

## *Research Article*

# **Mixed Convection Boundary Layer Flow towards a Vertical Plate with a Convective Surface Boundary Condition**

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The steady mixed convection flow towards an impermeable vertical plate with a convective surface boundary condition is investigated. The governing partial differential equations are first reduced to ordinary differential equations using a similarity transformation, before being solved numerically. The features of the flow and heat transfer characteristics for different values of the governing parameters are analyzed and discussed. Both assisting and opposing flows are considered. The results indicate that dual solutions exist for the opposing flow, whereas for the assisting flow, the solution is unique. Moreover, increasing the convective parameter is to increase the skin friction coefficient and the heat transfer rate at the surface.

## **1. Introduction**

The study of heat transfer of combined free and forced convection flow has attracted the interest of many researchers over the last few decades. Mixed convection flows are important when the buoyancy forces significantly affect the flow and the thermal fields due to the large temperature difference between the wall and the ambient fluid. One of the early investigations of mixed convection towards a vertical surface was made by Ramachandran et al. [1], who studied the two-dimensional stagnation flows considering both cases of arbitrary wall temperature and arbitrary surface heat flux variations. Ali and Al-Yousef [2] considered the laminar flow over a moving vertical surface with suction or injection when the buoyancy forces assist or oppose the flow. A similar problem was studied by Lin and Hoh [3], where, in addition, the flow also arises from the interaction of the flowing free stream. Partha et al.

[4] studied the mixed convection from an exponentially stretching surface by considering the effect of buoyancy and viscous dissipation. Some other related works can also be found in the papers by Chen [5], Ali [6, 7], Ishak [8], Bachok et al. [9], and Lok et al. [10].

The aim of this paper is to study the two-dimensional mixed convection flow on a vertical plate with a convective surface boundary condition. The boundary layer flow concerning a convective boundary condition for the Blasius flow has been discussed by Aziz [11]. Bataller [12] investigated a similar problem by considering radiation effects on the Blasius and Sakiadis flows. The effects of suction and injection on a similar problem has been studied by Ishak [13], while Yao et al. [14] studied the flow and heat transfer characteristics of a generalized stretching/shrinking wall with convective boundary conditions. Recently, Merkin and Pop [15] studied the forced convection flow of a uniform stream over a flat surface with a convective surface boundary condition.

In the present paper, the governing equations are transformed into a system of nonlinear ordinary differential equations, which are then solved numerically. Representative results for the velocity and temperature profiles as well as the skin friction coefficient and the local Nusselt number, which represents the heat transfer rate at the surface, are presented for some values of the governing parameters.

## 2. Problem Formulation

Consider a two-dimensional steady boundary layer flow towards a vertical plate immersed in a viscous fluid of ambient temperature  $T_\infty$ . The external velocity is prescribed as  $u_e(x) = a\sqrt{x}$ , where  $a$  is a constant. It is assumed that the left surface of the plate is heated by convection from a hot fluid at temperature  $T_f$ , which provides a heat transfer coefficient  $h_f$ . Under the Boussinesq and boundary layer approximations, the governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_e \frac{du_e}{dx} + \nu \frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty), \quad (2.2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}, \quad (2.3)$$

where  $u$  and  $v$  are the velocity components along the  $x$ - and  $y$ -directions, respectively,  $T$  is the fluid temperature in the boundary layer,  $g$  is the acceleration due to gravity,  $\alpha$  is the thermal diffusivity,  $\beta$  is the thermal expansion coefficient, and  $\nu$  is the kinematic viscosity. The boundary conditions may be written as (Aziz [11])

$$u = 0, \quad v = 0, \quad -k \frac{\partial T}{\partial y} = h_f(T_f - T_w) \quad \text{at } y = 0, \quad (2.4)$$

$$u \rightarrow u_e, \quad T \rightarrow T_\infty \quad \text{at } y \rightarrow \infty,$$

where  $k$  is the thermal conductivity of the fluid,  $T_w$  is the plate temperature, and  $T_f > T_w > T_\infty$ .

In order to solve (2.1)–(2.3) subject to the boundary conditions in (2.4), we introduce the following similarity transformation (see Aziz [11] and Ishak [13]):

$$\eta = \left( \frac{u_e}{\nu x} \right)^{1/2} y, \quad \psi = (\nu x u_e)^{1/2} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}, \quad (2.5)$$

where  $\eta$  is the similarity variable,  $f$  is the dimensionless stream function,  $\theta$  is the dimensionless temperature, and  $\psi$  is the stream function defined as  $u = \partial\psi/\partial y$  and  $v = -\partial\psi/\partial x$ , which identically satisfies (2.1). Substituting (2.5) into (2.2) and (2.3), we obtain the following nonlinear ordinary differential equations:

$$f''' + \frac{3}{4} f f'' - \frac{1}{2} f'^2 + \frac{1}{2} + \lambda \theta = 0, \quad (2.6)$$

$$\frac{1}{\text{Pr}} \theta'' + \frac{3}{4} f \theta' = 0, \quad (2.7)$$

which are subject to the boundary conditions

$$\begin{aligned} f(0) = 0, \quad f'(0) = 0, \quad \theta'(0) = -\gamma[1 - \theta(0)], \\ f'(\eta) \rightarrow 1, \quad \theta(\eta) \rightarrow 0 \quad \text{as } \eta \rightarrow \infty. \end{aligned} \quad (2.8)$$

Here, primes denote differentiation with respect to  $\eta$ , and  $\lambda$  (= constant) is the buoyancy parameter defined as  $\lambda = Gr_x / Re_x^2$ , with  $Gr_x = g\beta(T_f - T_\infty)x^3/\nu^2$  and  $Re_x = u_e x/\nu$  being the local Grashof and Reynolds numbers, respectively.

In (2.8),  $\gamma$  is given by

$$\gamma = \frac{h_f x^{1/4}}{k} \left( \frac{\nu}{a} \right)^{1/2}. \quad (2.9)$$

For the energy equation (2.7) to have a similarity solution, the quantity  $\gamma$  must be a constant and not a function of  $x$  as in (2.9). This condition can be met if the heat transfer coefficient  $h_f$  is proportional to  $x^{-1/4}$ . We, therefore assume that

$$h_f = cx^{-1/4}, \quad (2.10)$$

where  $c$  is a constant. Thus, we have

$$\gamma = \frac{c}{k} \left( \frac{\nu}{a} \right)^{1/2}. \quad (2.11)$$

With  $\gamma$  defined by (2.11), the solutions of (2.6) and (2.7) yield the similarity solutions, while with  $\gamma$  defined by (2.9), the generated solutions are local similarity solutions.

### 3. Numerical Method

The system of boundary value problem (BVP) of (2.6)–(2.8) was solved numerically via the shooting technique [16–21] by converting it into an equivalent initial value problem (IVP). In this technique, we choose a suitable finite value of  $\eta_\infty$  (where  $\eta_\infty$  corresponds to  $\eta \rightarrow \infty$ ), which depends on the values of the parameters used. First, the system of (2.6) and (2.7) is reduced to a first-order system (by introducing new variables) as follows:

$$\begin{aligned} f' &= p, & p' &= q, & q' + \frac{3}{4}fq - \frac{1}{2}p^2 + \frac{1}{2} + \lambda\theta &= 0, \\ \theta' &= r, & \frac{1}{\text{Pr}}r' + \frac{3}{4}fr &= 0, \end{aligned} \quad (3.1)$$

with the boundary conditions

$$\begin{aligned} f(0) &= 0, & p(0) &= 0, & r(0) &= -\gamma[1 - \theta(0)], \\ p(\eta_\infty) &= 1, & \theta(\eta_\infty) &= 0. \end{aligned} \quad (3.2)$$

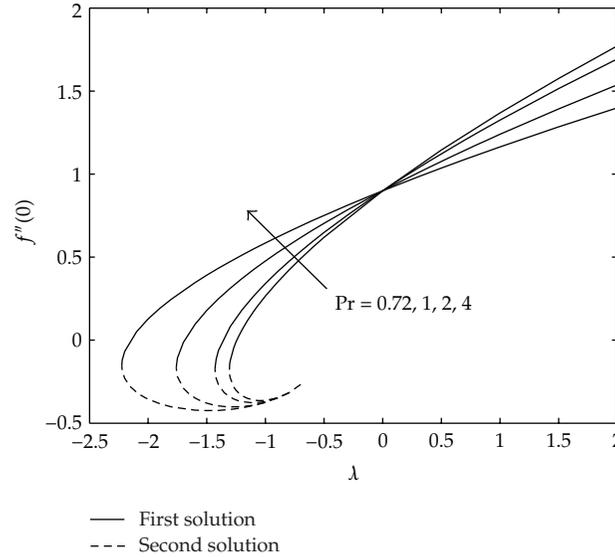
Now, we have a set of “partial” initial conditions

$$f(0) = 0, \quad p(0) = 0, \quad q(0) = \alpha_1, \quad \theta(0) = \alpha_2, \quad r(0) = -\gamma[1 - \alpha_2]. \quad (3.3)$$

A Runge-Kutta-Fehlberg method will be adopted to solve the applicable initial value problem. In order to integrate (3.1) as an IVP, we require a value for  $f''(0)$  and  $\theta(0)$ , that is,  $\alpha_1$  and  $\alpha_2$ , respectively. Since these values are not given in the boundary conditions in (3.2), a suitable guess values for  $f''(0)$  and  $\theta(0)$  are made, and integration is carried out. Then, we compare the calculated values for  $f'(\eta)$  and  $\theta(\eta)$  at  $\eta_\infty$  with the given boundary conditions  $f'(\eta_\infty) = 1$  and  $\theta(\eta_\infty) = 0$ , respectively, and adjust the estimated values of  $f''(0)$ ,  $\theta(0)$  and  $\eta_\infty$  to give a better approximation for the solution. This computation is done with the aid of shootlib file in Maple software. In this study, the boundary layer thickness  $\eta_\infty$  between 8 and 30 was used in the computation, depending on the values of the parameters considered, so that the boundary condition at “infinity” is achieved. For particular value of pertinent parameters, there is a possibility that two values of  $\eta_\infty$  are obtained, which gives two different velocity and temperature profiles that satisfy the boundary conditions. Consequently, this produces two different values of  $f''(0)$  and  $\theta(0)$ , respectively. As example for  $\text{Pr} = 0.72$ ,  $\gamma = 1$ ,  $\lambda = -1$ ,  $\eta_\infty \approx 11$  (small boundary layer thickness), and  $\eta_\infty \approx 30$  (large boundary layer thickness) were used to obtain first and second solutions, respectively. All velocity and temperature profiles for these two cases approached the infinity boundary conditions asymptotically, but with different boundary layer thicknesses.

### 4. Results and Discussion

The nonlinear ordinary differential equations (2.6) and (2.7) subject to the boundary conditions in (2.8) were solved numerically for some values of Prandtl number  $\text{Pr}$ , convective parameter  $\gamma$ , and buoyancy parameter  $\lambda$ . To validate the numerical results obtained, we also

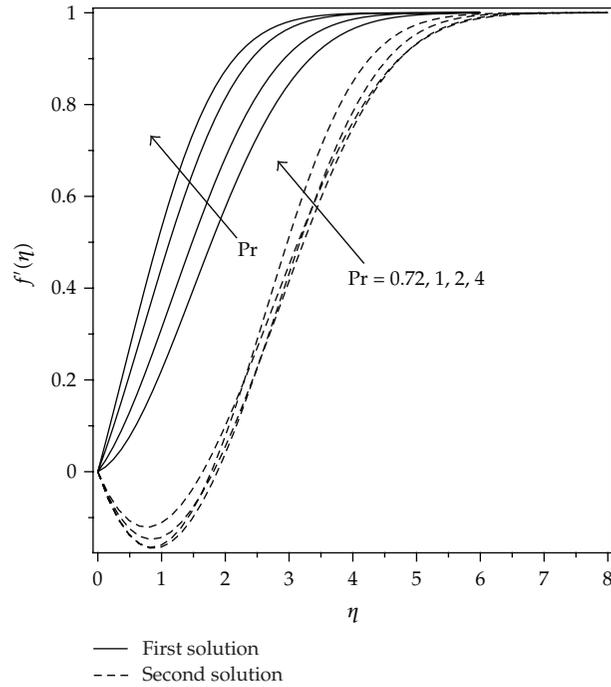


**Figure 1:** Variation of the skin friction coefficient  $f''(0)$  with  $\lambda$  for some values of Prandtl number  $Pr$  when  $\gamma = 1$ .

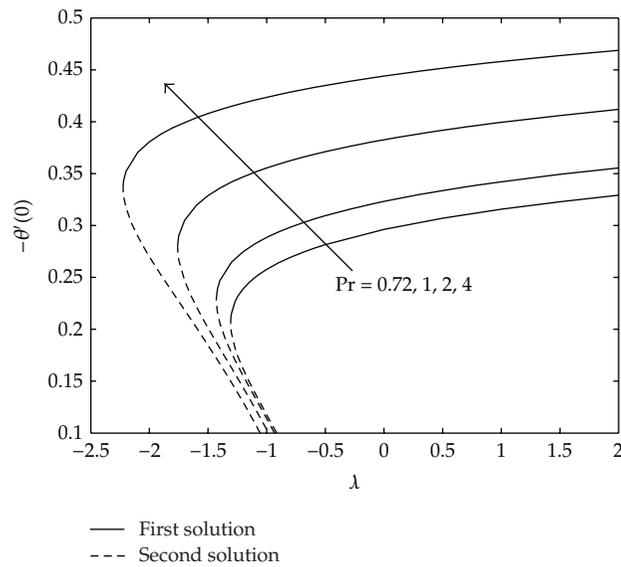
have solved this system of equations using `bvp4c` in Matlab software for certain values of parameters. The comparisons show excellent agreement between the two sets of results and so give confidence in our numerical approach.

The variation of the skin friction coefficient  $f''(0)$  with  $\lambda$  together with their velocity distributions for different values of  $Pr$  is shown in Figures 1 and 2, respectively, while the respective local Nusselt numbers  $-\theta'(0)$  together with their temperature distributions are shown in Figures 3 and 4. These velocity and temperature profiles support the validity of the numerical results obtained, besides supporting the dual nature of the solutions. In these figures, the solid lines and the dashed lines denote the first and second solutions, respectively. Figures 1 and 3 show that for the assisting flow ( $\lambda > 0$ ), there is a favorable pressure gradient due to the buoyancy force which increases the surface shear stress and the heat transfer rate at the surface. The increment is substantial in comparison to the no-buoyancy effect (forced convection). For the buoyancy-opposing flow ( $\lambda < 0$ ), dual solutions exist for certain range of the buoyancy parameter  $\lambda$ . The solution is unique for the assisting flow ( $\lambda > 0$ ). For each selected values of  $Pr$ , there is indeed a critical value  $\lambda_c$  of  $\lambda$  for which the solution exists. Based on our computations, we found that  $\lambda_c = -1.30718985, -1.42947390, -1.75963060,$  and  $-2.22432640$  for  $Pr = 0.72, 1, 2,$  and  $4$ , respectively. Therefore, the effect of the Prandtl number is to widen the range of the values of  $\lambda$  for which the solutions exist. It should be mentioned that the computations have been performed until the point where the solution does not converge, and the calculations were terminated at this location. It is worth mentioning that the existence of dual solutions in the mixed convection problems was also reported by Ramachandran et al. [1], Bachok et al. [9, 21], Lok et al. [10], Bhattacharyya and Layek [18], Bhattacharyya et al. [19], Afzal and Hussain [22], and Ishak et al. [23–26], among others.

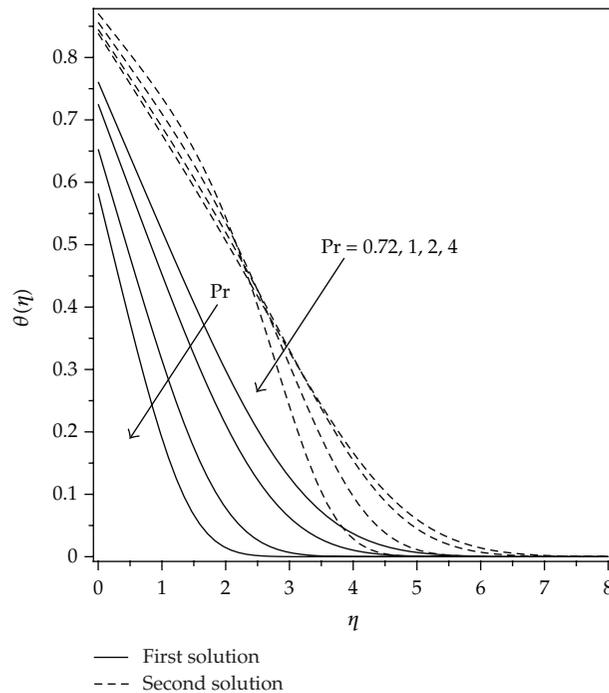
It is evident from Figures 2 and 4 that an increase in the Prandtl number results in an increase in both the skin friction coefficient and the local Nusselt number. This is because



**Figure 2:** Velocity profiles  $f'(\eta)$  for some values of Prandtl number  $Pr$  when  $\gamma = 1$  and  $\lambda = -1.2$ .



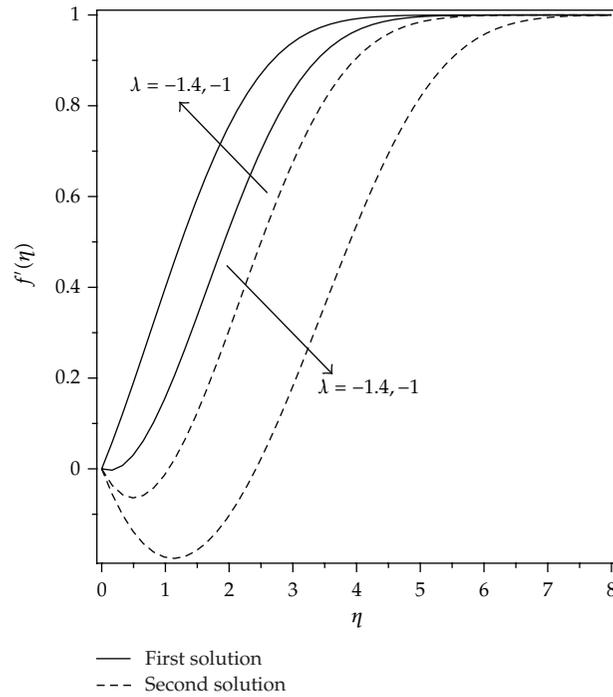
**Figure 3:** Variation of the heat transfer rate at the surface  $-\theta'(0)$  with  $\lambda$  for some values of Prandtl number  $Pr$  when  $\gamma = 1$ .



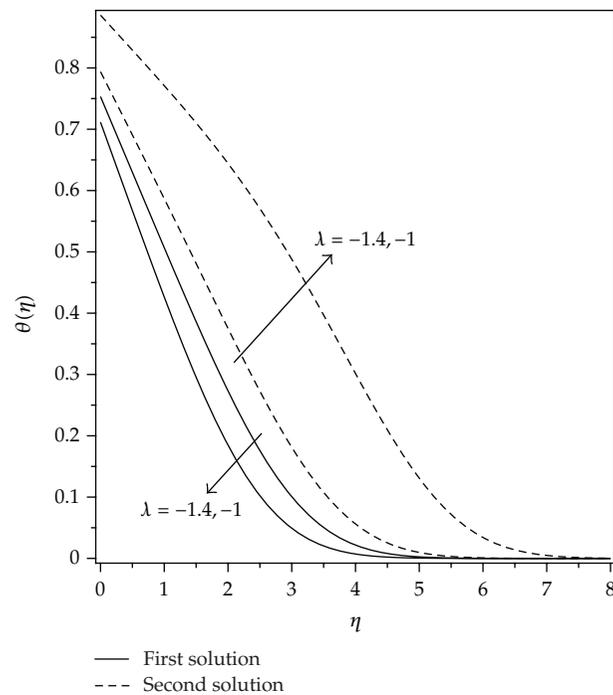
**Figure 4:** Temperature profiles  $\theta(\eta)$  for some values of Prandtl number  $Pr$  when  $\gamma = 1$  and  $\lambda = -1.2$ .

a higher Prandtl number fluid has a relatively low thermal conductivity, and thereby it reduces the thermal boundary layer thickness and in consequence increases the heat transfer rate at the surface (Char [27]). Moreover, the fluid on the right side of the plate is heated up by the hot fluid on the left surface of the plate, making it lighter and flow faster. These figures also show that the far-field boundary conditions (2.8) are satisfied asymptotically and hence support the validity of the numerical results obtained, besides supporting the existence of the dual solutions shown in Figures 1 and 3. It is interesting to note that from Figure 1, all curves intersect at  $f''(0) = 0.8997$ , that is, when  $\lambda = 0$  (forced convection flow). This is not surprising, since the flow field is uncoupled from the thermal field when  $\lambda = 0$ , which means that the Prandtl number does not affect the fluid velocity, hence the value of  $f''(0)$  remains the same when the buoyancy force is absent, which is clear from (2.6) and (2.7). From Figure 3, it is evident that the heat transfer rate at the surface is always greater than zero ( $-\theta'(0) > 0$ ), which means that the heat is transferred from the hot plate to the cool fluid on the right side.

Figures 5 and 6, respectively, present the velocity and temperature distributions for some values of buoyancy parameter ( $\lambda < 0$ ) when  $Pr = 1$ . It is obvious that the first solutions display a thinner boundary layer thickness compared to the second solutions. The effect of convective parameter  $\gamma$  on the velocity and temperature profiles of the impermeable plate when  $\lambda$  and  $Pr$  are set to unity can be seen in Figures 7 and 8, respectively. It is observed that a larger value of convective parameter  $\gamma$  produces a higher velocity and temperature gradients at the surface and therefore increasing the surface shear stress and the heat transfer rate at the surface. The temperature profiles are found to be qualitatively agreeing with those obtained by Aziz [11], who considered the boundary layer over a flat plate, and by Ishak [13], who reported the heat transfer over a static permeable flat plate. As reported by Aziz [11],



**Figure 5:** Velocity profiles  $f'(\eta)$  for some values of the buoyancy parameter  $\lambda$  when  $\gamma = 1$  and  $Pr = 1$ .



**Figure 6:** Temperature profiles  $\theta(\eta)$  for some values of the buoyancy parameter  $\lambda$  when  $\gamma = 1$  and  $Pr = 1$ .

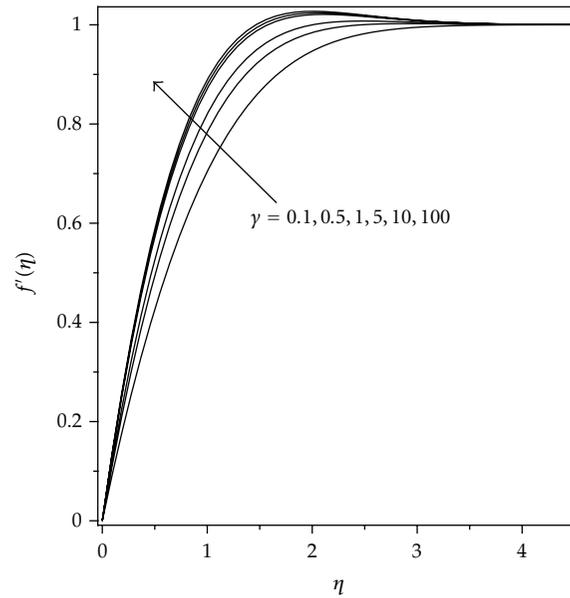


Figure 7: Velocity profiles  $f'(\eta)$  for some values of  $\gamma$  when  $Pr = 1$  and  $\lambda = 1$ .

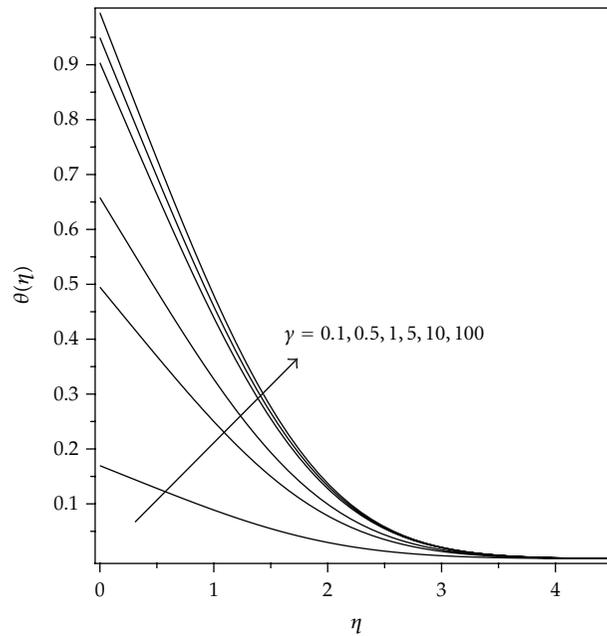


Figure 8: Temperature profiles  $\theta(\eta)$  for some values of  $\gamma$  when  $Pr = 1$  and  $\lambda = 1$ .

the parameter  $\gamma$  at any location  $x$  is proportional to the heat transfer coefficient associated with the hot fluid  $h_f$ . The thermal resistance on the hot fluid side is inversely proportional to  $h_f$ . Therefore, the hot fluid side convection resistance decreases as  $\lambda$  increases, and hence the surface temperature  $\theta(0)$  increases.

## 5. Conclusions

In this paper, the mixed convection boundary layer flow over an impermeable vertical plate with a convective surface boundary condition was studied. Similarity solutions for the flow and the thermal fields were obtained when the convective heat transfer from the left side of the plate is proportional to  $x^{-1/4}$ , where  $x$  is the distance from the leading edge. Using a numerical technique, the transformed governing equations were then solved to obtain the skin friction coefficient and the heat transfer rate at the surface as well as the velocity and temperature distributions for various values of the governing parameters, namely, Prandtl number  $Pr$ , buoyancy parameter  $\lambda$ , and convective parameter  $\gamma$ . It was found that both the skin friction coefficient and the heat transfer rate at the surface increase as  $\lambda$  increases for the selected values of  $Pr$  for the assisting flow ( $\lambda > 0$ ), while dual solutions were found to exist for the opposing flow ( $\lambda < 0$ ). Moreover, higher values of  $\gamma$  contribute to an increase in both the skin friction coefficient and the heat transfer rate at the surface.

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