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Research Article

Adaptive Control and Synchronization of the Shallow Water Model

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The shallow water model is one of the important models in dynamical systems. This paper investigates the adaptive chaos control and synchronization of the shallow water model. First, adaptive control laws are designed to stabilize the shallow water model. Then adaptive control laws are derived to chaos synchronization of the shallow water model. The sufficient conditions for the adaptive control and synchronization have been analyzed theoretically, and the results are proved using a Barbalat's Lemma.

1. Introduction

A dynamical system is a system that changes over time. Chaotic systems are dynamical systems that are highly sensitive to initial conditions. Chaos phenomena in weather models were first observed by Lorenz equation; a large number of chaos phenomena and chaos behavior have been discovered in physical, social, economical, biological, and electrical systems.

Atmosphere is a dynamical system. An atmospheric model is a set of equations that describes behavior of the atmosphere. The shallow water model is simple model for the atmosphere. Shallow water model is the set of the equations of motion that describes the evolution of a horizontal structure, hydrostatic homogeneous, and incompressible flow on the sphere [1].

The control of chaotic systems is to design state feedback control laws that stabilize the chaotic systems. Control theory is an interdisciplinary branch of engineering and mathematics that deals with the behavior of dynamical systems. The usual objective of control theory is to calculate solutions for the proper corrective action from the controller that result in system stability.

Synchronization of chaotic systems is phenomena that may occur when two or more chaotic oscillators are coupled or when a chaotic oscillator drives another chaotic oscillator, because of the butterfly effect, which causes the exponential divergence of the trajectories of two identical chaotic systems started with nearby the same initial conditions. Synchronizing two chaotic systems is seemingly a very challenging problem in chaos literature [2–6].

In 1990, Pecora and Caroll [7] introduced a method to synchronize two identical chaotic systems and showed that it was possible for some chaotic systems to be completely synchronized. From then on, chaos synchronization has been widely explored in variety of fields including physical system [8], chemical systems [9], ecological systems [10], secure communications [11, 12], and so forth.

In most of the chaos synchronization approaches, the drive-response formalism has been used. If a particular chaotic system is called the drive system and another chaotic system is called the response system, then the idea of synchronization is to use the output of the drive system to control the response system so that the output of the response system tracks the output of drive system asymptotically stable.

This paper is organized as follows. Section 2 gives notations and definitions of the stability in the chaotic system. Section 3 presents the adaptive control chaos of the shallow water model. Section 4 presents adaptive synchronization of the shallow water model. The conclusion discussion is in Section 5.

2. Notations and Definitions

X denotes an infinite dimensional Banach Space with the corresponding norm $\| \ \|$, R denotes the real line.

Consider a nonlinear nonautonomous differential equation of the general form

$$\begin{aligned} \dot{x}(t) &= f(t, x(t)), \quad t \geq t_0 \in R, \\ x(t_0) &= x_0, \end{aligned} \tag{2.1}$$

where the state $x(t)$ take values in X, $f(t,x) : R \times X \to X$ is a given nonlinear function and $f(t,0) = 0$, for all $t \in R$. The stability conditions were proposed and presented in [13].

Definition 2.1. The zero solution of (2.1) is said to be *stable* if for every $\varepsilon > 0, t_0 \in R$, there exists a number $\delta > 0$ (depending upon ε and t_0) such that for any solution $x(t)$ of (2.1) with $\|x_0\| < \delta$ implies $\|x(t)\| < \varepsilon$, for all $t \geq t_0$.

Definition 2.2. The zero solution of (2.1) is said to be *asymptotically stable* if it is stable and there is a number $\delta > 0$ such that any solution $x(t)$ with $\|x_0\| < \delta$ satisfies $\lim_{t \to \infty} \|x(t)\| = 0$.

Consider the control system

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t \geq 0, \tag{2.2}$$

where $u(t)$ is the external control input. The adaptive control is the control method to design state feedback control laws that stabilize the chaotic systems.

Definition 2.3. The control system (2.2) is *stabilizable* if there exists feedback control $u(t) = k(x(t))$ such that the system

$$\dot{x}(t) = f(t, x(t), k(x(t))), \quad t \geq 0, \tag{2.3}$$

is asymptotically stable.

Consider two nonlinear systems

$$\dot{x} = f(t, x(t)), \tag{2.4}$$

$$\dot{y} = g(t, y(t)) + u(t, x(t), y(t)), \tag{2.5}$$

where $x, y \in R$, $f, g \in C^r[R \times R, R]$, $u \in C^r[R \times R \times R, R]$, $r \geq 1$, R is the set of nonnegative real number. Assume that (2.4) is the drive system, (2.5) is the response system, and $u(t, x(t), y(t))$ is the control vector.

Definition 2.4. Response system and drive system are said to be *synchronic* if for any initial conditions $x(t_0)$, $y(t_0) \in R$, $\lim_{t \to \infty} \|x(t) - y(t)\| = 0$.

Lemma 2.5 (Barbalat's lemma as used in stability). *For nonautonomous system,*

$$\dot{x}(t) = f(t, x(t)) \tag{2.6}$$

If there exists a scalar function $V(x,t)$ such that

(1) *V has a lower bound,*

(2) $\dot{V} \leq 0$,

(3) *$\dot{V}(x,t)$ is uniformly continuous in time,*

then $\lim_{t \to \infty} \dot{V}(x,t) = 0$ *by applying the Barbalat's Lemma to stabilize the chaotic systems.*

3. Adaptive Control Chaos of the Shallow Water Model

A chaotic system has complex dynamical behaviors; those posses some special features, such as being extremely sensitive to tiny variations of initial conditions. In this section, adaptive control method is applied to control chaos shallow water model.

Shallow water model is the set of the equations of motion that describes the evolution of a horizontal structure, hydrostatic homogeneous, and incompressible flow on the sphere. Euler's equations of motion of an ideal fluid are as follows:

$$\begin{aligned}
\frac{Du}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv, \\
\frac{Dv}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu, \\
\frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,
\end{aligned} \tag{3.1}$$

where ρ is the density of the fluid, p is the pressure, g is the gravity, and f is coliolis parameter. Using the hydrostatic approximation,

$$\frac{\partial p}{\partial z} = -\rho g. \tag{3.2}$$

This implies $Dw/Dt = 0$. Assume the pressure p of fluid is constant, this implies that $\partial p/\partial t = 0$ and consider the continuity equation (or the incompressibility condition),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{3.3}$$

By solving for $\partial w/\partial z$ and integrating with respect to z, then w can be expressed as

$$\begin{gathered}\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \\ w = \int_0^h -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).\end{gathered} \tag{3.4}$$

The surface (of the fluid) boundary condition on w is that the fluid particles follow the surface (i.e., $Dh/Dt = w|_{\text{surface}}$). Thus

$$\frac{Dh}{Dt} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{3.5}$$

To get an expression for the pressure in the fluid, integrate the hydrostatic equation (3.2) from $p = 0$ at the top downward,

$$p(x,y,z) = \int_h^z -g\rho dz = (h-z)\rho g. \tag{3.6}$$

Take the partial derivatives of p (at the surface) with respect to x and y,

$$\begin{gathered}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}((h-z)\rho g) = \rho g \frac{\partial h}{\partial x} \Longrightarrow -\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial h}{\partial x}, \\ \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}((h-z)\rho g) = \rho g \frac{\partial h}{\partial y} \Longrightarrow -\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\frac{\partial h}{\partial y}.\end{gathered} \tag{3.7}$$

Taking (3.2)–(3.7) into (3.1), so the shallow water model in Cartesian coordinates is as follows:

$$\frac{Du}{Dt} = -g\frac{\partial h}{\partial x} + fv,$$
$$\frac{Dv}{Dt} = -g\frac{\partial h}{\partial y} - fu, \tag{3.8}$$
$$\frac{Dw}{Dt} = -h\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right].$$

In the vector form, the shallow water model is as follows:

$$\dot{V} = -fk \times V - \nabla\Phi,$$
$$\dot{\Phi} = -\Phi\nabla V, \tag{3.9}$$

where $\mathbf{V} = u\overrightarrow{i} + v\overrightarrow{j}$ is the horizontal velocity, $\Phi = gh$ is the geopotential height.

Consider the controlled system of (3.9) which has the form

$$\dot{V} = -fk \times V - \nabla\Phi + u_1,$$
$$\dot{\Phi} = -\Phi\nabla V + u_2, \tag{3.10}$$

where u_1, u_2 is external control input which will drag the chaotic trajectory (V, Φ) of the shallow water model to equilibrium point $E = (\overline{V}, \overline{\Phi})$ which is one of two steady states E_0, E_1.

In this case the control law is

$$u_1 = -g\left(V - \overline{V}\right), \qquad u_2 = -k\left(\Phi - \overline{\Phi}\right), \tag{3.11}$$

where k, g (estimate of k^*, g^*, resp.) are updated according to the following adaptive algorithm:

$$\dot{g} = \mu\left(V - \overline{V}\right)^2,$$
$$\dot{k} = \rho\left(\Phi - \overline{\Phi}\right)^2, \tag{3.12}$$

where μ, ρ is adaption gains. Then the controlled systems have the following form:

$$\dot{V} = -fk \times V - \nabla\Phi - g\left(V - \overline{V}\right), \tag{3.13}$$

$$\dot{\Phi} = -\Phi\nabla V - k\left(\Phi - \overline{\Phi}\right). \tag{3.14}$$

Theorem 3.1. *For $g < g^*$, $k < k^*$, the equilibrium point $E = (\overline{V}, \overline{\Phi})$ of the system* (3.13), (3.14) *is asymptotically stable.*

Proof. Let us consider the Lyapunov function

$$V(\xi_1, \xi_2, \xi_3) = \frac{1}{2}\left[\left(V - \overline{V}\right)^2 + \left(\Phi - \overline{\Phi}\right)^2 + \frac{1}{\mu}(g - g^*)^2 + \frac{1}{\rho}(k - k^*)^2\right]. \tag{3.15}$$

The time derivative of V is

$$\dot{V} = \left(V - \overline{V}\right)\dot{V} + \left(\Phi - \overline{\Phi}\right)\dot{\Phi} + \frac{1}{\mu}(g - g^*)\dot{g} + \frac{1}{\rho}(k - k^*)\dot{k}. \tag{3.16}$$

By substituting (3.13)-(3.14) in (3.16),

$$\begin{aligned}\dot{V} &= \left(V - \overline{V}\right)\left[-fk \times V - \nabla\Phi - g\left(V - \overline{V}\right)\right] + \left(\Phi - \overline{\Phi}\right)\left[-\Phi\nabla V - k\left(\Phi - \overline{\Phi}\right)\right] \\ &\quad + \frac{1}{\mu}(g - g^*)\mu\left(V - \overline{V}\right)^2 + \frac{1}{\rho}(k - k^*)\rho\left(\Phi - \overline{\Phi}\right)^2.\end{aligned} \tag{3.17}$$

Let $\eta_1 = (V - \overline{V})$, $\eta_2 = (\Phi - \overline{\Phi})$. Since $(\overline{V}, \overline{\Phi})$ is an equilibrium point of the uncontrolled system (3.9), $\dot{V}$ becomes

$$\begin{aligned}\dot{V} &= \eta_1\left[-fk \times V - \nabla\Phi - g\left(V - \overline{V}\right)\right] + \eta_2\left[-\Phi\nabla V - k\left(\Phi - \overline{\Phi}\right)\right] + (g - g^*)\eta_1^2 + (k - k^*)\eta_2^2 \\ &= (-fk \times V)\eta_1 - \nabla\Phi\eta_1 - g\eta_1^2 - \Phi\nabla V\eta_2 - k\eta_2^2 + (g - g^*)\eta_1^2 + (k - k^*)\eta_2^2.\end{aligned} \tag{3.18}$$

It is clear that if we choose $g < g^*$ and $k < k^*$, then $\dot{V}$ is negative semidefinite. Since V is positive definite and $\dot{V}$ is negative semidefinite, $\eta_1, \eta_2, g, k \in L_\infty$. From $\dot{V}(t) \leq 0$, we can easily show that the square of η_1, η_2 is integrable with respect to t, namely, η_1, $\eta_2 \in L_2$. From (3.13)-(3.14), for any initial conditions, we have $\dot{\eta}_1$, $\dot{\eta}_2 \in L_\infty$. By the well-known Barbalat's Lemma, we conclude that $\eta_1, \eta_2 \to (0, 0)$ as $t \to \infty$. Therefore, the equilibrium point $E = (\overline{V}, \overline{\Phi})$ of the system (3.13)-(3.14) is asymptotically stable. □

4. Adaptive Synchronization of the Shallow Water Model

In this section, the adaptive synchronization is introduced to make two of the shallow water model. The sufficient condition for the synchronization has been analyzed theoretically, and the result is proved using a Barbalat's Lemma. Assume that there are two shallow water models such that the drive system is to control the response system. The drive and response system are given as

$$\begin{aligned}\dot{V} &= -f_1 k_1 \times V_1 - \nabla\Phi_1, \\ \dot{\Phi} &= -\Phi_1 \nabla V_1, \\ \dot{V} &= -f_2 k_2 \times V_2 - \nabla\Phi_2 - u_1, \\ \dot{\Phi} &= -\Phi_2 \nabla V_2 - u_2\end{aligned} \tag{4.1}$$

where $u = [u_1, u_2]^T$ is the controller. We choose

$$\begin{aligned} u_1 &= k_1' e_V, \\ u_2 &= k_2' e_\Phi, \end{aligned} \tag{4.2}$$

where $k_1', k_2' \geq 0$ and e_V, e_Φ are the error states which are defined as follows

$$\begin{aligned} e_V &= V_2 - V_1, \\ e_\Phi &= \Phi_2 - \Phi_1. \end{aligned} \tag{4.3}$$

Theorem 4.1. *Let $k_1, f_1, k_1', k_2' \geq 0$ be property chosen so that the following matrix inequalities holds:*

$$P = \begin{pmatrix} k_1 f_1 + k_1' & 0 \\ 0 & k_2' \end{pmatrix} > 0, \tag{4.4}$$

then the two shallow water models (4.1) *can be synchronized under the adaptive control* (4.2).

Proof. It is easy to see from (4.1) that the error system is

$$\begin{aligned} \dot{e_V} &= -f_2 k_2 \times V_2 - \nabla\Phi_2 + f_1 k_1 \times V_1 + \nabla\Phi_1 - u_1, \\ \dot{e_\Phi} &= -\Phi_2 \nabla V_2 + \Phi_1 \nabla V_1 - u_2. \end{aligned} \tag{4.5}$$

Let $e_{kf} = k_2 f_2 - k_1 f_1$. Choose the Lyapunov function as follows:

$$V(t) = \frac{1}{2}\left[e_V^2 + e_\Phi^2\right]. \tag{4.6}$$

Then the differentiation of V along trajectories of (4.6) is

$$\begin{aligned} \dot{V}(t) &= e_V \dot{e_V} + e_\Phi \dot{e_\Phi} \\ &= e_V\left[-f_2 k_2 \times V_2 - \nabla\Phi_2 + f_1 k_1 \times V_1 + \nabla\Phi_1 - u_1\right] + e_\Phi\left[-\Phi_2 \nabla V_2 + \Phi_1 \nabla V_1 - u_2\right] \\ &= -e_V\left[f_2 k_2 \times V_2 + \nabla\Phi_2 - f_1 k_1 \times V_1 - \nabla\Phi_1 + u_1\right] - e_\Phi\left[\Phi_2 \nabla V_2 - \Phi_1 \nabla V_1 + u_2\right] \\ &= -e_V\left[f_2 k_2 \times V_2 - f_1 k_1 \times V_1 + f_1 k_1 \times V_2 - f_1 k_1 \times V_2\right] - e_V\left[\nabla\Phi_2 - \nabla\Phi_1\right] \\ &\quad - e_V u_1 - e_\Phi\left[\Phi_2 \nabla V_2 - \Phi_1 \nabla V_1 + \Phi_1 \nabla V_2 - \Phi_1 \nabla V_2\right] - e_\Phi u_8 \\ &= -e_V\left[e_{kf} \times V_2 + f_1 k_1 (V_2 - V_1)\right] - e_V \nabla(\Phi_2 - \Phi_1) - e_V k_1' e_V \\ &\quad - e_\Phi\left[(\Phi_2 - \Phi_1)\nabla V_2 + \Phi_1 \nabla(V_2 - V_1)\right] - e_\Phi k_2' e_\Phi \end{aligned}$$

$$
\begin{aligned}
&= -e_V\left[e_{kf} \times V_2 + f_1 k_1 e_V\right] - e_V \nabla e_\Phi - e_V^2 k_1' - e_\Phi\left[e_\Phi \nabla V_2 + \Phi_1 \nabla e_V\right] - e_\Phi^2 k_2' \\
&= -e_V e_{kf} \times V_2 + f_1 k_1 e_V^2 - e_V \nabla e_\Phi - e_V^2 k_1' - e_\Phi^2 \nabla V_2 - e_\Phi \Phi_1 \nabla e_V - e_\Phi^2 k_2' \\
&\le -f_1 k_1 e_V^2 - e_V^2 k_1' - e_\Phi^2 k_2' \\
&\le -\left(f_1 k_1 + k_1'\right) e_V^2 - k_2' e_\Phi^2 \\
&= -e^T P e,
\end{aligned}
\tag{4.7}
$$

where P is as in (4.4). Since $V(t)$ is positive definite and $\dot{V}(t)$ is negative semidefinite, it follows that $e_V, e_\Phi, k_1, f_1, k_1', k_2' \in L_\infty$. From $\dot{V}(t) \le -e^T P e$, we can easily show that the square of e_V, e_Φ is integrable with respect to t, namely, $e_V, e_\Phi \in L_2$. From (4.5), for any initial conditions, we have $\dot{e}_{V(t)}, \dot{e}_{\Phi(t)} \in L_\infty$. By the well-known Barbalat's Lemma, we conclude that $(e_V, e_\Phi) \to (0,0)$ as $t \to \infty$. Therefore, in the closed-loop system, $V_2(t) \to V_1(t), \Phi_2(t) \to \Phi_1(t)$ as $t \to \infty$. This implies that the two shallow water models have synchronized under the adaptive controls (4.2). □

5. Conclusions

In this paper, we applied adaptive control theory for the chaos control and synchronization of the shallow water model. First, we designed adaptive control laws to stabilize the shallow water model based on the adaptive control theory and stability theory. Then, we derived adaptive synchronization to the shallow water model. The sufficient conditions for the adaptive control and synchronization of the shallow water model have been analyzed theoretically, and the results are proved using a Barbalat's Lemma.

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