Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 746872, 16 pages
doi:10.1155/2012/746872

Research Article

Periodic Boundary Value Problems for Semilinear Fractional Differential Equations

Jia Mu and Yongxiang Li

Department of Mathematics, Northwest Normal University, Lanzhou, Gansu 730000, China

Correspondence should be addressed to Jia Mu, mujia88@163.com

Received 27 September 2011; Accepted 5 December 2011

Academic Editor: Kwok W. Wong

We study the periodic boundary value problem for semilinear fractional differential equations in an ordered Banach space. The method of upper and lower solutions is then extended. The results on the existence of minimal and maximal mild solutions are obtained by using the characteristics of positive operators semigroup and the monotone iterative scheme. The results are illustrated by means of a fractional parabolic partial differential equations.

1. Introduction

In this paper, we consider the periodic boundary value problem (PBVP) for semilinear fractional differential equation in an ordered Banach space X,

$$\begin{gathered} D^{\alpha}u(t) + Au(t) = f(t, u(t)), \quad t \in I, \\ u(0) = u(\omega), \end{gathered} \tag{1.1}$$

where D^{α} is the Caputo fractional derivative of order $0 < \alpha < 1$, $I = [0, \omega]$, $-A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup (i.e., strongly continuous semigroup) $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X, and $f : I \times X \to X$ is a continuous function.

Fractional calculus is an old mathematical concept dating back to the 17th century and involves integration and differentiation of arbitrary order. In a later dated 30th of September 1695, L'Hospital wrote to Leibniz asking him about the differentiation of order 1/2. Leibniz' response was "an apparent paradox from which one day useful consequences will be drawn." In the following centuries, fractional calculus developed significantly within pure mathematics. However, the applications of fractional calculus just emerged in the last few decades. The advantage of fractional calculus becomes apparent in science and engineering. In recent years, fractional calculus attracted engineers' attention, because it can describe the behavior of real dynamical systems in compact expressions, taking into account nonlocal characteristics like infinite memory [1–3]. Some instances are thermal diffusion phenomenon [4], botanical electrical impedances [5], model of love between humans [6], the relaxation of water on a porous dyke whose damping ratio is independent of the mass of moving water [7], and so forth. On the other hand, directing the behavior of a process with fractional-order controllers would be an advantage, because the responses are not restricted to a sum of exponential functions; therefore, a wide range of responses neglected by integer-order calculus would be approached [8]. For other advantages of fractional calculus, we can see real materials [9–13], control engineering [14, 15], electromagnetism [16], biosciences [17], fluid mechanics [18], electrochemistry [19], diffusion processes [20], dynamic of viscoelastic materials [21], viscoelastic systems [22], continuum and statistical mechanics [23], propagation of spherical flames [24], robotic manipulators [25], gear transmissions [26], and vibration systems [27]. It is well known that the fractional-order differential and integral operators are nonlocal operators. This is one reason why fractional differential operators provide an excellent instrument for description of memory and hereditary properties of various physical processes.

In recent years, there have been some works on the existence of solutions (or mild solutions) for semilinear fractional differential equations, see [28–36]. They use mainly Krasnoselskii's fixed-point theorem, Leray-Schauder fixed-point theorem, or contraction mapping principle. They established various criteria on the existence and uniqueness of solutions (or mild solutions) for the semilinear fractional differential equations by considering an integral equation which is given in terms of probability density functions and operator semigroups. Many partial differential equations involving time-variable t can turn to semilinear fractional differential equations in Banach spaces; they always generate an unbounded closed operator term A, such as the time fractional diffusion equation of order $\alpha \in (0,1)$, namely,

$$\partial_t^{\alpha} u(y,t) = Au(y,t), \quad t \geq 0,\ y \in R, \tag{1.2}$$

where A may be linear fractional partial differential operator. So, (1.1) has the extensive application value.

However, to the authors' knowledge, no studies considered the periodic boundary value problems for the abstract semilinear fractional differential equations involving the operator semigroup generator $-A$. Our results can be considered as a contribution to this emerging field. We use the method of upper and lower solutions coupled with monotone iterative technique and the characteristics of positive operators semigroup.

The method of upper and lower solutions has been effectively used for proving the existence results for a wide variety of nonlinear problems. When coupled with monotone iterative technique, one obtains the solutions of the nonlinear problems besides enabling the study of the qualitative properties of the solutions. The basic idea of this method is that using the upper and lower solutions as an initial iteration, one can construct monotone sequences, and these sequences converge monotonically to the maximal and minimal solutions. In some papers, some existence results for minimal and maximal solutions are obtained by establishing comparison principles and using the method of upper and lower solutions and the monotone iterative technique. The method requires establishing comparison theorems which play an important role in the proof of existence of minimal and maximal solutions. In abstract semilinear fractional differential equations, positive operators semigroup can play this role, see Li [37–41].

In Section 2, we introduce some useful preliminaries. In Section 3, in two cases: $T(t)$ is compact or noncompact, we establish various criteria on existence of the minimal and maximal mild solutions of PBVP (1.1). The method of upper and lower solutions coupled with monotone iterative technique, and the characteristics of positive operators semigroup are applied effectively. In Section 4, we give also an example to illustrate the applications of the abstract results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

If $-A$ is the infinitesimal generator of a C_0-semigroup in a Banach space, then $-(A + qI)$ generates a uniformly bounded C_0-semigroup for $q > 0$ large enough. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of a C_0-semigroup to the case in which the semigroup is uniformly bounded. Hence, for convenience, throughout this paper, we suppose that $-A$ is the infinitesimal generator of a uniformly bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$. This means that there exists $M \geq 1$ such that

$$\|T(t)\| \leq M, \quad t \geq 0. \tag{2.1}$$

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1 (see [9, 32]). The fractional integral of order α with the lower limit zero for a function $f \in AC[0,\infty)$is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,\ 0 < \alpha < 1, \tag{2.2}$$

provided the right side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 (see [9, 32]). The Riemann-Liouville derivative of order α with the lower limit zero for a function $f \in AC[0,\infty)$ can be written as

$${}^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\alpha}} ds, \quad t > 0,\ 0 < \alpha < 1. \tag{2.3}$$

Definition 2.3 (see [9, 32]). The Caputo fractional derivative of order α for a function $f \in AC[0,\infty)$ can be written as

$$D^{\alpha} f(t) = {}^{L}D^{\alpha}\big(f(t) - f(0)\big), \quad t > 0,\ 0 < \alpha < 1. \tag{2.4}$$

Remark 2.4 (see [32]). (i) If $f \in C^1[0,\infty)$, then

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds, \quad t > 0,\ 0 < \alpha < 1. \tag{2.5}$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then the integrals and derivatives which appear in Definitions 2.1–2.3 are taken in Bochner's sense.

For more fractional theories, one can refer to the books [9, 42–44].

Throughout this paper, let X be an ordered Banach space with norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $P = \{y \in X \mid y \geq \theta\}$ (θ is the zero element of X) is normal with normal constant N. X_1 denotes the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. Let $C(I,X)$ be the Banach space of all continuous X-value functions on interval I with norm $\|u\|_C = \max_{t\in I}\|u(t)\|$. For $u, v \in C(I,X)$, $u \leq v$ if $u(t) \leq v(t)$ for all $t \in I$. For $v, w \in C(I,X)$, denote the ordered interval $[v,w] = \{u \in C(I,X) \mid v \leq u \leq w\}$ and $[v(t), w(t)] = \{y \in X \mid v(t) \leq y \leq w(t)\}$, $t \in I$. Set $C^\alpha(I,X) = \{u \in C(I,X) \mid D^\alpha u$ exists and $D^\alpha u \in C(I,X)\}$.

Definition 2.5. If $v_0 \in C^\alpha(I,X) \cap C(I,X_1)$ and satisfies

$$\begin{gathered} D^\alpha v_0(t) + Av_0(t) \leq f(t, v_0(t)), \quad t \in I, \\ v_0(0) \leq v(\omega), \end{gathered} \tag{2.6}$$

then v_0 is called a lower solution of PBVP (1.1); if all inequalities of (2.6) are inverse, one calls it an upper solution of PBVP (1.1).

Definition 2.6 (see [29, 45]). If $h \in C(I,X)$, by the mild solution of LIVP,

$$\begin{gathered} D^\alpha u(t) + Au(t) = h(t), \quad t \in I, \\ u(0) = x_0 \in X, \end{gathered} \tag{2.7}$$

one means that the function $u \in C(I,X)$ and satisfies

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{\alpha-1} V(t-s)h(s)ds, \tag{2.8}$$

where

$$U(t) = \int_0^\infty \zeta_\alpha(\theta) T(t^\alpha\theta) d\theta, \qquad V(t) = \alpha \int_0^\infty \theta\zeta_\alpha(\theta) T(t^\alpha\theta) d\theta, \tag{2.9}$$

$$\begin{gathered} \zeta_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\rho_\alpha\left(\theta^{-1/\alpha}\right), \\ \rho_\alpha(\theta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty), \end{gathered} \tag{2.10}$$

and $\zeta_\alpha(\theta)$ is a probability density function defined on $(0,\infty)$.

Remark 2.7. (i) [29–31] $\zeta_\alpha(\theta) \geq 0, \theta \in (0,\infty)$, $\int_0^\infty \zeta_\alpha(\theta)d\theta = 1$, and $\int_0^\infty \theta\zeta_\alpha(\theta)d\theta = 1/\Gamma(1+\alpha)$.

(ii) [33, 34, 46, 47] The Laplace transform of ζ_α is given by

$$\int_0^\infty e^{-p\theta}\zeta_\alpha(\theta)d\theta = \sum_{n=0}^\infty \frac{(-p)^n}{\Gamma(1+n\alpha)} = E_\alpha(-p), \tag{2.11}$$

where $E_\alpha(\cdot)$ is Mittag-Leffler function (see [42]).

(iii) [48] For $p < 0$, $0 < E_\alpha(p) < E_\alpha(0) = 1$.

Lemma 2.8. *If $\{T(t)\}_{t\geq 0}$ is an exponentially stable C_0-semigroup, there are constants $N \geq 1$ and $\delta > 0$, such that*

$$\|T(t)\| \leq Ne^{-\delta t}, \quad t \geq 0, \tag{2.12}$$

then the linear periodic boundary value problem (LPBVP)

$$\begin{gathered} D^\alpha u(t) + Au(t) = h(t), \quad t \in I, \\ u(0) = u(\omega) \end{gathered} \tag{2.13}$$

has a unique mild solution

$$(Ph)(t) = U(t)B(h) + \int_0^t (t-s)^{\alpha-1}V(t-s)h(s)ds, \tag{2.14}$$

where $U(t)$ and $V(t)$ are given by (2.9)

$$B(h) = (I - U(\omega))^{-1}\int_0^\omega (\omega-s)^{\alpha-1}V(\omega-s)h(s)ds. \tag{2.15}$$

Proof. In X, give equivalent norm $|\cdot|$ by

$$|x| = \sup_{t\geq 0}\left\|e^{\delta t}T(t)x\right\|, \tag{2.16}$$

then $\|x\| \leq |x| \leq N\|x\|$. By $|T(t)|$, we denote the norm of $T(t)$ in $(X, |\cdot|)$, then for $t \geq 0$,

$$\begin{aligned} |T(t)x| &= \sup_{s\geq 0}\left\|e^{\delta s}T(s)T(t)x\right\| \\ &= e^{-\delta t}\sup_{s\geq 0}\left\|e^{\delta(s+t)}T(s+t)x\right\| \\ &= e^{-\delta t}\sup_{\eta\geq t}\left\|e^{\delta\eta}T(\eta)x\right\| \\ &\leq e^{-\delta t}|x|. \end{aligned} \tag{2.17}$$

Thus, $|T(t)| \leq e^{-\delta t}$. Then by Remark 2.7,

$$\begin{aligned} |U(\omega)| &= \left| \int_0^\infty \zeta_\alpha(\theta) T(\omega^\alpha \theta) d\theta \right| \\ &\leq \int_0^\infty \zeta_\alpha(\theta) e^{-\delta \omega^\alpha \theta} d\theta \\ &= E_\alpha(-\delta \omega^\alpha) < 1. \end{aligned} \tag{2.18}$$

Therefore, $I - U(\omega)$ has bounded inverse operator and

$$(I - U(\omega))^{-1} = \sum_{n=0}^{\infty} (U(\omega))^n. \tag{2.19}$$

Set

$$x_0 = (I - U(\omega))^{-1} \int_0^\omega (\omega - s)^{\alpha-1} V(\omega - s) h(s) ds, \tag{2.20}$$

then

$$u(t) = U(t)x_0 + \int_0^t (t - s)^{\alpha-1} V(t - s) h(s) ds \tag{2.21}$$

is the unique mild solution of LIVP (2.7) and satisfies $u(0) = u(\omega)$. So set

$$\begin{aligned} B(h) &= (I - U(\omega))^{-1} \int_0^\omega (\omega - s)^{\alpha-1} V(\omega - s) h(s) ds, \\ (Ph)(t) &= U(t)B(h) + \int_0^t (t - s)^{\alpha-1} V(t - s) h(s) ds, \end{aligned} \tag{2.22}$$

then Ph is the unique mild solution of LPBVP (2.13). □

Remark 2.9. For sufficient conditions of exponentially stable C_0-semigroup, one can see [49].

Definition 2.10. A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called a compact semigroup if $T(t)$ is compact for $t > 0$.

Definition 2.11. A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called an equicontinuous semigroup if $T(t)$ is continuous in the uniform operator topology (i.e., uniformly continuous) for $t > 0$.

Remark 2.12. Compact semigroups, differential semigroups, and analytic semigroups are equicontinuous semigroups, see [50]. In the applications of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroups are analytic semigroups.

Definition 2.13. A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called a positive semigroup if $T(t)x \geq \theta$ for all $x \geq \theta$ and $t \geq 0$.

Remark 2.14. From Definition 2.13, if $h \geq \theta$, $x_0 \geq \theta$, and $T(t)(t \geq 0)$ is a positive C_0-semigroup generated by $-A$, the mild solution $u \in C(I, X)$ given by (2.8) satisfies $u \geq \theta$. For the applications of positive operators semigroup, we can see [37–41]. It is easy to see that positive operators semigroup can play the role as the comparison principles.

Definition 2.15. A bounded linear operator K on X is called to be positive if $Kx \geq \theta$ for all $x \geq \theta$.

Lemma 2.16. *The operators U and V given by* (2.9) *have the following properties:*

(i) *For any fixed $t \geq 0$, $U(t)$ and $V(t)$ are linear and bounded operators, that is, for any $x \in X$,*

$$\|U(t)x\| \leq M\|x\|, \qquad \|V(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\|, \tag{2.23}$$

(ii) *$\{U(t)\}_{t\geq 0}$ and $\{V(t)\}_{t\geq 0}$ are strongly continuous,*

(iii) *$\{U(t)\}_{t\geq 0}$ and $\{V(t)\}_{t\geq 0}$ are compact operators if $\{T(t)\}_{t\geq 0}$ is a compact semigroup,*

(iv) *$U(t)$ and $V(t)$ are continuous in the uniform operator topology (i.e., uniformly continuous) for $t > 0$ if $\{T(t)\}_{t\geq 0}$ is an equicontinuous semigroup,*

(v) *$U(t)$ and $V(t)$ are positive for $t \geq 0$ if $\{T(t)\}_{t\geq 0}$ is a positive semigroup,*

(vi) *$(I - U(\omega))^{-1}$ is a positive operator if $\{T(t)\}_{t\geq 0}$ is an exponentially and positive semigroup.*

Proof. For the proof of (i)–(iii), one can refer to [29, 31]. We only check (iv), (v), and (vi) as follows.

(iv) For $0 < t_1 \leq t_2$, we have

$$\begin{aligned} \|U(t_2) - U(t_1)\| &\leq \int_0^\infty \zeta_\alpha(\theta)\left\|T(t_2^\alpha\theta) - T(t_1^\alpha\theta)\right\|d\theta, \\ \|V(t_2) - V(t_1)\| &\leq \alpha\int_0^\infty \theta\zeta_\alpha(\theta)\left\|T(t_2^\alpha\theta) - T(t_1^\alpha\theta)\right\|d\theta. \end{aligned} \tag{2.24}$$

Since $T(t)$ is continuous in the uniform operator topology for $t > 0$, by Lebesque-dominated convergence theorem and Remark 2.7 (i), $U(t)$ and $V(t)$ are continuous in the uniform operator topology for $t > 0$.

(v) By Remark 2.7 (i), the proof is then complete.

(vi) By (v), (2.18), and (2.19), the proof is then complete.

□

3. Main Results

Case 1. $\{T(t)\}_{t\geq 0}$ is compact.

Theorem 3.1. *Assume that $\{T(t)\}_{t\geq 0}$ is a compact and positive semigroup in X, PBVP* (1.1) *has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$ and satisfies the following.*

(H) *There exists a constant* $C > 0$ *such that*

$$f(t, x_2) - f(t, x_1) \geq -C(x_2 - x_1), \tag{3.1}$$

for any $t \in I$, *and* $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$, *that is,* $f(t, x) + Cx$ *is increasing in* x *for* $x \in [v_0(t), w_0(t)]$.

Then PBVP (1.1) *has the minimal and maximal mild solutions between* v_0 *and* w_0, *which can be obtained by a monotone iterative procedure starting from* v_0 *and* w_0, *respectively.*

Proof. It is easy to see that $-(A + CI)$ generates an exponentially stable and positive compact semigroup $S(t) = e^{-Ct}T(t)$. By (2.1), $\|S(t)\| \leq M$. Let $\Phi(t) = \int_0^\infty \zeta_\alpha(\theta) S(t^\alpha \theta) d\theta, \Psi(t) = \alpha \int_0^\infty \theta \zeta_\alpha(\theta) S(t^\alpha \theta) d\theta$. By Remark 2.7 (i), we have that

$$\|\Phi(t)\| \leq M, \quad \|\Psi(t)\| \leq \frac{\alpha}{\Gamma(1 + \alpha)} M, \quad t \geq 0. \tag{3.2}$$

From Lemma 2.8, $(I - \Phi(\omega))$ has bounded inverse operator and

$$(I - \Phi(\omega))^{-1} = \sum_{n=0}^{\infty} (\Phi(\omega))^n. \tag{3.3}$$

By Lemma 2.16 (v) and (vi), $\Phi(t)$ and $\Psi(t)$ are positive for $t \geq 0$, and $(I - \Phi(\omega))^{-1}$ is positive.

Let $D = [v_0, w_0]$, then we define a mapping $Q : D \to C(I, X)$ by

$$Qu(t) = \Phi(t) B_1(u) + \int_0^t (t - s)^{\alpha - 1} \Psi(t - s) [f(s, u(s)) + Cu(s)] ds, \quad t \in I, \tag{3.4}$$

where

$$B_1(u) = (I - \Phi(\omega))^{-1} \int_0^\omega (\omega - s)^{\alpha - 1} \Psi(\omega - s) [f(s, u(s)) + Cu(s)] ds. \tag{3.5}$$

By the continuity of f and Lemma 2.16 (ii), $Q : D \to C(I, X)$ is continuous. By Lemma 2.8, $u \in D$ is a mild solution of PBVP (1.1) if and only if

$$u = Qu. \tag{3.6}$$

For $u_1, u_2 \in D$ and $u_1 \leq u_2$, from (H), the positivity of operators $(I - \Phi(\omega))^{-1}$, $\Phi(t)$, and $\Psi(t)$, we have that

$$Qu_1 \leq Qu_2. \tag{3.7}$$

Now, we show that $v_0 \le Qv_0$, $Qw_0 \le w_0$. Let $D^\alpha v_0(t) + Av_0(t) + Cv_0(t) \triangleq \sigma(t)$, by Definition 2.5, the positivity of operator $\Psi(t)$, we have that

$$
\begin{aligned}
v_0(t) &= \Phi(t)v_0(0) + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\sigma(s)ds \\
&\le \Phi(t)v_0(0) + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[f(s,v_0(s)) + Cv_0(s)\right]ds, \quad t \in I.
\end{aligned}
\tag{3.8}
$$

In particular,

$$
v_0(\omega) \le \Phi(\omega)v_0(0) + \int_0^\omega (\omega-s)^{\alpha-1}\Psi(\omega-s)\left[f(s,v_0(s)) + Cv_0(s)\right]ds. \tag{3.9}
$$

By Definition 2.5, $v_0(0) \le v(\omega)$, and by the positivity of operator $(I-\Phi(\omega))^{-1}$, we have that

$$
v_0(0) \le (I-\Phi(\omega))^{-1}\int_0^\omega (\omega-s)^{\alpha-1}\Psi(\omega-s)\left[f(s,v_0(s)) + Cv_0(s)\right]ds = B_1(v_0). \tag{3.10}
$$

Then by (3.8) and the positivity of operator $\Phi(t)$,

$$
\begin{aligned}
v_0(t) &\le \Phi(t)B_1(v_0) + \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[f(s,v_0(s)) + Cv_0(s)\right]ds \\
&= (Qv_0)(t), \quad t \in I,
\end{aligned}
\tag{3.11}
$$

namely, $v_0 \le Qv_0$. Similarly, we can show that $Qw_0 \le w_0$. For $u \in D$, in view of (3.7), then $v_0 \le Qv_0 \le Qu \le Qw_0 \le w_0$. Thus, $Q : D \to D$ is a continuous increasing monotonic operator. We can now define the sequences

$$
v_n = Qv_{n-1}, \quad w_n = Qw_{n-1}, \quad n = 1, 2, \ldots, \tag{3.12}
$$

and it follows from (3.7) that

$$
v_0 \le v_1 \le \cdots v_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0. \tag{3.13}
$$

In the following, we prove that $\{v_n\}$ and $\{w_n\}$ are convergent in $C(I,X)$. First, we show that $QD = \{Qu \mid u \in D\}$ is precompact in $C(I,X)$. Let

$$
(Wu)(t) = \int_0^t (t-s)^{\alpha-1}\Psi(t-s)\left[f(s,u(s)) + Cu(s)\right]ds, \quad t \in I, \tag{3.14}
$$

then we prove that for all $0 < t \leq \omega$, $(WD)(t) = \{(Wu)(t) \mid u \in D\}$ is precompact in X. For $0 < \varepsilon < t$, let

$$\begin{aligned}
(W_\varepsilon u)(t) &= \int_0^{t-\varepsilon} (t-s)^{\alpha-1}\Psi(t-s)\left[f(s,u(s)) + Cu(s)\right]ds \\
&= \int_0^{t-\varepsilon} (t-s)^{\alpha-1}\left(\alpha\int_0^\infty \theta\zeta_\alpha(\theta)S\left((t-s)^\alpha\theta\right)d\theta\right)\left[f(s,u(s)) + Cu(s)\right]ds \\
&= S(\varepsilon)\int_0^{t-\varepsilon} (t-s)^{\alpha-1}\left(\alpha\int_0^\infty \theta\zeta_\alpha(\theta)S\left((t-s)^\alpha\theta-\varepsilon\right)d\theta\right)\left[f(s,u(s)) + Cu(s)\right]ds.
\end{aligned} \tag{3.15}$$

For $u \in D$, by (H), $f(t,v_0(t)) + Cv_0(t) \leq f(t,u(t)) + Cu(t) \leq f(t,w_0(t)) + Cw_0(t)$ for $0 \leq t \leq \omega$. By the normality of the cone P, there is $M_1 > 0$ such that

$$\left\|f(t,u(t)) + Cu(t)\right\| \leq M_1, \quad 0 \leq t \leq \omega. \tag{3.16}$$

Thus, by (3.16) and Remark 2.7 (i), we have

$$\begin{aligned}
&\left\|\int_0^{t-\varepsilon} (t-s)^{\alpha-1}\left(\alpha\int_0^\infty \theta\zeta_\alpha(\theta)S\left((t-s)^\alpha\theta-\varepsilon\right)d\theta\right)\left[f(s,u(s)) + Cu(s)\right]ds\right\| \\
&\quad\leq M_1\int_0^{t-\varepsilon} (t-s)^{\alpha-1}\left(\alpha\int_0^\infty \theta\zeta_\alpha(\theta)\left\|S\left((t-s)^\alpha\theta-\varepsilon\right)\right\|d\theta\right)\,ds \\
&\quad\leq MM_1\int_0^{t-\varepsilon} (t-s)^{\alpha-1}\left(\alpha\int_0^\infty \theta\zeta_\alpha(\theta)d\theta\right)ds \\
&\quad= MM_1\frac{\alpha}{\Gamma(1+\alpha)}\int_0^{t-\varepsilon} (t-s)^{\alpha-1}ds \\
&\quad= MM_1\frac{(t^\alpha-\varepsilon^\alpha)}{\Gamma(1+\alpha)}, \quad 0 < t \leq \omega.
\end{aligned} \tag{3.17}$$

Then by (3.15), (3.17) and the compactness of $S(\varepsilon)$, for $0 < t \leq \omega$, $(W_\varepsilon D)(t) = \{(W_\varepsilon u)(t) \mid u \in D\}$ is precompact in X. Furthermore, by (3.16) and Lemma 2.16 (i), we have

$$\begin{aligned}
\|(Wu)(t) - (W_\varepsilon u)(t)\| &= \left\|\int_{t-\varepsilon}^{t} (t-s)^{\alpha-1}\Psi(t-s)\left[f(s,u(s)) + Cu(s)\right]ds\right\| \\
&\leq MM_1\frac{\alpha}{\Gamma(1+\alpha)}\int_{t-\varepsilon}^{t} (t-s)^{\alpha-1}ds \\
&= MM_1\frac{\varepsilon^\alpha}{\Gamma(1+\alpha)}.
\end{aligned} \tag{3.18}$$

Therefore, for $0 < t \leq \omega$, $(WD)(t)$ is precompact in X. In particular, $(WD)(\omega)$ is precompact in X, and then $B_1(D) = (I-\Phi(\omega))^{-1}(WD)(\omega)$ is precompact. Then in view of Lemma 2.16 (i), $(QD)(t) = \{(Qu(t)) \mid u \in D\} = \Phi(t)B_1(D) + (WD)(t)$ is precompact in X for $0 \leq t \leq \omega$.

Furthermore, for $0 \le t_1 < t_2 \le \omega$, by (3.16) and Lemma 2.16 (i) we have that

$$\begin{aligned}
\|(Wu)(t_2) - (Wu)(t_1)\| &= \left\| \int_0^{t_2} (t_2 - s)^{\alpha-1}\Psi(t_2 - s)\left[f(s,u(s)) + Cu(s)\right]ds \right. \\
&\qquad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1}\Psi(t_1 - s)\left[f(s,u(s)) + Cu(s)\right]ds \right\| \\
&\le M_1 \int_0^{t_1} \left\| (t_2 - s)^{\alpha-1}\Psi(t_2 - s) - (t_1 - s)^{\alpha-1}\Psi(t_1 - s) \right\| ds \\
&\quad + MM_1 \frac{\alpha}{\Gamma(1+\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
&\le M_1 \int_0^{t_1} (t_2 - s)^{\alpha-1} \|\Psi(t_2 - s) - \Psi(t_1 - s)\| ds \\
&\quad + M_1 \int_0^{t_1} \left\| \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right]\Psi(t_1 - s) \right\| ds + \frac{MM_1}{\Gamma(1+\alpha)}(t_2 - t_1)^{\alpha} \\
&\le M_1 (t_2 - t_1)^{\alpha-1} \int_0^{t_1} \|\Psi(t_2 - s) - \Psi(t_1 - s)\| ds \\
&\quad + \frac{MM_1}{\Gamma(1+\alpha)} \left| t_1^{\alpha} + (t_2 - t_1)^{\alpha} - t_2^{\alpha} \right| + \frac{MM_1}{\Gamma(1+\alpha)}(t_2 - t_1)^{\alpha} \\
&\le M_1 (t_2 - t_1)^{\alpha-1} \int_0^{t_1} \|\Psi(t_2 - s) - \Psi(t_1 - s)\| ds + \frac{2MM_1}{\Gamma(1+\alpha)}(t_2 - t_1)^{\alpha} \\
&\quad + \frac{MM_1}{\Gamma(1+\alpha)}\left(t_2^{\alpha} - t_1^{\alpha}\right).
\end{aligned} \tag{3.19}$$

By Remark 2.12 and Lemma 2.16 (iv), $\Psi(t)$ is continuous in the uniform operator topology for $t > 0$. Then by Lebesque-dominated convergence theorem, WD is equicontinuous in $C(I,X)$. By Lemma 2.16 (ii), $\{\Psi(t)\}_{t\ge 0}$ is strongly continuous. So, QD is equicontinuous in $C(I,X)$.

Then by Ascoli-Arzela's theorem, $QD = \{Qu \mid u \in D\}$ is precompact in $C(I,X)$. By (3.12) and (3.13), $\{v_n\}$ has a convergent subsequence in $C(I,X)$. Combining this with the monotonicity of $\{v_n\}$, it is itself convergent in $C(I,X)$. Using a similar argument to that for $\{v_n\}$, we can prove that $\{w_n\}$ is also convergent in $C(I,X)$. Set

$$\underline{u} = \lim_{n\to\infty} v_n, \qquad \overline{u} = \lim_{n\to\infty} w_n. \tag{3.20}$$

Let $n \to \infty$, by the continuity of Q and (3.12), we have

$$\underline{u} = Q\underline{u}, \qquad \overline{u} = Q\overline{u}. \tag{3.21}$$

By (3.7), if $u \in D$ is a fixed-point of Q, then $v_1 = Qv_0 \le Qu = u \le Qw_0 = w_1$. By induction, $v_n \le u \le w_n$. By (3.13) and taking the limit as $n \to \infty$, we conclude that $v_0 \le \underline{u} \le u \le \overline{u} \le w_0$. This means that $\underline{u}, \overline{u}$ are the minimal and maximal fixed-points of Q on $[v_0, w_0]$, respectively. By (3.6), they are the minimal and maximal mild solutions of PBVP (1.1) on $[v_0, w_0]$, respectively. □

Theorem 3.2. *Assume that $\{T(t)\}_{t\geq 0}$ is a compact and positive semigroup in X, $f(t,\theta) \geq \theta$ for $t \in I$. If there is $y \in X$ such that $y \geq \theta$, $Ay \geq f(t,y)$ for $t \in I$, and f satisfies the following:*

(H_1) *There exists a constant $C_1 > 0$ such that*

$$f(t,x_2) - f(t,x_1) \geq -C_1(x_2 - x_1), \tag{3.22}$$

for any $t \in I$, and $\theta \leq x_1 \leq x_2 \leq y$, that is, $f(t,x) + C_1 x$ is increasing in x for $x \in [\theta, y]$.

Then PBVP (1.1) *has a positive mild solution u: $\theta \leq u \leq y$.*

Proof. Let $v_0 = \theta$ and $w_0 = y$, by Theorem 3.1, PBVP (1.1) has mild solution on $[v_0, w_0]$. □

Case 2. $\{T(t)\}_{t\geq 0}$ is noncompact.

Theorem 3.3. *Assume that the positive cone P is regular, $\{T(t)\}_{t\geq 0}$ is an equicontinuous and positive semigroup in X, PBVP* (1.1) *has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$, and (H) holds, then PBVP* (1.1) *has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. By the proof of Theorem 3.1, (3.2)–(3.13) and (3.19) are valid. By Lemma 2.16 (iv), $\Psi(t)$ is continuous in the uniform operator topology for $t > 0$. Then by Lebesque-dominated convergence theorem, WD is equicontinuous in $C(I,X)$. From Lemma 2.16 (ii), $\{\Psi(t)\}_{t\geq 0}$ is strongly continuous. So, QD is equicontinuous in $C(I,X)$. Thus, $\{Qv_n\}$ is equicontinuous in $C(I,X)$.

For $0 \leq t \leq \omega$, by (3.7) and (3.13), $\{(Qv_n)(t)\}$ is monotone in X. Since the cone P is regular, then $\{(Qv_n)(t)\}$ is convergent in X.

By Ascoli-Arzela's theorem, $\{Qv_n\}$ is precompact in $C(I,X)$ and $\{Qv_n\}$ has a convergent subsequence in $C(I,X)$. Combining this with the monotonicity of $\{Qv_n\}$, it is itself convergent in $C(I,X)$. Using a similar argument to that for $\{Qw_n\}$, we can prove that $\{Qw_n\}$ is also convergent in $C(I,X)$. Let

$$\underline{u} = \lim_{n\to\infty} v_n = \lim_{n\to\infty} Qv_{n-1}, \qquad \overline{u} = \lim_{n\to\infty} w_n = \lim_{n\to\infty} Qw_{n-1}, \tag{3.23}$$

then it is similar to the proof of Theorem 3.1 that $\underline{u}$ and $\overline{u}$ are the minimal and maximal mild solutions of PBVP (1.1) on $[v_0, w_0]$, respectively. □

Corollary 3.4. *Let X be an ordered and weakly sequentially complete Banach space. Assume that $\{T(t)\}_{t\geq 0}$ is an equicontinuous and positive semigroup in X, PBVP* (1.1) *has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$, and (H) holds, then PBVP* (1.1) *has the minimal and maximal mild solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. In an ordered and weakly sequentially complete Banach space, the normal cone P is regular. Then the proof is complete. □

Corollary 3.5. *Let X be an ordered and reflective Banach space. Assume that $\{T(t)\}_{t\geq 0}$ is an equicontinuous and positive semigroup in X, PBVP* (1.1) *has a lower solution v_0 and an upper solution w_0 with $v_0 \leq w_0$, and (H) holds, then PBVP* (1.1) *has the minimal and maximal mild* *solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. In an ordered and reflective Banach space, the normal cone P is regular. Then the proof is complete. □

By Theorem 3.3, Corollaries 3.4 and 3.5, we have the following.

Corollary 3.6. *Assume that $\{T(t)\}_{t\geq 0}$ is an equicontinuous and positive semigroup in X, $f(t,\theta) \geq \theta$ for $t \in I$. If there is $y \in X$ such that $y \geq \theta$, $Ay \geq f(t,y)$ for $t \in I$, f satisfies (H_1) and one of the following conditions:*

(i) *X is an ordered Banach space, whose positive cone P is regular,*

(ii) *X is an ordered and weakly sequentially complete Banach space,*

(iii) *X is an ordered and reflective Banach space.*

then PBVP (1.1) *has positive mild solution u: $\theta \leq u \leq y$.*

4. Examples

Example 4.1. Consider the following periodic boundary value problem for fractional parabolic partial differential equations in X:

$$\begin{gathered}\partial_t^\alpha u + A(x,D)u = g(x,t,u), \quad (x,t) \in \Omega \times I, \\ Bu = 0, \quad (x,t) \in \partial\Omega \times I, \\ u(x,0) = u(x,\omega), \quad x \in \Omega,\end{gathered} \tag{4.1}$$

where ∂_t^α is the Caputo fractional partial derivative with order $0 < \alpha < 1$, $I = [0,\omega]$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, $g : \overline{\Omega} \times I \times \mathbb{R} \to \mathbb{R}$ is continuous, $Bu = b_0(x)u + \delta(\partial u/\partial n)$ is a regular boundary operator on $\partial\Omega$, and

$$A(x,D)u = -\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial y_i}\right) \tag{4.2}$$

is a symmetrical strong elliptic operator of second order, whose coefficient functions are Hölder continuous in Ω.

Let $X = L^p(\Omega)(p \geq 2)$, $P = \{v \mid v \in L^p(\Omega), v(x) \geq 0 \text{ a.e. } x \in \Omega\}$, then X is a Banach space, and P is a regular cone in X. Define the operator A as follows:

$$D(A) = \left\{u \in W^{2,p}(\Omega) \mid Bu = 0\right\}, \qquad Au = A(x,D)u. \tag{4.3}$$

Then $-A$ generates a uniformly bounded analytic semigroup $T(t)(t \geq 0)$ in X (see [39]). By the maximum principle, we can easily find that $T(t)(t \geq 0)$ is positive (see [39]). Let

$u(t) = u(\cdot, t)$, $f(t, u) = g(\cdot, t, u(\cdot, t))$, then the problem (4.1) can be transformed into the following problem:

$$\begin{gathered} D^{\alpha} u(t) + Au(t) = f(t, u(t)), \quad t \in I, \\ u(0) = u(\omega). \end{gathered} \tag{4.4}$$

Theorem 4.2. *Let $f(x, t, 0) \geq 0$. If there exists $w_0(x, t) \in C^{2,\alpha}(\Omega \times I)$ such that*

$$\begin{gathered} \partial_t^{\alpha} w_0 + A(x, D) w_0 \geq g(x, t, w_0), \quad (x, t) \in \Omega \times I, \\ Bw = 0, \quad (x, t) \in \partial\Omega \times I, \\ w_0(x, 0) \geq w_0(x, \omega), \quad x \in \Omega, \end{gathered} \tag{4.5}$$

and g satisfies the following:

(H_4) *there exists a constant $C_2 \geq 0$ such that*

$$g(x, t, \xi_2) - g(x, t, \xi_1) \geq -C_2(\xi_2 - \xi_1), \tag{4.6}$$

for any $t \in I$, and $0 \leq \xi_1 \leq \xi_2 \leq w_0$.

Then PBVP (4.1) *has a mild solution $u : 0 \leq u \leq w_0$.*

Proof. Set $v_0 = 0$, by Theorem 3.3, PBVP (4.1) has the minimal and maximal solutions between 0 and w_0. □

Acknowledgments

This research was supported by NNSFs of China (nos. 10871160, 11061031) and Project of NWNU-KJCXGC-3-47.

References

[1] I. Petráš, "A note on the fractional-order cellular neural networks," in *the International Joint Conference on Neural Networks (IJCNN '06)*, pp. 1021–1024, July 2006.
[2] L. Dorcak, I. Petras, I. Kostial, and J. Terpak, "Fractional-order state space models," in *Proceedings of the International Carpathian Control Conference*, pp. 193–198, 2002.
[3] D. Cafagna, "Past and present—fractional calculus: a mathematical tool from the past for present engineers," *IEEE Industrial Electronics Magazine*, vol. 1, no. 2, pp. 35–40, 2007.
[4] A. Benchellal, T. Poinot, and J.-C. Trigeassou, "Fractional modelling and identification of a thermal process," in *the 2nd IFAC Workshop on Fractional Differentiation and Its Applications (FDA '06)*, vol. 2, pp. 248–253, July 2006.
[5] I. S. Jesus, J. A. Tenreiro Machado, and J. Boaventura Cunha, "Fractional electrical dynamics in fruits and vegetables," in *the 2nd IFAC Workshop on Fractional Differentiation and its Applications (FDA '06)*, vol. 2, pp. 308–313, July 2006.
[6] W. M. Ahmad and R. El-Khazali, "Fractional-order dynamical models of love," *Chaos, Solitons and Fractals*, vol. 33, no. 4, pp. 1367–1375, 2007.
[7] A. Oustaloup, J. Sabatier, and X. Moreau, "From fractal robustness to the CRONE approach," in *Systèmes Différentiels Fractionnaires (Paris, 1998)*, vol. 5 of *ESAIM Proc.*, pp. 177–192, Soc. Math. Appl. Indust., Paris, Farance, 1998.

[8] I. Podlubny, “The laplace transform method for linear differential equations of the fractional order,” Tech. Rep., Slovak Academy of Sciences, Institute of Experimental Physics, 1994.

[9] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[10] J. A. Tenreiro MacHado, M. F. Silva, R. S. Barbosa et al., “Some applications of fractional calculus in engineering,” *Mathematical Problems in Engineering*, vol. 2010, Article ID 639801, 34 pages, 2010.

[11] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Mathematics in Science and Engineering, Vol. 11, Academic Press, London, UK, 1974.

[12] M. F. Silva, J. A. T. Machado, and A. M. Lopes, “Comparison of fractional and integer order control of an hexapod robot,” in *Proceedings of the ASME Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, vol. 5, pp. 667–676, Chicago, Ill, USA, September 2003.

[13] R. E. Gutiérrez, J. M. Rosário, and J. Tenreiro MacHado, “Fractional order calculus: basic concepts and engineering applications,” *Mathematical Problems in Engineering*, vol. 2010, Article ID 375858, 19 pages, 2010.

[14] F. B. M. Duarte and J. A. Tenreiro Machado, “Chaotic phenomena and fractional-order dynamics in the trajectory control of redundant manipulators,” *Nonlinear Dynamics*, vol. 29, no. 1–4, pp. 315–342, 2002.

[15] O. P. Agrawal, “A general formulation and solution scheme for fractional optimal control problems,” *Nonlinear Dynamics*, vol. 38, no. 1–4, pp. 323–337, 2004.

[16] N. Engheta, “On fractional calculus and fractional multipoles in electromagnetism,” *IEEE Transactions on Antennas and Propagation*, vol. 44, no. 4, pp. 554–566, 1996.

[17] R. L. Magin, “Fractional calculus models of complex dynamics in biological tissues,” *Computers & Mathematics with Applications*, vol. 59, no. 5, pp. 1586–1593, 2010.

[18] V. V. Kulish and J. L. Lage, “Application of fractional calculus to fluid mechanics,” *Journal of Fluids Engineering*, vol. 124, no. 3, pp. 803–806, 2002.

[19] K. B. Oldham, “Fractional differential equations in electrochemistry,” *Advances in Engineering Software*, vol. 41, no. 1, pp. 9–12, 2010.

[20] V. Gafiychuk, B. Datsko, and V. Meleshko, “Mathematical modeling of time fractional reaction-diffusion systems,” *Journal of Computational and Applied Mathematics*, vol. 220, no. 1-2, pp. 215–225, 2008.

[21] C. Lederman, J.-M. Roquejoffre, and N. Wolanski, “Mathematical justification of a nonlinear integrodifferential equation for the propagation of spherical flames,” *Annali di Matematica Pura ed Applicata. Series IV*, vol. 183, no. 2, pp. 173–239, 2004.

[22] R. L. Bagley and P. J. Torvik, “A theoretical basis for the application of fractional calculus to viscoelasticity,” *Journal of Rheology*, vol. 27, no. 3, pp. 201–210, 1983.

[23] F. Mainardi, “Fractional calculus: some basic problems in continuum and statistical mechanics,” in *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi, Eds., vol. 378 of *CISM Courses and Lectures*, pp. 291–348, Springer, Vienna, Austria, 1997.

[24] F. C. Meral, T. J. Royston, and R. Magin, “Fractional calculus in viscoelasticity: an experimental study,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 4, pp. 939–945, 2010.

[25] E. J. Solteiro Pires, J. A. Tenreiro Machado, and P. B. De Moura Oliveira, “Fractional order dynamics in a GA planner,” *Signal Processing*, vol. 83, no. 11, pp. 2377–2386, 2003.

[26] K. Hedrih and V. Nikoli-Stanojevi, “A model of gear transmission: fractional order system dynamics,” *Mathematical Problems in Engineering*, vol. 2010, Article ID 972873, 23 pages, 2010.

[27] J. Cao, C. Ma, H. Xie, and Z. Jiang, “Nonlinear dynamics of duffing system with fractional order damping,” *Journal of Computational and Nonlinear Dynamics*, vol. 5, no. 4, Article ID 041012, 6 pages, 2010.

[28] J. Wang, Y. Zhou, and W. Wei, “A class of fractional delay nonlinear integrodifferential controlled systems in Banach spaces,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 10, pp. 4049–4059, 2011.

[29] J. Wang and Y. Zhou, “A class of fractional evolution equations and optimal controls,” *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 262–272, 2011.

[30] J. Wang, Y. Zhou, W. Wei, and H. Xu, “Nonlocal problems for fractional integrodifferential equations via fractional operators and optimal controls,” *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1427–1441, 2011.

[31] Y. Zhou and F. Jiao, "Existence of mild solutions for fractional neutral evolution equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1063–1077, 2010.

[32] Y. Zhou and F. Jiao, "Nonlocal Cauchy problem for fractional evolution equations," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 4465–4475, 2010.

[33] M. M. El-Borai, "Some probability densities and fundamental solutions of fractional evolution equations," *Chaos, Solitons and Fractals*, vol. 14, no. 3, pp. 433–440, 2002.

[34] M. M. El-Borai, "The fundamental solutions for fractional evolution equations of parabolic type," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 2004, no. 3, pp. 197–211, 2004.

[35] M. M. El-Borai and A. Debbouche, "Almost periodic solutions of some nonlinear fractional differential equations," *International Journal of Contemporary Mathematical Sciences*, vol. 4, no. 25–28, pp. 1373–1387, 2009.

[36] A. Debbouche and M. M. El-Borai, "Weak almost periodic and optimal mild solutions of fractional evolution equations," *Electronic Journal of Differential Equations*, no. 46, pp. 1–8, 2009.

[37] Y. X. Li, "Existence and uniqueness of positive periodic solutions for abstract semilinear evolution equations," *Journal of Systems Science and Mathematical Sciences*, vol. 25, no. 6, pp. 720–728, 2005.

[38] Y. X. Li, "Existence of solutions to initial value problems for abstract semilinear evolution equations," *Acta Mathematica Sinica*, vol. 48, no. 6, pp. 1089–1094, 2005.

[39] Y. X. Li, "Periodic solutions of semilinear evolution equations in Banach spaces," *Acta Mathematica Sinica*, vol. 41, no. 3, pp. 629–636, 1998.

[40] Y. X. Li, "Global solutions of initial value problems for abstract semilinear evolution equations," *Acta Analysis Functionalis Applicata*, vol. 3, no. 4, pp. 339–347, 2001.

[41] Y. X. Li, "Positive solutions of abstract semilinear evolution equations and their applications," *Acta Mathematica Sinica*, vol. 39, no. 5, pp. 666–672, 1996.

[42] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier, Amsterdam, The Netherlands, 2006.

[43] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1993.

[44] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.

[45] J. Mu, "Monotone iterative technique for fractional evolution equations in banach spaces," *Journal of Applied Mathematics*, vol. 2011, Article ID 767186, 13 pages, 2011.

[46] W. Feller, *An Introduction to Probability Theory and Its Applications. Vol. II*, John Wiley & Sons, New York, NY, USA, 2nd edition, 1971.

[47] W. R. Schneider and W. Wyss, "Fractional diffusion and wave equations," *Journal of Mathematical Physics*, vol. 30, no. 1, pp. 134–144, 1989.

[48] Z. Wei, W. Dong, and J. Che, "Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 10, pp. 3232–3238, 2010.

[49] F. L. Huang, "Spectral properties and stability of one-parameter semigroups," *Journal of Differential Equations*, vol. 104, no. 1, pp. 182–195, 1993.

[50] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.