+}$ could be an empty set.

Now let $F(\omega)$, $G(\omega)$, and $H(\omega)$ be the polynomials defined above. Define the sets

$$
\begin{aligned}
K^+ &= \{F(\omega_l) \mid \omega_l \in R(H)_{\mathbb{R}^+} \cup \{0\},\ F(\omega_l) > 0\}, \\
K^- &= \{F(\omega_l) \mid \omega_l \in R(H)_{\mathbb{R}^+} \cup \{0\},\ F(\omega_l) < 0\}.
\end{aligned}
\tag{2.12}
$$

If there is no elements in $R(H)_{\mathbb{R}^+} \cup \{0\}$ such that $F(\omega_l) > 0$ then we will define $K^+ = \{0^+\}$. Similarly, if there is no elements in $R(H)_{\mathbb{R}^+} \cup \{0\}$ such that $F(\omega_l) < 0$ then define $K^- = \{0^-\}$. Note that only can happen either $K^- = \{0^-\}$ or $K^+ = \{0^+\}$ but both at the same time never since we can always evaluate in $\omega = 0$. That is, we always have an extreme.

Therefore we have the following alternative method for calculating the maximal stability interval.

Theorem 2.12. *Consider the polynomial family $p_c(t) = p_0(t) + kp_1(t)$ with $p_0(t)$ Hurwitz stable and having positive coefficients, and let $F(\omega), G(\omega)$, and $H(\omega)$ be the polynomials defined above. Then the maximal interval of stability for $p_c(t)$ is described by*

$$
\begin{aligned}
k_{\min}^- &= \max\left\{-\frac{G(\omega_l)}{F(\omega_l)} \mid F(\omega_l) \in K^+\right\}, \\
k_{\max}^+ &= \min\left\{-\frac{G(\omega_l)}{F(\omega_l)} \mid F(\omega_l) \in K^-\right\}.
\end{aligned}
\tag{2.13}
$$

Remark 2.13. A difference with the Biała s method is that in our approach it is not necessary to calculate the inverse of any matrix. Other difference is that in the Biała s method the roots of an n-degree polynomial must be found while in our approach we have that if degree of both (n and m, resp., $n > m$) $p_0(t)$ and $p_1(t)$ is either even or odd then $\deg H(\omega) = n + m - 2$ and in the other cases $\deg H(\omega) = n + m - 1$. Therefore, by symmetry of $H(\omega)$ we have to find the roots of a polynomial with degree $(n + m - 2)/2$ or $(n + m - 1)/2$ both less than or equal to $n - 1$.

Example 2.14. Consider the polynomial $p_0(t) = t^3 + 6t^2 + 12t + 6$ and $p_1(t) = t^2 - 2t + 1$, for the polynomial family $p(t,k) = p_0(t) + kp_1(t)$. We will verify the maximal stability interval by Biała s method. First we have

$$H(p_0(t)) = \begin{pmatrix} 6 & 6 & 0 \\ 1 & 12 & 0 \\ 0 & 6 & 6 \end{pmatrix}, \qquad H(p_1(t)) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}. \tag{2.14}$$

Next,

$$
\begin{aligned}
-H(p_0(t))^{-1}H(p_1(t)) &= -\begin{pmatrix} \dfrac{2}{11} & -\dfrac{1}{11} & 0 \\ -\dfrac{1}{66} & \dfrac{1}{11} & 0 \\ \dfrac{1}{66} & -\dfrac{1}{11} & \dfrac{1}{6} \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\[2mm]
&= \begin{pmatrix} -\dfrac{2}{11} & -\dfrac{4}{11} & 0 \\ \dfrac{1}{66} & \dfrac{13}{66} & 0 \\ -\dfrac{1}{66} & -\dfrac{4}{11} & -\dfrac{1}{6} \end{pmatrix},
\end{aligned}
\tag{2.15}
$$

whose characteristic polynomial is $\lambda^3 - (2/11)\lambda^2 - (1/36)\lambda + (1/198)$ which is a 3-degree polynomial and has as roots $\lambda_1 = \lambda_2 = -1/6$ and $\lambda_3 = 2/11$. Thus $\lambda_{\max}^+(H(p_0)^{-1}H(p_1)) = 2/11$ and $\lambda_{\min}^-(H(p_0)^{-1}H(p_1)) = -1/6$. Then $p(t,k) = p_0(t) + kp_1(t)$ is robustly stable in $[k_{\min}^-, k_{\max}^+] = [-6, 11/2]$.

Now with our proposed method. We see

$$
\begin{aligned}
p_0(i\omega) &= \left(6 - 6\omega^2\right) + i\omega\left(12 - \omega^2\right) \\
&= P\left(\omega^2\right) + i\omega Q\left(\omega^2\right), \\
p_1(i\omega) &= \left(1 - \omega^2\right) + i\omega(-2) \\
&= p\left(\omega^2\right) + i\omega q\left(\omega^2\right).
\end{aligned}
\tag{2.16}
$$

Thus,

$$F(\omega) = 8\omega^4 - 36\omega^2 + 6,$$
$$G(\omega) = \omega^6 + 12\omega^4 + 72\omega^2 + 36, \tag{2.17}$$
$$H(\omega) = -\omega^4 + 25\omega^2 - 24.$$

Thus $R(H)_{\mathbb{R}^+} = \{1, \sqrt{24}\}$. Now,

$$F(1) = -22 < 0,$$
$$F\left(\sqrt{24}\right) = 3750 > 0,$$
$$F(0) = 6 > 0,$$
$$G(1) = 121, \tag{2.18}$$
$$G\left(\sqrt{24}\right) = 22527,$$
$$G(0) = 36.$$

Therefore

$$k_{\max}^+ = -\frac{G(1)}{F(1)} = \frac{11}{2},$$
$$k_{\min}^- = \max\left\{ -\frac{G\left(\sqrt{24}\right)}{F\left(\sqrt{24}\right)}, -\frac{G(0)}{F(0)} \right\} = -6. \tag{2.19}$$

Note that we just only need to compute a 2-degree polynomial root in our method, while in Białas one it is a 3-degree polynomial.

Example 2.15. Consider the linear control system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -7 & -2 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} (-3k, -2k, -k, 0)x. \tag{2.20}$$

From this system we have that $p_0(t) = t^4 + t^3 + 7t^2 + 2t + 3$ and $p_1(t) = t^2 + 2t + 3$ and their Hurwitz matrices are

$$
H(p_0(t)) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 7 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 7 & 3 \end{pmatrix}, \qquad
H(p_1(t)) = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \tag{2.21}
$$

where the inverse of $H(p_0)$ is

$$
H(p_0(t))^{-1} = \begin{pmatrix} \dfrac{11}{2} & -\dfrac{4}{7} & \dfrac{6}{7} & 0 \\[2mm] -\dfrac{2}{7} & \dfrac{2}{7} & -\dfrac{3}{7} & 0 \\[2mm] \dfrac{1}{7} & -\dfrac{1}{7} & \dfrac{5}{7} & 0 \\[2mm] -\dfrac{5}{21} & \dfrac{5}{21} & -\dfrac{32}{21} & \dfrac{1}{3} \end{pmatrix}. \tag{2.22}
$$

Hence, the 4-degree characteristic polynomial of the matrix

$$
-H(p_0(t))^{-1}H(p_1(t)) = -\begin{pmatrix} \dfrac{11}{2} & -\dfrac{4}{7} & \dfrac{6}{7} & 0 \\[2mm] -\dfrac{2}{7} & \dfrac{2}{7} & -\dfrac{3}{7} & 0 \\[2mm] \dfrac{1}{7} & -\dfrac{1}{7} & \dfrac{5}{7} & 0 \\[2mm] -\dfrac{5}{21} & \dfrac{5}{21} & -\dfrac{32}{21} & \dfrac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & -\dfrac{18}{7} & -\dfrac{24}{7} & 0 \\[2mm] 0 & \dfrac{2}{7} & \dfrac{12}{7} & 0 \\[2mm] 0 & -\dfrac{1}{7} & -\dfrac{13}{7} & 0 \\[2mm] 0 & \dfrac{5}{21} & \dfrac{24}{7} & 1 \end{pmatrix} \tag{2.23}
$$

is $t^4 + (12/7)t^3 + (3/7)t^2 - (2/7)t$ and has as roots $\lambda_1 = 0$, $\lambda_2 = 2/7$ and $\lambda_{3,4} = -1$. Then $\lambda^+_{\max}(H(p_0)^{-1}H(p_1)) = 2/7$ and $\lambda^-_{\min}(H(p_0)^{-1}H(p_1)) = -1$. Therefore, system (2.20) is robustly stable in $[-1, 7/2]$.

Now with our method. By evaluating $p_0(t)$ and $p_1(t)$ in $i\omega$ we have

$$
\begin{aligned}
p_0(i\omega) &= \left(\omega^4 - 7\omega^2 + 3\right) + i\omega\left(-\omega^2 + 2\right) \\
&= P\left(\omega^2\right) + i\omega Q\left(\omega^2\right), \\
p_1(i\omega) &= \left(-\omega^2 + 3\right) + i\omega(2) \\
&= p\left(\omega^2\right) + i\omega q\left(\omega^2\right).
\end{aligned}
\tag{2.24}
$$

Thus,

$$
\begin{aligned}
F(\omega) &= -\omega^6 + 8\omega^4 - 20\omega^2 + 9, \\
G(\omega) &= \omega^8 - 13\omega^6 + 51\omega^4 - 38\omega^2 + 9, \\
H(\omega) &= \omega^2\left(-9 + \omega^2\right).
\end{aligned}
\tag{2.25}
$$

It is not hard to see that $R(H)_{\mathbb{R}^+} = \{0, 3\}$. Next, $F(0) = 9 > 0$, $F(3) = -252 < 0$, and $G(0) = 9$, $G(3) = 882$. Therefore,

$$
\begin{aligned}
k_{\max}^+ &= -\frac{G(3)}{F(3)} = \frac{7}{2} \\
k_{\min}^- &= -\frac{G(0)}{F(0)} = -1.
\end{aligned}
\tag{2.26}
$$

Note the easiness of roots finding for our polynomial $H(\omega)$.

As we see, in our method computations are easier operatively speaking since matrices inverse and roots of bigger degree polynomials have been found in Białas method.

3. Proofs of the Theorems

Before start with the proofs of the theorems we present the following lemma wich can be found in [3].

Lemma 3.1 (continuous root dependence). *Consider the family of polynomials $\mathcal{P}$ described by $P(\lambda, t) = \sum_{i=0}^{n} a_i(\lambda)t^i$ and $\lambda \in \Omega$ under Assumption 2.1. Then the roots of $P(\lambda, t)$ vary continuously with respect to $\lambda \in \Omega$. That is, there exist continuously mappings $t_i : \Omega \rightarrow \mathbb{C}$ for $i = 1, 2, \ldots, n$ such that $t_1(\lambda), \ldots, t_n(\lambda)$ are the roots of $P(\lambda, t)$.*

3.1. Proof of Theorem 2.4

Since $P(\lambda, t)$ satisfies Assumption 2.1, by Lemma 3.1 there exist n continuous function roots of $P(\lambda, t)$, say $t_1(\lambda), \ldots, t_n(\lambda)$, $\lambda \in [a, b]$. Let us denote $\alpha_j(\lambda) = \Re(t_j(\lambda))$ as the real parts of the roots. Without loss of generality we can suppose that for $j = 1, \ldots, n_1$, $\alpha_j(a) \in \mathbb{R}^-$ and

for $j = n_1 + 1, \ldots, n$, $\alpha_j(a) \in \mathbb{R}^+$, while for $\lambda = b$ at most $n_1 - 1$ $\alpha_j(b)$'s belong to $\mathbb{R}^-$ and at least $n - n_1 + 1$ belong to $\mathbb{R}^+$. Then there exists at least one $t_j(\lambda)$ such that $\alpha_j(a) < 0$ and $\alpha_j(b) > 0$. Let $\alpha_{j_1}(\lambda), \ldots, \alpha_{j_m}(\lambda)$ be such functions. Then by continuity and the intermediate value theorem we have that for each $1 \leq r \leq m$ there exists $\rho_r \in (a, b]$ such that $\alpha_{j_r}(\rho_r) = 0$. Define $\rho = \min\{\rho_r \mid r = 1, \ldots, m\}$. Therefore, for $\lambda = \rho$ at least one $\alpha_{j_r}(\rho) = 0$. Thus $P(\rho, t)$ has n_1 roots in $\mathbb{C}^- \cup i\mathbb{R}$ with at least one root in $i\mathbb{R}$, as we claim.

The proof of Theorem 2.5 is similar.

3.2. Proof of Theorem 2.6

($\Rightarrow$) If all of the elements of the family have n_1 roots in $\mathbb{C}^-$ and $n - n_1$ roots in $\mathbb{C}^+$ the it is clear that $P(\lambda, i\omega) \neq 0$ for all $\omega \in \mathbb{R}$ and for all $\lambda \in \Omega$.

($\Leftarrow$) Suppose that $P(\lambda, i\omega) \neq 0$ for all $\omega \in \mathbb{R}$ and for all $\lambda \in \Omega$. If there is $\lambda_0 \in \Omega$ such that the polynomial $P(\lambda_0, t)$ has m_1 roots in $\mathbb{C}^-$ and $n - m_1$ roots in $\mathbb{C}^+$ with $n_1 \neq m_1$, the from Theorem 2.5 there exsists ρ such that $P(\rho, i\omega) = 0$ for some $\omega \in \mathbb{R}$, which is a contradiction.

3.3. Proof of Theorem 2.12

By generalization of zero exclusion principle, the polynomial $p_c(t)p_0(-t)$ has n roots in $\mathbb{C}^-$ and n roots in $\mathbb{C}^+$ if and only if

$$p_c(i\omega)p_0(-i\omega) \neq 0 \tag{3.1}$$

for all $\omega \in \mathbb{R}$. Thus, if k satisfies

$$p_c(i\omega)p_0(-i\omega) = 0 \tag{3.2}$$

for some $\omega \in \mathbb{R}$, then

$$G(\omega) + kF(\omega) + ik\omega H(\omega) = 0 \tag{3.3}$$

for some $\omega \in \mathbb{R}$. Consequently

$$\begin{aligned} \omega H(\omega) &= 0, \\ G(\omega) + kF(\omega) &= 0. \end{aligned} \tag{3.4}$$

And this system is satisfied if $k = -G(\omega_l)/F(\omega_l)$, where $\omega_l = 0$ or $\omega_l \in R(H)$. Since $G(\omega) > 0$ for all $\omega \in \mathbb{R}$ and we want to know the minimum $k > 0$ and the maximum $k < 0$ where it does not happen, then

$$\begin{aligned} k_{\min}^- &= \max\left\{ -\frac{G(\omega_l)}{F(\omega_l)} \mid F(\omega_l) > 0, \ \omega_l = 0 \text{ or } \omega_l \in R(H) \right\}, \\ k_{\max}^+ &= \min\left\{ -\frac{G(\omega_l)}{F(\omega_l)} \mid F(\omega_l) < 0, \ \omega_l = 0 \text{ or } \omega_l \in R(H) \right\}. \end{aligned} \tag{3.5}$$

Now by symmetry of $F(w)$ and $H(w)$ we will just consider real positive roots of $H(w)$. Thus

$$k_{\min}^- = \max\left\{ -\frac{G(w_l)}{F(w_l)} \mid F(w_l) \in K^+ \right\},$$

$$k_{\max}^+ = \min\left\{ -\frac{G(w_l)}{F(w_l)} \mid F(w_l) \in K^- \right\}. \tag{3.6}$$

This ends the proof.

Remark 3.2. In the proof it is not necessary to consider cases when $F(w_l) = 0$, since in other case in the system (3.4) we would have $G(w_l) = 0$, but

$$G(w_l) = \left| p_0(iw_l) \right|^2$$

$$= p_0(iw_l)p_0(-iw_l) \tag{3.7}$$

$$= 0,$$

which is impossible since $p_0(t)$ is Hurwitz. However, if occurs either $K^+ = \{0^+\}$ or $K^- = \{0^-\}$, then we evaluate in $w = 0$ and depending on sign of $F(0)$ we will get either

$$k_{\min}^- = \lim_{r \to 0^+} -\frac{G(0)}{r} = -\infty \tag{3.8}$$

or

$$k_{\max}^+ = \lim_{r \to 0^-} -\frac{G(0)}{r} = +\infty. \tag{3.9}$$

The following example illustrates the second part of Remark 3.2.

Example 3.3. Consider the linear control system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left(-\frac{13}{2}k, -11k, -5k \right) x. \tag{3.10}$$

Then we have that $p_0(t) = t^3 + 6t^2 + 11t + 6$ and $p_1(t) = 5t^2 + 11t + (13/2)$. First, for Białas's method we have that the Hurwitz matrices of $p_0(t)$ and $p_1(t)$ are

$$H(p_0(t)) = \begin{pmatrix} 6 & 6 & 0 \\ 1 & 11 & 0 \\ 0 & 6 & 6 \end{pmatrix}, \qquad H(p_1(t)) = \begin{pmatrix} 5 & \dfrac{13}{2} & 0 \\ 0 & 11 & 0 \\ 0 & 5 & \dfrac{13}{2} \end{pmatrix}. \tag{3.11}$$

Hence,

$$
\begin{aligned}
-H(p_0(t))^{-1}H(p_1(t)) &= \begin{pmatrix} \dfrac{11}{60} & -\dfrac{1}{10} & 0 \\[6pt] -\dfrac{1}{60} & \dfrac{1}{10} & 0 \\[6pt] \dfrac{1}{60} & -\dfrac{1}{10} & \dfrac{1}{6} \end{pmatrix} \begin{pmatrix} 5 & \dfrac{13}{2} & 0 \\[6pt] 0 & 11 & 0 \\[6pt] 0 & 5 & \dfrac{13}{2} \end{pmatrix} \\[12pt]
&= \begin{pmatrix} -\dfrac{11}{12} & -\dfrac{11}{120} & 0 \\[6pt] \dfrac{1}{12} & -\dfrac{119}{120} & 0 \\[6pt] -\dfrac{1}{12} & \dfrac{19}{120} & -\dfrac{13}{12} \end{pmatrix},
\end{aligned}
\tag{3.12}
$$

whose characteristic polynomial is $\lambda^3 - (359/120)\lambda^2 + (4297/1140)\lambda - (143/144)$ which is a 3-degree polynomial and its roots are $\lambda_{1,2} = -(229 \pm i\sqrt{359})/240$ and $\lambda_3 = -13/12$. Hence $\lambda_{\max}^+(-H(p_0)^{-1}H(p_1)) = 0^-$ and $\lambda_{\min}^-(-H(p_0)^{-1}H(p_1)) = -13/12$. Then $p(t,k) = p_0(t) + kp_1(t)$ is robustly stable for k in $[-12/13, \infty)$.

With our proposed method we can see that

$$
\begin{aligned}
p_0(i\omega) &= \left(6 - 6\omega^2\right) + i\omega\left(11 - \omega^2\right) \\[6pt]
&= P\left(\omega^2\right) + i\omega Q\left(\omega^2\right), \\[6pt]
p_1(i\omega) &= \left(\frac{13}{2} - 5\omega^2\right) + 11i\omega \\[6pt]
&= p\left(\omega^2\right) + i\omega q\left(\omega^2\right).
\end{aligned}
\tag{3.13}
$$

Thus,

$$
\begin{aligned}
F(\omega) &= 19\omega^4 + 112\omega^2 + 39, \\[6pt]
G(\omega) &= \omega^6 - 94\omega^4 + 49\omega^2 + 36, \\[6pt]
H(\omega) &= -5\omega^4 - 9\omega^2 - 11.
\end{aligned}
\tag{3.14}
$$

Note that $F(\omega) > 0$ for all $\omega \in \mathbb{R}$ and by a single test for second-order equations, $H(\omega)$ have no real roots. Therefore by Remark 3.2 $k_{\max} = +\infty$ and $k_{\min} = -G(0)/F(0) = -12/13$. Consequently, $p_0(t) + kp_1(t)$ is stable for all $k \in [-12/13, \infty)$.

4. Conclusions

In this paper, first we obtain generalizations of the Boundary Crossing Theorem and the Zero Exclusion Principle, which are results that allow to obtain important results about stability of families of polynomials. Next we use such generalizations for calculating the maximal

interval of stability, which is a different approach to the Białas method. Since in Białas method we have to find the inverse of the matrix $H(p_0)$ and the roots of the n-degree characteristic polynomial of $-H(p_0)^{-1}H(p_1)$, we have found that in our approach easier computations have arisen due if degree of both $p_0(t)$ and $p_1(t)$ is either even or odd then $\deg H(\omega) = n + m - 2$ and in the other cases $\deg H(\omega) = n + m - 1$ and by symmetry of $H(\omega)$ we must to find the roots of a polynomial with degree $(n + m - 2)/2$ or $(n + m - 1)/2$ both less than or equal to $n - 1$, respectively.

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