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Research Article

A Stabilized Low Order Finite-Volume Method for the Three-Dimensional Stationary Navier-Stokes Equations

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This paper proposes and analyzes a stabilized finite-volume method (FVM) for the three-dimensional stationary Navier-Stokes equations approximated by the lowest order finite element pairs. The method studies the new stabilized FVM with the relationship between the stabilized FEM (FEM) and the stabilized FVM under the assumption of the uniqueness condition. The results have three prominent features in this paper. Firstly, the error analysis shows that the stabilized FVM provides an approximate solution with the optimal convergence rate of the same order as the usual stabilized FEM solution solving the stationary Navier-Stokes equations. Secondly, superconvergence results on the solutions of the stabilized FEM and stabilized FVM are derived on the H^1-norm and the L^2-norm for the velocity and pressure. Thirdly, residual technique is applied to obtain the L^2-norm error for the velocity without additional regular assumption on the exact solution.

1. Introduction

Recently, the development of stable mixed FEMs is a fundamental component in the search for the efficient numerical methods for solving the Navier-Stokes equations governing the flow of an incompressible fluid by using a primitive variable formulation. The object of this work is to analyze the stabilized finite volume method for solving the three-dimensional stationary Navier-Stokes equations.

The importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying the so-called inf-sup condition is widely understood. The numerous mixed finite elements satisfying the inf-sup condition have been proposed over the years. However, elements not satisfying the inf-sup condition may also work well. So far, the most convenient choice of the finite element space from an implementational point of view would be the elements of the low polynomial order in the velocity and the pressure with an identical degree distribution for both the velocity and the pressure.

This paper focuses on the stabilized method called local polynomial pressure projection for the three-dimensional Navier-Stokes equations [1–5]. The proposed method is characterized by the following features. First, the method does not require approximation of derivatives, specification of mesh-dependent parameters, edge-based data structures, and a nonstandard assembly procedure. Second, this method is completely local at the element level.

On the other hand, FVM has become an active area in numerical analysis. The most attractive things are that FVM can keep local conservation and have the advantages of FVM and finite difference methods. The FVM is also termed the control volume method, the covolume method, or the first-order generalized difference method. Nowadays, it is difficult in analyzing FVM to obtain L^2-norm error estimates because trail functions and test functions are derived from different spaces. Many papers were devoted to its error analysis for second-order elliptic and parabolic partial differential problems [6–10]. Error estimates of optimal order in the H^1-norm are the same as those for the linear FEM [9, 11]. Error estimates of optimal order in the L^2-norm can be obtained as well [8, 9]. Moreover, the FVM for generalized Stokes problems was studied by many people [11–13]. They analyzed this method by using a relationship between it and the FEM and obtained its error estimates through those known for the latter method. Also, it still requires H^3 smoothness assumption of the exact solution to obtain $O(h^2)$ error bound in most previous literatures. However, for the Stokes problems only the finite elements that satisfy the discrete *inf-sup* condition have been studied.

The work of [14, 15] for the two-dimensional stationary Stokes equations is extended in this paper for the three-dimensional stationary Navier-Stokes equations approximated by lowest equal-order finite elements. Following the abstract framework of the relationship between the stabilized FEM and stabilized FVM [14, 15], the stabilized FVM is studied, and the optimal error estimate of the stabilized FVM is obtained for the three-dimensional stationary Navier-Stokes equations relying on the uniqueness condition. As far as known, there still requires much research on FVM results [16] about the velocity in L^2-norm and superconvergence result between FEM solution and FVM solution of the three-dimensional Navier-Stokes equations.

The remainder of the paper is organized as follows. In Section 2, an abstract functional setting of the three-dimensional Navier-Stokes problem is given with some basic assumptions. In Section 3, the stability of the stabilized FVM is analyzed and provided by Brouwer's fixed-point theorem. In Section 4, the optimal error estimates of the stabilized finite volume approximation for the three-dimensional stationary Navier-Stokes equations are obtained.

2. FVM Formulation

Let Ω be a bounded domain in R^3, assumed to have a Lipschitz-continuous boundary Γ and to satisfy a further condition stated in $(A1)$ below. The three-dimensional stationary Navier-Stokes equations are considered as follows:

$$-\nu\Delta u + \nabla p + (u\cdot\nabla)u = f, \quad \text{in}\,\Omega, \tag{2.1a}$$

$$\operatorname{div} u = 0, \quad \text{in}\,\Omega, \tag{2.1b}$$

$$u|_{\partial\Omega} = 0, \quad \text{on}\ \partial\Omega, \tag{2.1c}$$

where $\nu > 0$ is the viscosity, $u = (u_1(x), u_2(x), u_3(x))$ represents the velocity vector, $p = p(x)$ the pressure, and $f = (f_1(x), f_2(x), f_3(x))$ the prescribed body force.

In order to introduce a variational formulation, we set [17]

$$X = \left[H_0^1(\Omega)\right]^3, \qquad Y = \left[L^2(\Omega)\right]^3, \qquad M = L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q dx = 0\right\},$$
$$D(A) = \left[H^2(\Omega)\right]^3 \cap X. \tag{2.2}$$

As mentioned above, a further assumption on Ω is presented.

(A1) Assume that Ω is regular so that the unique solution $(v, q) \in (X, M)$ of the steady Stokes problem

$$-\Delta v + \nabla q = g, \quad \operatorname{div} v = 0 \quad \text{in}\, \Omega,\ v|_{\partial\Omega} = 0, \tag{2.3}$$

for a prescribed $g \in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \le c\|g\|_0, \tag{2.4}$$

where $c > 0$ is a general constant depending on Ω. Here and after, $\|\cdot\|_i$ and $|\cdot|_i$ denote the usual norm and seminorm of the Sobolev space $H^i(\Omega)$ or $H^i(\Omega)^3$ for $i = 0, 1, 2$.

We denote by $(\cdot,\cdot)$ the inner product on $L^2(\Omega)$ or Y. The space $H_0^1(\Omega)$ and X are equipped with their equivalent scalar product and norm [17]

$$((u, v)) = (\nabla u, \nabla v), \quad \|\nabla u\|_0 = ((u, u))^{1/2}. \tag{2.5}$$

It is well known [18] that for each $v \in X$ there hold the following inequalities:

$$\|v\|_{L^4} \le 2^{1/2}\|v\|_0^{1/4}\|\nabla v\|_0^{3/4}, \quad \|v\|_0 \le \gamma\|v\|_1, \tag{2.6}$$

where γ is a positive constant depending only on Ω.

The continuous bilinear form $a(\cdot,\cdot)$ on $X \times X$ and $d(\cdot,\cdot)$ on $X \times M$, respectively, are defined by

$$a(u, v) = ((u, v)), \quad \forall u, v \in X, \qquad d(v, q) = -(v, \nabla q) = (q, \operatorname{div} v), \quad \forall v \in X,\ q \in M. \tag{2.7}$$

Also, the trilinear term is defined by

$$b(u, v, w) = ((u\cdot\nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w)$$
$$= \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v), \quad \forall u, v, w \in X \tag{2.8}$$

and satisfies

$$b(u,v,w) \le c_0\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0. \tag{2.9}$$

Then the mixed variational form of (2.1a)–(2.1c) is to seek $(u,p) \in (X,M)$ such that

$$a(u,v) - d(v,p) + d(u,q) + b(u,u,v) = (f,v), \quad \forall (v,q) \in X \times M. \tag{2.10}$$

The existence and uniqueness results are classical and can be found in [18–20].

We introduce the finite-dimensional subspace $(X_h, M_h) \subset (X,M)$, which is characterized by τ_h with mesh scale h, a partitioning of Ω into tetrahedron or hexahedron, assumed to be regular in the usual sense(see [20–22]).

Here, the space (X_h, M_h) satisfies the following approximation properties. For each $v \in D(A)$, $p \in H^1(\Omega)$, there exist approximations $I_h v \in X_h$ and $J_h q \in M_h$ such that

$$\|u - I_h u\|_0 + h\left(\|\nabla(u - I_h u)\|_0 + \|p - J_h p\|_0\right) \le ch^2\left(\|u\|_2 + \|p\|_1\right), \tag{2.11}$$

together with the inverse inequality

$$\|\nabla v_h\|_0 \le c_1 h^{-1}\|v_h\|_0, \qquad \|u_h\|_{L^\infty} \le c_2 h^{-1/2}\|\nabla u_h\|_0. \tag{2.12}$$

The stable and accurate finite element approximational solution of (2.10) requires that (X_h, M_h) satisfies the discrete inf-sup condition

$$\sup_{v_h \in X_h} \frac{d(v_h, q_h)}{\|\nabla v_h\|_0} \ge \beta\|q_h\|_0, \tag{2.13}$$

where β is positive constant independent of h.

The main purpose of this paper is to study a stabilized FVM for the stationary 3D Navier-Stokes equations. We follow [23, 24] to obtain the dual partition $\widetilde{K}_h$. We first choose an arbitrary point Q in the interior of each tetrahedron $\widetilde{K}$ and then connect Q with the barycenters Q_{ijk} of its 2D faces $\Delta P_i P_j P_k$ by straight lines (see Figure 1). On each face $\Delta P_i P_j P_k$, we connect by straight lines Q_{ijk} with the middle points of the segments P_iP_j, P_jP_k, and P_kP_i, respectively. Then the contribution of $\widetilde{\mathrm{K}}$ to the control volume $\widetilde{K}$ of a vertex P of $\widetilde{K}$ is the volume surrounding P by these straight lines, for example, the contribution from one simplex to the control volume $\widetilde{K}$ with the interfaces γ_{12} and γ_{13}.

Then, the dual finite element space can be constructed for the FVM as follows:

$$\widetilde{X}_h = \left\{ \tilde{v} \in \left[L^2(\Omega)\right]^3 : \tilde{v}|_{\widetilde{K}} \in P_0\left(\widetilde{K}\right)\ \forall \widetilde{K} \in \widetilde{K}_h;\ \tilde{v}|_{\partial\widetilde{K}} = 0 \right\}. \tag{2.14}$$

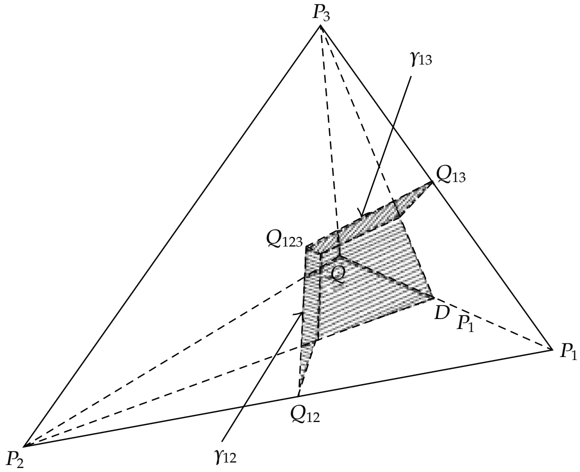

Figure 1: Control volumes in three-dimensional case.

Obviously, the dimensions of X_h and $\widetilde{X}_h$ are the same. Furthermore, there exists an invertible linear mapping $\Gamma_h : X_h \to \widetilde{X}_h$ such that for

$$v_h(x) = \sum_{j=1}^{N} v_h(P_j)\phi_j(x), \quad x \in \Omega,\ v_h \in X_h, \tag{2.15}$$

with

$$\Gamma_h v_h(x) = \sum_{j=1}^{N} v_h(P_j)\chi_j(x), \tag{2.16}$$

where $\{\phi_j\}$ indicates the basis for the finite element space X_h, and $\{\chi_j\}$ denotes the basis for the finite volume space $\widetilde{X}_h$ that are the characteristic functions associated with the dual partition $\widetilde{K}_h$:

$$\chi_j(x) = \begin{cases} 1 & \text{if } x \in \widetilde{K}_j \in \widetilde{K}_h, \\ 0 & \text{otherwise}. \end{cases} \tag{2.17}$$

The above idea of connecting the trial and test spaces in the Petrov-Galerkin method through the mapping Γ_h was first introduced in [25, 26] in the context of elliptic problems. Furthermore, the mapping Γ_h satisfies the following properties [26].

Lemma 2.1. *Let $K \in K_h$. If $v_h \in X_h$ and $1 \le r \le \infty$, then*

$$\int_K (v_h - \Gamma_h v_h)dx = 0, \tag{2.18}$$

$$\|\Gamma_h v_h\|_0 \le c_3 \|v_h\|_0, \qquad \|v_h - \Gamma_h v_h\|_{L^r(K)} \le c_4 h_K \|\nabla v_h\|_{L^r(K)}, \tag{2.19}$$

where h_K is the diameter of the element K.

Multiplying (2.1a) by $\Gamma_h v_h \in \widetilde{X}_h$ and integrating over the dual elements $\widetilde{K} \in \widetilde{K}_h$, (2.1b) by $q_h \in M_h$ and over the primal elements $K \in K_h$, and applying Green's formula, we define the following bilinear forms for the FVM:

$$\begin{aligned}
A(u_h, \Gamma_h v_h) &= -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \widetilde{K}_j} \frac{\partial u_h}{\partial n} ds, \quad u_h, v_h \in X_h, \\
D(\Gamma_h v_h, p_h) &= \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \widetilde{K}_j} p_h n \, ds, \quad p_h \in M_h, \\
b(u_h, v_h, \Gamma_h w_h) &= ((u_h \cdot \nabla) v_h, \Gamma_h w_h) + \frac{1}{2}(\operatorname{div} u_h v_h, \Gamma_h w_h), \\
(f, \Gamma_h v_h) &= \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\widetilde{K}_j} f \, dx, \quad v_h \in X_h,
\end{aligned} \tag{2.20}$$

where n is the unit normal outward to $\partial \widetilde{K}_j$ and these terms are well posed.

As noted above, this paper forces on a class of unstable velocity-pressure pairs consisting of the lowest equal-order finite elements

$$\begin{aligned}
X_h &= \left\{ v \in X : v|_K \in [R_1(K)]^3, \ \forall K \in \tau_h \right\}, \\
M_h &= \left\{ q \in M : q|_K \in R_i(K), \ i = 0, 1, \ \forall K \in \tau_h \right\},
\end{aligned} \tag{2.21}$$

where $R_i(K)$, $i = 0, 1$ represent piecewise constant range and continuous range on set K, R_i, $i = 0,1$ are spaces of polynomials, the maximum degree of which is bounded uniformly with respect to $K \in \tau_h$ and h. The corresponding stabilized FEM is formulated as follows [3]:

$$a(\overline{u}_h, v_h) - d(v_h, \overline{p}_h) + d(\overline{u}_h, q_h) + G(\overline{p}_h, q_h) + b(\overline{u}_h, \overline{u}_h, v_h) = (f, v_h), \quad \forall (v_h, q_h) \in (X_h, M_h). \tag{2.22}$$

Also, the corresponding stabilized FVM is defined for the solution $(u_h, p_h) \in (X_h, M_h)$ as follows:

$$\mathcal{C}_h((u_h, p_h); (v_h, q_h)) + b(u_h, u_h, \Gamma_h v_h) = (f, \Gamma_h v_h) \quad \forall (v_h, q_h) \in (X_h, M_h), \tag{2.23}$$

where

$$\mathcal{C}_h((u_h, p_h); (v_h, q_h)) = A(u_h, \Gamma_h v_h) + D(\Gamma_h v_h, p_h) + d(u_h, q_h) + G(p_h, q_h). \tag{2.24}$$

Obviously, the bilinear form $G(\cdot,\cdot)$ can be defined by the following symmetry form: [1]

$$G(p, q) = (p - \Pi_h p, q - \Pi_h q). \tag{2.25}$$

Note that

$$\Pi_h = \begin{cases} L^2(\Omega) \longrightarrow R_0 & \text{if } i = 1, \\ L^2(\Omega) \longrightarrow R_1 & \text{if } i = 0. \end{cases} \tag{2.26}$$

Here, the operator Π_h satisfies the following properties: [1, 4]

$$(p, q_h) = (\Pi_h p, q_h) \quad \forall p \in M,\ q_h \in R_i, \tag{2.27}$$

$$\|\Pi_h p\|_0 \leq c_5 \|p\|_0 \quad \forall p \in M, \tag{2.28}$$

$$\|p - \Pi_h p\|_{L^p} \leq c_6 h \|p\|_{H^{1,p}} \quad \forall p \in H^1(\Omega) \cap M. \tag{2.29}$$

In particular, the L^2-projection operator Π_h can be extended to the vector case.

This section concentrates on the study of a relationship between the FEM and FVM for the Stokes equations.

Lemma 2.2. *It holds that* [11–13]

$$A(u_h, \Gamma_h v_h) = a(u_h, v_h) \quad \forall u_h,\ v_h \in X_h, \tag{2.30}$$

with the following properties:

$$\begin{gathered} A(u_h, \Gamma_h v_h) = A(v_h, \Gamma_h u_h), \\ |A(u_h, \Gamma_h v_h)| \leq c_7 \|\nabla u_h\|_0 \|\nabla v_h\|_0, \\ |A(v_h, \Gamma_h v_h)| \geq \nu \|\nabla v_h\|_0^2. \end{gathered} \tag{2.31}$$

Moreover, the bilinear form $D(\cdot,\cdot)$ satisfies [14]

$$D(\Gamma_h v_h, q_h) = -d(v_h, q_h) \quad \forall (v_h, q_h) \in (X_h, M_h). \tag{2.32}$$

Based on detailed results on existence, uniqueness, and regularity of the solution for the FVM (2.23), the following result establishes its continuity and weak coercivity.

Theorem 2.3. *It holds that [14]*

$$\left|\mathcal{C}_h((u_h,p_h),(v_h,q_h))\right| \le c(\|u_h\|_1+\|p_h\|_0)(\|v_h\|_1+\|q_h\|_0) \quad \forall (u_h,p_h),(v_h,q_h) \in (X_h,M_h). \tag{2.33}$$

Moreover,

$$\sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\left|\mathcal{C}_h((u_h,p_h),(v_h,q_h))\right|}{\|v_h\|_1+\|q_h\|_0} \ge \beta(\|u_h\|_1+\|p_h\|_0) \quad \forall (u_h,p_h)\in(X_h,M_h), \tag{2.34}$$

where β is independent of h.

3. Stability

In this section, we analyze the results of FVM for the three-dimensional stationary Navier-Stokes equations. Firstly, we are now in a position to show the well-posedness of system (2.23)

$$h_0 = \frac{4c_2c_3c_4\gamma h^{1/2}\|f\|_0}{\nu^2}. \tag{3.1}$$

Theorem 3.1 (stability). *For each $h>0$ such that*

$$0 < h_0 \le \frac{1}{2}, \tag{3.2}$$

system (2.23) *admits a solution $(u_h,p_h)\in(X_h,M_h)$. Moreover, if the viscosity $\nu>0$, the body force $f\in Y$, and the mesh size $h>0$ satisfy*

$$0<h_0\le\frac{1}{4}, \qquad 1-2c_0c_5\gamma\nu^{-2}\|f\|_0>0, \tag{3.3}$$

then the solution $(u_h,p_h)\in(X_h,M_h)$ is unique. Furthermore, it satisfies

$$\|\nabla u_h\|_0 \le \frac{2c_3\gamma}{\nu}\|f\|_0, \qquad \|p_h\|_0 \le \beta^{-1}c_3\gamma\|f\|_0(4c_3c_8\nu^{-2}\gamma\|f\|_0+1). \tag{3.4}$$

Proof. For fixed $f\in Y$, we introduce the set

$$B_M=\left\{(v_h,q_h)\in(X_h,M_h):\|\nabla u_h\|_0\le\frac{2c_3\gamma}{\nu}\|f\|_0, \|p_h\|_0\le\beta^{-1}c_3\gamma\|f\|_0(4c_3c_8\nu^{-2}\gamma\|f\|_0+1)\right\}. \tag{3.5}$$

Then we define the mapping $T_h : (X_h, M_h) \to (X_h, M_h)$ by [19]

$$A(P_h w_h, \Gamma_h v_h) + D(\Gamma_h v_h, p_h) + d(T_h w_h, q_h) + G(p_h, q_h) + b(w_h, T_h w_h, \Gamma_h v_h) = (f, \Gamma_h v_h) \quad \forall (v_h, q_h) \in (X_h, M_h), \tag{3.6}$$

where $T_h(w_h, p_h) :\equiv (T_1 w_h, T_2 p_h) = (u_h, p_h)$. We will prove that T_h maps B_M into B_M.

First, taking $(v_h, q_h) = (u_h, p_h) \in (X_h, M_h)$ in (3.6), and using (2.12) and (2.19), we see that for for all $(v_h, q_h) \in (X_h, M_h)$

$$\begin{aligned} \nu \|\nabla u_h\|_0^2 &\le A(u_h, \Gamma_h u_h) + G(p_h, p_h) \\ &\le \|f\|_0 \|\Gamma_h u_h\|_0 + 2c_2 c_4 h^{1/2} \|\nabla w_h\|_0^2 \|\nabla u_h\|_0^2 \\ &\le c_3 \gamma \|f\|_0 \|\nabla u_h\|_0 + \frac{4c_2 c_4 c_3 \gamma h^{1/2} \|f\|_0}{\nu} \|\nabla u_h\|_0^2 \end{aligned} \tag{3.7}$$

since

$$\begin{aligned} b(w_h, u_h, \Gamma_h u_h) &= b(w_h, u_h, \Gamma_h u_h - u_h) \\ &\le \left(\|w_h\|_{L^\infty} \|\nabla u_h\|_0 + \frac{\sqrt{3}}{2} \|u_h\|_{L^\infty} \|\nabla w_h\|_0 \right) \|\Gamma_h u_h - u_h\|_0 \\ &\le 2c_2 c_4 h^{1/2} \|\nabla w_h\|_0 \|\nabla u_h\|_0^2. \end{aligned} \tag{3.8}$$

Thus, we have

$$\nu \left(1 - \frac{4c_2 c_3 c_4 \gamma h^{1/2} \|f\|_0}{\nu^2} \right) \le c_3 \gamma \|f\|_0 \|\nabla u_h\|_0, \tag{3.9}$$

which implies

$$\|\nabla u_h\|_0 \le \frac{2c_3 \gamma}{\nu} \|f\|_0. \tag{3.10}$$

Then, using the definition of $b(\cdot;\cdot,\cdot)$ and $\mathcal{C}_h(\cdot,\cdot)$, (2.19), setting $c_8 = 2\max\{2c_2 c_4 h^{1/2}, c_0\}$, and the same approach as above gives that

$$\begin{aligned} \frac{\mathcal{C}_h((u_h, p_h), (v_h, q_h))}{\|\nabla v_h\|_0 + \|q_h\|_0} &\ge \beta \|p_h\|_0, \\ |b(w_h; u_h, \Gamma_h v_h)| &= |b(w_h; u_h, \Gamma_h v_h - v_h) + b(w_h; u_h, v_h)| \\ &\le c_8 \|\nabla w_h\|_0 \|\nabla u_h\|_0 \|\nabla v_h\|_0, \\ |(f, \Gamma_h v_h)| &\le \|f\|_0 \|\Gamma_h v_h\|_0 \\ &\le c_3 \gamma \|f\|_0 \|\nabla v_h\|_0, \end{aligned} \tag{3.11}$$

which, together with (3.10), gives

$$\|p_h\|_0 \le \beta^{-1} c_3 \gamma \|f\|_0 \left(4 c_3 c_8 \nu^{-2} \gamma \|f\|_0 + 1\right). \tag{3.12}$$

Since the mapping T_h is well defined, it follows from Brouwer's fixed-point theorem that there exists a solution to system (2.23).

To prove uniqueness, assume that (u_1, p_1) and (u_2, p_2) are two solutions to (2.23). Then we see that

$$\mathcal{C}_h\big((u_1 - u_2, p_1 - p_2), (v_h, q_h)\big) + b(u_1 - u_2; u_1, \Gamma_h v_h) + b(u_2; u_1 - u_2, \Gamma_h v_h) = 0. \tag{3.13}$$

Letting $(v_h, q_h) = (u_1 - u_2, p_1 - p_2) = (e, \eta)$, we obtain

$$\begin{aligned} \mathcal{C}_h\big((e,\eta),(e,\eta)\big) &\ge \nu \|\nabla e\|_0^2, \\ |b(e; u_1, \Gamma_h e) + b(u_2; e, \Gamma_h e)| &= |b(e; u_1, \Gamma_h e - e) + b(e; u_1, e) + b(u_2; e, \Gamma_h e - e)| \\ &\le 2 c_2 c_4 h^{1/2} (\|\nabla u_1\|_0 + \|\nabla u_2\|_0) \|\nabla e\|_0^2 + c_0 \|\nabla u_1\|_0 \|\nabla e\|_0^2 \\ &\le 2\left(\nu h_0 + c_0 c_5 \gamma \nu^{-1} \|f\|_0\right) \|\nabla e\|_0^2, \end{aligned} \tag{3.14}$$

which together with (3.3) and (3.13), gives

$$0 \le \nu (1 - 2 c_0 c_5 \gamma \nu^{-2} \|f\|_0) \|\nabla e\|_0^2 \le 0, \tag{3.15}$$

which shows that $e = 0$ by (3.15); that is, $u_1 = u_2$. Next, applying (3.3) to (3.13) and (2.34) yields that $p_1 = p_2$. Therefore, it follows that (2.23) has a unique solution. □

4. Optimal Error Estimates

Theorem 4.1 (optimal error and superconvergent results). *Assume that $h > 0$ satisfies* (3.2) *and $f \in Y$ and $\nu > 0$ satisfy* (3.2). *Let $(u, p) \in (X, M)$ and $(u_h, p_h) \in (X_h, M_h)$ be the solution of* (2.10) *and* (2.23), *respectively. Then it holds*

$$\|u - u_h\|_1 + \|p - p_h\|_0 \le \kappa h (\|u\|_2 + \|p\|_1 + \|f\|_0). \tag{4.1}$$

Also, if $f \in [H^1(\Omega)]^3$, there holds for the solution $(\overline{u}_h, \overline{p}_h)$ of (2.22) that

$$\|u_h - \overline{u}_h\|_1 + \|p_h - \overline{p}_h\|_0 \le \kappa h^{3/2} (\|u\|_2 + \|p\|_1 + \|f\|_1). \tag{4.2}$$

Proof. Subtracting (2.10) from (2.23) gives that

$$\mathcal{C}_h\big((e,\eta); (v_h, q_h)\big) + b(e, \overline{u}_h, \Gamma_h v_h) + b(u_h, e, v_h) + b(u_h, u_h, v_h - \Gamma_h v_h) = 0, \tag{4.3}$$

with $(e,\eta) = (\overline{u}_h - u_h, \overline{p}_h - p_h)$. By $(v_h, q_h) = (e,\eta)$, it follows that

$$\nu\|\nabla e\|_0^2 + G(\eta,\eta) + b(e,u_h,e) + b(u_h,u_h,\Gamma_h e - e) = (f, v_h - \Gamma_h v_h). \tag{4.4}$$

Using Theorem 3.1, (2.12), (2.23), and (2.25) gives

$$\begin{aligned}
b(e,u_h,e) &\le c_0\|\nabla u\|_0\|\nabla e\|_0^2 \\
&\le 2c_2c_4h^{1/2}\|\nabla u_h\|_0\|\nabla e\|_0^2 \\
&\le \frac{4c_2c_4c_5\gamma h^{1/2}\|f\|_0}{\nu^2}\|\nabla e\|_0^2 \\
&= \nu h_0\|\nabla e\|_0^2, \\
b(u_h,u_h,\Gamma_h e - e) &\le \left|\left(((u_h - \Pi_h u_h)\cdot\nabla)u_h + \frac{1}{2}\operatorname{div} u_h(u_h - \Pi_h u_h), e - \Gamma_h e\right)\right| \\
&\le \|\nabla u_h\|_\infty\|u_h - \Pi_h u_h\|_0\|e - \Gamma_h e\|_0 \\
&\le ch^{3/2}\|\nabla u_h\|_0\|\nabla e\|_0^2.
\end{aligned} \tag{4.5}$$

Similarly, by Lemma 2.1 and (2.25), we have

$$\begin{aligned}
|(f, e - \Gamma_h e)| &= |(f - \Pi_h f, e - \Gamma_h e)| \\
&\le Ch^i\|f\|_i\|e - \Gamma_h e\|_0 \\
&\le ch^{2(i+1)}\|f\|_i^2 + \frac{\nu_0}{4}\|e\|_1^2, \quad i = 0,1.
\end{aligned} \tag{4.6}$$

Combining the above inequalities with (4.3) gives

$$\|e\|_1 \le c\left(h^{3/2} + h^{i+1}\right)\|f\|_i, \quad i = 0,1. \tag{4.7}$$

In the same argument, it follows from (2.34) that

$$\|\eta\|_0 \le c\left(h^{3/2} + h^{i+1}\right)\|f\|_i, \quad i = 0,1. \tag{4.8}$$

Noting that [3]

$$\|u - u_h\|_1 + \|p - p_h\|_0 \le ch(\|u\|_2 + \|p\|_1 + \|f\|_0), \tag{4.9}$$

(4.6)–(4.8), and using a triangle inequality completes the proof of Theorem 4.1. □

As noted above, it is still difficult to achieve an optimal error estimate for the velocity in the L^2-norm for the three-dimensional stationary Navier-Stokes equations. Here, the following dual problem is proposed and analyzed:

$$a(v,\Phi) + d(v,\Psi) - d(\Phi,q) + b(u;v,\Phi) + b(v;u,\Phi) = (u - u_h, v). \tag{4.10}$$

Because of convexity of the domain Ω, this problem has a unique solution that satisfies the regularity property [18]

$$\|\Phi\|_2 + \|\Psi\|_1 \leq C\|u - u_h\|_0. \tag{4.11}$$

Below set $(\Phi_h, \Psi_h) = (I_h\Phi, J_h\Psi) \in (X_h, M_h)$, which satisfies, by (3.2),

$$\|\Phi - \Phi_h\|_0 + h(\|\Phi - \Phi_h\|_1 + \|\Psi - \Psi_h\|_0) \leq Ch^2(\|\Phi\|_2 + \|\Psi\|_1). \tag{4.12}$$

Theorem 4.2 (optimal L^2-error for the velocity). *Let (u, p) be the solution of* (2.1a)–(2.1c) *and let (u_h, p_h)be the solution of* (4.3). *Then, under the assumptions of Theorem 4.1, it holds*

$$\|u - u_h\|_0 \leq Ch^2\left(\|u\|_2 + \|p\|_1 + \|f\|_1\right). \tag{4.13}$$

Proof. Multiplying (2.1a) and (2.1b) by $\Gamma_h\Phi_h \in \widetilde{X}_h$ and $\Psi_h \in M_h$ and integrating over the dual elements $\widetilde{K}$ and the primary elements K, respectively, and adding the resulting equations to (2.23) with $(v_h, q_h) = (\Phi_h, \Psi_h)$, we see that

$$\begin{aligned} &A(e,\Gamma_h\Phi_h) + D\left(\Gamma_h\Phi_h,\eta\right) + d(e,\Psi_h) + G\left(\eta,\Psi_h\right) \\ &\quad + b(e;u,\Gamma_h\Phi_h) + b(u;e,\Gamma_h\Phi_h) - b(e;e,\Gamma_h\Phi_h) = G\left(p,\Psi_h\right), \end{aligned} \tag{4.14}$$

where $(e,\eta) = (u - u_h, p - p_h)$. Subtracting (4.14) from (4.10) with $(v,q) = (e,\eta)$ to obtain

$$\begin{aligned} \|e\|_0^2 &= a(e,\Phi - \Phi_h) + d(e,\Psi - \Psi_h) - d\left(\Phi - \Phi_h,\eta\right) - G\left(\eta,\Psi_h\right) + G\left(p,\Psi_h\right) + a(e,\Phi_h) \\ &\quad - A(e,\Gamma_h\Phi_h) - d\left(\Phi_h,\eta\right) - D\left(\Gamma_h\Phi_h,\eta\right) + b(u;e,\Phi - \Gamma_h\Phi_h) + b(e;u,\Phi - \Gamma_h\Phi_h) \\ &\quad + b(e;e,\Gamma_h\Phi_h) \\ &= a(e,\Phi - \Phi_h) + d(e,\Psi - \Psi_h) - d\left(\Phi - \Phi_h,\eta\right) - G\left(\eta,\Psi_h\right) + G\left(p,\Psi_h\right) + b(e;e,\Gamma_h\Phi_h) \\ &\quad + b(u;e,\Phi - \Gamma_h\Phi_h) + b(e;u,\Phi - \Gamma_h\Phi_h) + \left(f - (u\cdot\nabla)u,\Phi_h - \Gamma_h\Phi_h\right). \end{aligned} \tag{4.15}$$

Obviously, we deduce from Theorem 3.1, (2.27)–(2.29), (4.11), the inverse inequality (2.12), and the Cauchy inequality that

$$\begin{aligned}
\left|a(e,\Phi-\Phi_h)+d(e,\Psi-\Psi_h)-d(\Phi-\Phi_h,\eta)\right| &\le c\left(\|e\|_1+\|\eta\|_0\right)\left(\|\Phi-\Phi_h\|_1+\|\Psi-\Psi_h\|_0\right)\\
&\le ch^2\left(\|u\|_2+\|p\|_1\right)\left(\|\Phi\|_2+\|\Psi\|_1\right)\\
&\le ch^2\left(\|u\|_2+\|p\|_1\right)\|e\|_0,\\
\left|G(\eta,\Psi_h)-G(p,\Psi_h)\right| &\le ch\left(\|p-\Pi p\|_0+\|\eta\|_0\right)\|\Psi\|_1\\
&\le ch^2\left(\|u\|_2+\|p\|_1\right)\|e\|_0,\\
\left|(f-(u\cdot\nabla)u,\Phi_h-\Gamma_h\Phi_h)\right| &= \big|\big([f-\Pi_h f]\\
&\qquad -[(u\cdot\nabla)u-\Pi_h(u\cdot\nabla)u],\Phi_h-\Gamma_h\Phi_h\big)\big|\\
&\le ch^2\left(\|f\|_1+\|\nabla[(u\cdot\nabla)u]\|_0\right)\|\Phi_h\|_1\\
&\le ch^2\Big(\|f\|_1+\|u\|_0^{1/2}\|u\|_2^{3/2}+\|u\|_{1,4}^2\Big)\|e\|_0.\\
|b(u;e,\Phi-\Gamma_h\Phi_h)+b(e;u,\Phi-\Gamma_h\Phi_h)| &\le c\|u\|_2\|e\|_1\left(\|\Phi_h-\Gamma_h\Phi_h\|_0+\|\Phi-\Phi_h\|_0\right)\\
&\le ch^2\left(\|u\|_2+\|p\|_1\right)\|\Phi\|_1\\
&\le ch^2\left(\|u\|_2+\|p\|_1\right)\|e\|_0,\\
|b(e;e,\Gamma_h\Phi_h)| &= |b(e;e,\Gamma_h\Phi_h-\Phi_h)+b(e;e,\Phi_h)|\\
&\le c\Big(\|e\|_{0,4}\|e\|_1\|\Gamma_h\Phi_h-\Phi_h\|_{0,4}+\|e\|_1^2\|\Phi_h\|_1\Big)\\
&\le ch\|e\|_0^{1/4}\|e\|_1^{7/4}\|\nabla\Phi_h\|_{0,4}+c\|e\|_1^2\|\Phi_h\|_1\\
&\le ch^2\left(\|u\|_2+\|p\|_1\right)\|e\|_0.
\end{aligned} \tag{4.16}$$

Combining all these inequalities with (4.15) yields (4.13). □

In this paper, we have obtained optimal and convergent results of the stabilized mixed finite volume method for the stationary Navier-Stokes equations approximated by the low order finite elements. Furthermore, we could apply the same technique presented to develop and obtain the corresponding results of other (stabilized) mixed finite volume methods in two or three dimensions.

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