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Research Article

On the General Solution of the Ultrahyperbolic Bessel Operator

Rattapan Damkengpan[1] and Kamsing Nonlaopon[1,2]

[1] *Department of Mathematics, Khon Kaen University, Khon Kaen 40002, Thailand*
[2] *Centre of Excellence in Mathematics, Commission on Higher Education (CHE), Si Ayuthaya road, Bangkok 10400, Thailand*

Correspondence should be addressed to Kamsing Nonlaopon, nkamsi@kku.ac.th

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We study the general solution of equation $\square_{B,c}^{k} u(x) = f(x)$, where $\square_{B,c}^{k}$ is the ultrahyperbolic Bessel operator iterated k-times and is defined by $\square_{B,c}^{k} = [(1/c^2)(B_{x_1} + B_{x_2} + \cdots + B_{x_p}) - (B_{x_{p+1}} + \cdots + B_{x_{p+q}})]^k$, $p + q = n$, n is the dimension of $\mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ {}_x > 0, \ldots, x_n > 0\}$, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\beta_i + 1$, $\beta_i > -1/2$, $x_i > 0$ $(i = 1, 2, \ldots, n)$, $f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, k is a nonnegative integer, c is a positive constant, and $x \in \mathbb{R}_n^+$.

1. Introduction

The n-dimensional ultrahyperbolic operator $\square^k$ iterated k-times is defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{1.1}$$

where $p + q = n$, n is the dimension of space $\mathbb{R}^n$, and k is a nonnegative integer.

Consider the linear differential equation of the form

$$\square^k u(x) = f(x), \tag{1.2}$$

where $u(x)$ and $f(x)$ are generalized functions and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Gel'fand and Shilov [1] first introduced the fundamental solution of (1.2), which is a complicated form. Later, Trione [2] has shown that the generalized function $R_{2k}(x)$, defined by (2.8) with $|v| = 0$, is a unique fundamental solution of (1.2) and Téllez [3] also proved that $R_{2k}(x)$ exists only in the case when p is odd with n odd or even and $p+q = n$. A wealth of some effective works on the fundamental solution of the n-dimensional classical ultrahyperbolic operator have, presented by Kananthai and Sritanratana [4–9].

In 2004, Yildirim et al. [10] have introduced the Bessel ultrahyperbolic operator iterated k-times with $x \in \mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ x_1 > 0, \ldots, x_n > 0\}$,

$$\square_B^k = \left(B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - \cdots - B_{x_{p+q}}\right)^k, \tag{1.3}$$

where $p + q = n$, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\beta_i + 1$, $\beta_i > -1/2$ [11], k is a nonnegative integer, and n is the dimension of $\mathbb{R}_n^+$. They also have studied the fundamental solution of Bessel ultrahyperbolic operator.

In 2007, Sarikaya and Yildirim [12] have studied the weak solution of the compound Bessel ultrahyperbolic equation and also studied the Bessel ultrahyperbolic heat equation [13].

In 2009, Saglam et al. [14] have developed the operator of (1.3), defined by (1.6), and it is called the ultrahyperbolic Bessel operator iterated k-times. They have also studied the product of the ultrahyperbolic Bessel operator related to elastic waves.

Next, Srisombat and Nonlaopon [15] have studied the weak solution of

$$\square_{B,c}^k u(x) = f(x), \tag{1.4}$$

where $u(x)$ and $f(x)$ are some generalized functions. They have developed (1.4) into the form

$$\sum_{k=0}^{m} C_k \square_{B,c}^k u(x) = f(x), \tag{1.5}$$

which is called the compound ultrahyperbolic Bessel equation. In finding the solution of (1.5), they have used the properties of B-convolution for the generalized functions.

The purpose of this study is to find the general solution of equation $\square_{B,c}^k u(x) = f(x)$, where $\square_{B,c}^k$ is the ultrahyperbolic Bessel operator iterated k-times and is defined by

$$\square_{B,c}^k = \left[\frac{1}{c^2}\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right) - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)\right]^k \tag{1.6}$$

$p + q = n$, n is the dimension of $\mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ x_1 > 0, \ldots, x_n > 0\}$, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\beta_i + 1$, $\beta_i > -1/2$, $x_i > 0$ $(i = 1, 2, \ldots, n)$, $f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, k is a nonnegative integer, c is a positive constant, and $x \in \mathbb{R}_n^+$.

2. Preliminaries

Let T_x^y be the generalized shift operator acting on the function φ, according to the law [11, 16]:

$$T_x^y\varphi(x) = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi\left(\sqrt{x_1^2 + y_1^2 - 2x_1y_1\cos\theta_1}, \ldots, \sqrt{x_n^2 + y_n^2 - 2x_ny_n\cos\theta_n}\right) \times \left(\prod_{i=1}^n \sin^{2v_i-1}\theta_i\right) d\theta_1 \cdots d\theta_n, \tag{2.1}$$

where $x, y \in \mathbb{R}_n^+$ and $C_v^* = \prod_{i=1}^n (\Gamma(v_i+1)/\Gamma(1/2)\Gamma(v_i))$. We remark that this shift operator is closely connected to the Bessel differential operator [11]:

$$\begin{gathered} \frac{d^2U}{dx^2} + \frac{2v}{x}\frac{dU}{dx} = \frac{d^2U}{dy^2} + \frac{2v}{y}\frac{dU}{dy}, \\ U(x,0) = f(x), \\ U_y(x,0) = 0. \end{gathered} \tag{2.2}$$

The convolution operator is determined by the T_x^y as follows:

$$(f * \varphi)(y) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy. \tag{2.3}$$

The convolution (2.3) is known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator.

(a) $T_x^y \cdot 1 = 1$.

(b) $T_x^0 \cdot f(x) = f(x)$.

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function all $x > 0$, and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^n x_i^{2v_i}\right) dx < \infty, \tag{2.4}$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy. \tag{2.5}$$

(d) From (c), we have the following equality for $g(x) = 1$:

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^n y_i^{2v_i}\right) dy. \tag{2.6}$$

(e) $(f * g)(x) = (g * f)(x)$.

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional space $\mathbb{R}_n^+$. Denote the nondegenerated quadratic form by

$$V = c^2\left(x_1^2 + x_2^2 + \cdots + x_p^2\right) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.7}$$

where $p + q = n$. The interior of the forward cone is defined by $\Gamma_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}_n^+ : x_i > 0,\ i = 1, \ldots, n \text{ and } V > 0\}$, where $\overline{\Gamma}_+$ designates its closure. For any complex number α, we define

$$R_{\alpha,c}^H(x) = \begin{cases} \dfrac{V^{(\alpha-n-2|v|)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.8}$$

where

$$K_n(\alpha) = \frac{\pi^{(n+2|v|-1)/2}\Gamma((2+\alpha-n-2|v|)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p-2|v|)/2)\Gamma((p+2|v|-\alpha)/2)}. \tag{2.9}$$

The function $R_{\alpha,c}^H(x)$ is introduced by [10, 12, 17, 18]. It is well known that $R_{\alpha,c}^H(x)$ is an ordinary function if $\mathrm{Re}(\alpha) \geq n$ and is the distribution of α if $\mathrm{Re}(\alpha) < n$. Let supp $R_{\alpha,c}^H(x) \subset \overline{\Gamma}_+$, where supp $R_{\alpha,c}^H(x)$ denotes the support of $R_{\alpha,c}^H(x)$.

By putting $p = c = 1$ into (2.7), (2.8), and (2.9), and using the Legendre's duplication of $\Gamma(z)$,

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{2.10}$$

the formula (2.8) is reduced to

$$M_\alpha^H(x) = \begin{cases} \dfrac{V^{((\alpha-n-2|v|)/2)}}{H_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.11}$$

where $V = x_1^2 - x_2^2 - \cdots - x_n^2$ and

$$H_n(\alpha) = \pi^{(n+2|v|-1)/2}2^{\alpha-1}\Gamma\left(\frac{2+\alpha-n-2|v|}{2}\right)\Gamma\left(\frac{\alpha}{2}\right). \tag{2.12}$$

Note that the function $M_\alpha^H(x)$ is precisely the Bessel hyperbolic kernel of Marcel Riesz.

Lemma 2.2. *Given the equation*

$$\Box_{B,c}^k u(x) = \delta(x), \tag{2.13}$$

where $\square_{B,c}^{k}$ *is defined by* (1.6) *and* $x \in \mathbb{R}_n^+$, *then we obtain* $u(x) = R_{2k,c}^{H}(x)$ *as a fundamental solution of* (2.13), *where* $R_{2k,c}^{H}(x)$ *is defined by* (2.8).

The proof of this Lemma is given in [14].

Lemma 2.3. *The B-convolutions of tempered distributions.*

(a) $(\square_{B,c}^{k}\delta) * u(x) = \square_{B,c}^{k}u(x)$, *where* $u(x)$ *is any tempered distribution.*

(b) *Let* $R_{2k,c}^{H}(x)$ *and* $R_{2m,c}^{H}(x)$ *be defined by* (2.8); *then* $R_{2k,c}^{H}(x) * R_{2m,c}^{H}(x)$ *exists and is a tempered distribution.*

(c) *Let* $R_{2k,c}^{H}(x)$ *and* $R_{2m,c}^{H}(x)$ *be defined by* (2.8); *then* $R_{2k,c}^{H}(x) * R_{2m,c}^{H}(x) = R_{2k+2m,c}^{H}(x)$, *where* k *and* m *are nonnegative integers.*

The proof of this Lemma is given in [15].

Lemma 2.4. *Given that* P *is a hypersurface*

$$P\delta^{(m)}(P) + mP\delta^{(m-1)}(P) = 0, \tag{2.14}$$

where $\delta^{(m)}$ *is the Dirac-delta distribution with* m *derivatives.*

The proof of this Lemma is given in [1].

Lemma 2.5. *Given the equation*

$$\square_{B,c}^{k}u(x) = 0, \tag{2.15}$$

where $\square_{B,c}^{k}$ *is the ultrahyperbolic Bessel operator iterated* k*-times, as defined by* (1.6), *and* $x \in \mathbb{R}_n^+$, *then*

$$u(x) = \left[R_{2(k-1),c}^{H}(x)\right]^{(m)}, \tag{2.16}$$

defined by (2.8) *with* m *derivatives, as a solution of* (2.15) *with* $m = ((n+2|v|-4)/2)$, $n+2|v| \geq 4$ *and* n *is an even dimension.*

Proof. We first show that the generalized function $\delta^{(m)}(c^2r^2 - s^2)$, where $r^2 = x_1^2 + x_2^2 + \cdots + x_p^2$, $s^2 = x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2$, $p+q = n$, is a solution of

$$\square_{B,c}u(x) = 0, \tag{2.17}$$

and $\square_{B,c}$ is defined by (1.6) with $k = 1$ and $x \in \mathbb{R}_n^+$. Now for $1 \leq i \leq p$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}\delta^{(m)}\left(c^2r^2 - s^2\right) &= 2c^2x_i\delta^{(m+1)}\left(c^2r^2 - s^2\right),\\
\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(c^2r^2 - s^2\right) &= 2c^2\delta^{(m+1)}\left(c^2r^2 - s^2\right) + 4c^4x_i^2\delta^{(m+2)}\left(c^2r^2 - s^2\right).
\end{aligned} \tag{2.18}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{c^2}\sum_{i=1}^{p}\left[\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(c^2r^2-s^2\right)+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\delta^{(m)}\left(c^2r^2-s^2\right)\right]\\
&\quad=2p\delta^{(m+1)}\left(c^2r^2-s^2\right)+4c^2r^2\delta^{(m+2)}\left(c^2r^2-s^2\right)+4\left|v'\right|\delta^{(m+1)}\left(c^2r^2-s^2\right)\\
&\quad=\left(2p+4\left|v'\right|\right)\delta^{(m+1)}\left(c^2r^2-s^2\right)+4\left(c^2r^2-s^2\right)\delta^{(m+2)}\left(c^2r^2-s^2\right)+4s^2\delta^{(m+2)}\left(c^2r^2-s^2\right)\\
&\quad=\left(2p+4\left|v'\right|\right)\delta^{(m+1)}\left(c^2r^2-s^2\right)-4(m+2)\delta^{(m+1)}\left(c^2r^2-s^2\right)+4s^2\delta^{(m+2)}\left(c^2r^2-s^2\right)\\
&\quad=\left[2p+4\left|v'\right|-4(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)+4s^2\delta^{(m+2)}\left(c^2r^2-s^2\right)
\end{aligned}\tag{2.19}$$

by applying Lemma 2.4 with $P=c^2r^2-s^2$, where $|v'|=v_1+v_2+\cdots+v_p$.

Similarly, we have

$$\begin{aligned}
&\sum_{i=p+1}^{p+q}\left[\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(c^2r^2-s^2\right)+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\delta^{(m)}\left(c^2r^2-s^2\right)\right]\\
&\quad=\left[-\left(2q+4\left|v''\right|\right)+4(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)+4c^2r^2\delta^{(m+2)}\left(c^2r^2-s^2\right)
\end{aligned}\tag{2.20}$$

by applying Lemma 2.4 with $P=c^2r^2-s^2$, where $|v''|=v_{p+1}+v_{p+2}+\cdots+v_{p+q}$.

Thus, we have

$$\begin{aligned}
\Box_{B,c}\delta^{(m)}\left(c^2r^2-s^2\right)&=\frac{1}{c^2}\sum_{i=1}^{p}\left[\frac{\partial^2}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\right]\delta^{(m)}\left(c^2r^2-s^2\right)\\
&\quad-\sum_{i=p+1}^{p+q}\left[\frac{\partial^2}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\right]\delta^{(m)}\left(c^2r^2-s^2\right)\\
&=\left[2(p+q+2|v|)-8(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)\\
&\quad-4\left(c^2r^2-s^2\right)\delta^{(m+2)}\left(c^2r^2-s^2\right)\\
&=\left[2(n+2|v|)-8(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)+4(m+2)\delta^{(m+1)}\left(c^2r^2-s^2\right)\\
&=\left[2(n+2|v|)-4(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)
\end{aligned}\tag{2.21}$$

by applying Lemma 2.4 with $P=c^2r^2-s^2$, where $|v|=|v'|+|v''|$.

If $[2(n+2|v|)-4(m+2)]=0$, we obtain

$$\Box_{B,c}\delta^{(m)}\left(c^2r^2-s^2\right)=0.\tag{2.22}$$

That is, $u(x) = \delta^{(m)}(c^2r^2 - s^2)$ is a solution of (2.15) with $m = (n+2|v|-4)/2,\ n+2|v| \geq 4$, and n is an even dimension. Now $\Box_{B,c}^{k} u(x) = 0$ can be written in the form

$$\Box_{B,c}\left(\Box_{B,c}^{k-1} u(x)\right) = 0. \tag{2.23}$$

From (2.17), we have

$$\Box_{B,c}^{k-1} u(x) = \delta^{(m)}\left(c^2r^2 - s^2\right) \tag{2.24}$$

with $m = (n + 2|v| - 4)/2, n + 2|v| \geq 4$, and n being an even dimension. By Lemma 2.3(a), we can write (2.24) in the from

$$\Box_{B,c}^{k-1} \delta * u(x) = \delta^{(m)}\left(c^2r^2 - s^2\right). \tag{2.25}$$

B-convolving both sides of the above equation with the function $R_{2(k-1),c}^{H}(x)$, we obtain

$$\begin{gathered} R_{2(k-1),c}^{H}(x) * \Box_{B,c}^{k-1} \delta * u(x) = R_{2(k-1),c}^{H}(x) * \delta^{(m)}(c^2r^2 - s^2), \\ \Box_{B,c}^{k-1}\left[R_{2(k-1),c}^{H}(x)\right] * u(x) = \left[R_{2(k-1),c}^{H}(x)\right]^{(m)}, \\ \delta * u(x) = u(x) = \left[R_{2(k-1),c}^{H}(x)\right]^{(m)}, \end{gathered} \tag{2.26}$$

by Lemma 2.2.

It follows that $u(x) = [R_{2(k-1),c}^{H}(x)]^{(m)}$ is a solution of (2.15) with $m = (n+2|v|-4)/2,\ n+2|v| \geq 4$ and n is an even dimension.

The generalized function $\delta^{(m)}(c^2r^2-s^2)$ mentioned in Lemma 2.5 has been also studied on the aspect of multiplicative product, distributional product and applications, for more details, see [19–23]. □

3. Main Result

Theorem 3.1. *Given the equation*

$$\Box_{B,c}^{k} u(x) = f(x), \tag{3.1}$$

where $\Box_{B,c}^{k}$ is the ultrahyperbolic Bessel operator iterated k-times and is defined by (1.6), *$f(x)$ is a generalized function, $u(x)$ is an unknown generalized function, $x \in \mathbb{R}_n^+$, and n is an even, then* (3.1) *has the general solution*

$$u(x) = \left[R_{2(k-1),c}^{H}(x)\right]^{(m)} + R_{2k,c}^{H}(x) * f(x), \tag{3.2}$$

where $[R_{2k,c}^{H}(x)]^{(m)}$ is a function defined by (2.8) *with m derivatives.*

Proof. B-convolving both sides of (3.1) with $R_{2k,c}^{H}(x)$, we obtain

$$R_{2k,c}^{H}(x) * \left(\square_{B,c}^{k} u(x)\right) = R_{2k,c}^{H}(x) * f(x). \tag{3.3}$$

By Lemma 2.2, we have

$$\square_{B,c}^{k}\left(R_{2k,c}^{H}(x)\right) * u(x) = \delta * u(x) = R_{2k,c}^{H}(x) * f(x). \tag{3.4}$$

So, we obtain that

$$u(x) = R_{2k,c}^{H}(x) * f(x) \tag{3.5}$$

is the solution of (3.1).

For a homogeneous equation $\square_{B,c}^{k} u(x) = 0$, we have a solution

$$u(x) = \left[R_{2(k-1),c}^{H}(x)\right]^{(m)} \tag{3.6}$$

by Lemma 2.5. Thus the general solution of (3.1) is

$$u(x) = \left[R_{2(k-1),c}^{H}(x)\right]^{(m)} + R_{2k,c}^{H}(x) * f(x). \tag{3.7}$$

This completes the proof. □

By putting $c = 1$, (3.1) becomes the Bessel ultrahyperbolic equation

$$\square_{B}^{k} w(x) = f(x), \tag{3.8}$$

where $\square_{B}^{k}$ is the Bessel ultrahyperbolic operator iterated k-times, and is defined by (1.3), $f(x)$ is a generalized function and $w(x)$ is an unknown generalized function. From (3.5) we have that

$$w(x) = R_{2k}^{H}(x) * f(x) \tag{3.9}$$

is a solution of (3.8), where $R_{2k}^{H}(x) = R_{2k,1}^{H}(x)$ defined by (2.8).

From (3.2), we obtain that the general solution of the Bessel ultrahyperbolic equation is

$$w(x) = \left[R_{2(k-1)}^{H}(x)\right]^{(m)} + R_{2k}^{H}(x) * f(x). \tag{3.10}$$

Moreover, if we put $k = 1,\ p = 1$ and $x_1 = t$(times), then (3.8) is reduced to the Bessel wave equation

$$\square_B w(x) = \left(B_t - \sum_{i=2}^{n} B_{x_i}\right) w(x) = f(x), \tag{3.11}$$

where

$$\square_B = B_t - \sum_{i=2}^{n} B_{x_i} \tag{3.12}$$

is the Bessel wave operator and $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$.

Thus, we obtain $w(x) = M_2(x) * f(x)$ as a solution of the Bessel wave equation, since $R_2^H(x)$ becomes $M_2^H(x)$, where $M_2^H(x)$ is the Bessel ultrahyperbolic kernel of Marcel Riesz, and is defined by (2.11) with $\alpha = 2$. And from (3.2), we obtain the general solution of Bessel wave equation as

$$w(x) = \delta^{(m)}(x) + M_2^H(x) * f(x), \tag{3.13}$$

where $\delta^{(m)}(x)$ is a solution of

$$\left(B_t - \sum_{i=2}^{n} B_{x_i}\right) w(x) = 0. \tag{3.14}$$

Now we put $V = t^2 - x_2^2 - x_3^2 - \cdots - x_n^2$ and $s^2 = x_2^2 + x_3^2 + \cdots + x_n^2$. By [24], we obtain that

$$w(x,t) = \delta^{(m)}\left(t^2 - s^2\right) \tag{3.15}$$

is the solution of (3.14) with the initial conditions $w(x,0) = 0$ and $\partial w(x,0)/\partial t = (-1)^m 2\pi^{m+1}\delta(x)$ at $t = 0$ and $x = (x_2, x_3, \ldots, x_n) \in \mathbb{R}_{n-1}^+$.

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