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Research Article

A Tikhonov-Type Regularization Method for Identifying the Unknown Source in the Modified Helmholtz Equation

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An inverse source problem in the modified Helmholtz equation is considered. We give a Tikhonov-type regularization method and set up a theoretical frame to analyze the convergence of such method. A priori and a posteriori choice rules to find the regularization parameter are given. Numerical tests are presented to illustrate the effectiveness and stability of our proposed method.

1. Introduction

The Helmholtz equation often arises in the study of physical problems involving partial differential equations (PDEs) in both space and time. It has a wide range of applications, for example, radar, sonar, geographical exploration, and medical imaging. A kind of important equations similar with the Helmholtz equation in science and engineering is

$$\Delta u(x,y) - k^2 u(x,y) = f(x), \tag{1.1}$$

where the constant $k > 0$ is the wave number and $f(x)$ is the source term. This equation is called the modified Helmholtz equation. It appears, for example, in the semi-implicit temporal discretization of the heat or the Navier-Stokes equations [1] and in the linearized Poisson-Boltzmann equation.

Inverse source problems have attracted great attention of many researchers over recent years because of their applications to many practical problems such as crack determination

[2, 3], heat source determination [4–6], inverse heat conduction [7–11], pollution source identification [12], electromagnetic source identification [13], Stefan design problems [14], sound source reconstruction [15], and identification of current dipolar sources in the so-called inverse electroencephalography/magnetoencephalography (EEG/MEG) problems [16, 17]. Theoretical investigation on the inverse source identification problems can be found in the works of [18–23].

The main difficulty of inverse source identification problems is that they are typically ill posed in the sense of Hadamard [24]. In other words, any small error in the scattered measurement data may induce enormous error to the solution. In general, the unknown source can only be recovered from boundary measurements if some a priori knowledge is assumed. For instances, if one of the products in the separation of variables is known [25, 26], the base area of a cylindrical source is known [25], or a nonseparable type is in the form of a moving front [26], then the boundary condition plus some scattered boundary measurements can uniquely determine the unknown source term. Furthermore, when the unknown source term is relatively smooth, some regularization techniques can be employed, see [5, 27–29] for more details. In addition, due to the complexity and ill posedness of the inverse source identification problems, some of the variational methods [6, 30] are also employed to deal with them.

In this paper, we will consider the following problem (see [29]):

$$\begin{gathered}
\Delta u(x,y) - k^2 u(x,y) = f(x), \quad 0 < x < \pi, 0 < y < \infty, \\
u(0,y) = u(\pi,y) = 0, \quad 0 \le y < \infty, \\
u(x,0) = 0, \quad 0 \le x \le \pi, \\
u(x,y)|_{y\to\infty} \text{ bounded}, \quad 0 \le x \le \pi, \\
u(x,1) = g(x), \quad 0 \le x \le \pi,
\end{gathered} \tag{1.2}$$

for determining the source term $f(x)$ such that the solution $u(x,y)$ of the modified Helmholtz equation satisfies the given supplementary condition $u(x,1) = g(x)$, where the constant $k > 0$ is the wave number. In practice, the data $g(x)$ is usually obtained through measurement and the measured data is denoted by $g^\delta(x)$.

To determine the source term $f(x)$, we require the following assumptions:

(A) $f(x) \in L^2(0,\pi)$ and $g(x) \in L^2(0,\pi)$;

(B) there exists a relation between the function $g(x)$ and the measured data $g^\delta(x)$:

$$\left\| g - g^\delta \right\|_{L^2} \le \delta, \tag{1.3}$$

where $\|\cdot\|_{L^2}$ denotes the norm in the space $L^2(0,\pi)$ and $\delta > 0$ is the noise level.

(C) The source term $f(x)$ satisfies the a priori bound

$$\|f\|_{H^p} \le E, \quad p \ge 0, \tag{1.4}$$

where E is a positive constant, and $\|\cdot\|_{H^p}$ denotes the norm in Sobolev space $H^p(0,\pi)$ which is defined by [31] as follows:

$$\|f(\cdot)\|_{H^p} = \left(\sum_{n=1}^{\infty}\left(1+n^2\right)^p \left|\langle f, X_n\rangle\right|^2\right)^{1/2}. \tag{1.5}$$

We can refer to [31] for the details of the Sobolev space $H^p(0,\pi)$.

Using the separation of variables, we can obtain the explicit solution of the modified Helmholtz equation:

$$u(x,y) = -\sum_{n=1}^{\infty} \frac{1-e^{-\sqrt{n^2+k^2}y}}{n^2+k^2} f_n X_n(x), \tag{1.6}$$

where

$$\left\{ X_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n = 1,2,\ldots \right\} \tag{1.7}$$

is an orthogonal basis in $L^2(0,\pi)$, and

$$f_n = \sqrt{\frac{2}{\pi}}\int_0^{\pi} f(x)\sin(nx)dx. \tag{1.8}$$

According to (1.6) and the supplementary condition $u(x,1) = g(x)$, we have

$$g(x) = -\sum_{n=1}^{\infty} \frac{1-e^{-\sqrt{n^2+k^2}}}{n^2+k^2} f_n X_n(x). \tag{1.9}$$

Based on this relation, we can define an operator K on the space $L^2(0,\pi)$ as

$$(Kf)(x) = -\sum_{n=1}^{\infty} \frac{1-e^{-\sqrt{n^2+k^2}}}{n^2+k^2} f_n X_n(x). \tag{1.10}$$

Thus, our inverse source problem is formulated as follows: give the data $g(x)$ and the operator K and then determine the unknown source term $f(x)$ such that (1.10) holds.

It is straightforward that the operator K is invertible and then the exact solution of (1.2) is

$$f(x) = \left(K^{-1}g\right)(x) = -\sum_{n=1}^{\infty} \frac{n^2+k^2}{1-e^{-\sqrt{n^2+k^2}}} g_n X_n(x), \tag{1.11}$$

with

$$g_n = \sqrt{\frac{2}{\pi}} \int_0^{\pi} g(x) \sin(nx) dx. \tag{1.12}$$

Note that the factor $(n^2 + k^2)/(1 - e^{-\sqrt{n^2+k^2}})$ increases rapidly and tends to infinity as $n \to \infty$, so a small perturbation in the data $g(x)$ may cause a dramatically large error in the solution $f(x)$. Therefore, this inverse source problem is mildly ill posed. It is impossible to obtain the unknown source using classical methods as above.

In [29], the simplified Tikhonov regularization method was given for (1.2). In this method, the regularization parameter is a priori chosen. It is well known that the ill-posed problem is usually sensitive to the regularization parameter and the a priori bound is usually difficult to be obtained precisely in practice. So the a priori choice rule of the regularization parameter is unreliable in practical problems. In this paper, we will present a Tikhonov-type regularization method to deal with (1.2) and show that the regularization parameter can be chosen by an a posteriori rule based on the discrepancy principle in [27].

The rest of this paper is organized as follows. In Section 2, we establish a quasinormal equation, which is crucial for proving the convergence of the Tikhonov-type regularization method. In Section 3, we give the Tikhonov-type regularization method and then prove the convergence of such method. Also, we give a priori and a posteriori choice rules to find the regularization parameter in the regularization method. In Section 4, we demonstrate a numerical example to illustrate the effectiveness of the method. In Section 5, we give some conclusions.

2. Preparation

In this section, we give an auxiliary result which will be used in this paper.

We first define an operator T on $L^2(0, \pi)$ as follows:

$$(Tf)(x) = \sum_{n=1}^{\infty} \left(1 + n^2\right)^p f_n X_n(x). \tag{2.1}$$

Let us observe that the operator T is well defined and is a self-adjoint linear operator.

Next we give a lemma, which is important for discussing the regularization method. For simplicity, we denote the spaces $H^p(0, \pi)$ and $L^2(0, \pi)$ by X and Y, respectively.

Lemma 2.1. *Let $A : X \to Y$ be a linear and bounded operator between two Hilbert spaces, and let T be defined as in* (2.1), $\alpha > 0$. *Then for any $x \in X$ the following Tikhonov functional*

$$J_\alpha(x) := \|Ax - y\|_{L^2}^2 + \alpha \|x\|_{H^p}^2 \tag{2.2}$$

has a unique minimum $x^\alpha \in X$, and this minimum x^α is the unique solution of the quasinormal equation $A^* A x^\alpha + \alpha T^2 x^\alpha = A^* y$.

Proof. We divide the proof into three steps.

Step 1. The existence of a minimum of $J_\alpha(x)$ is proved. Let $\{x_n\}_{n=1}^{\infty} \subset X$ be a minimizing sequence; that is, $J_\alpha(x_n) \to I := \inf_{x\in X} J_\alpha(x)$ as $n \to \infty$. We first need to show that $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence. According to the definition of $J_\alpha(x)$, we have

$$\begin{aligned}
&J_\alpha(x_n) + J_\alpha(x_m) - 2J_\alpha\left(\frac{x_n + x_m}{2}\right) \\
&= \|Ax_n - y\|_{L^2}^2 + \alpha\|x_n\|_{H^p}^2 + \|Ax_m - y\|_{L^2}^2 + \alpha\|x_m\|_{H^p}^2 - 2\left\|A\left(\frac{x_n+x_m}{2}\right) - y\right\|_{L^2}^2 - 2\alpha\left\|\frac{x_n+x_m}{2}\right\|_{H^p}^2 \\
&= \langle Ax_n - y, Ax_n - y\rangle_{L^2} + \langle Ax_m - y, Ax_m - y\rangle_{L^2} - 2\left\langle A\left(\frac{x_n+x_m}{2}\right) - y, A\left(\frac{x_n+x_m}{2}\right) - y\right\rangle_{L^2} \\
&\quad + \alpha\|x_n\|_{H^p}^2 + \alpha\|x_m\|_{H^p}^2 - 2\alpha\left\|\frac{x_n + x_m}{2}\right\|_{H^p}^2 \\
&= \frac{1}{2}\|A(x_n - x_m)\|_{L^2}^2 + \frac{\alpha}{2}\left(2\|x_n\|_{H^p}^2 + 2\|x_m\|_{H^p}^2 - \|x_n + x_m\|_{H^p}^2\right) \\
&= \frac{1}{2}\|A(x_n - x_m)\|_{L^2}^2 + \frac{\alpha}{2}\|x_n - x_m\|_{H^p}^2.
\end{aligned} \tag{2.3}$$

This implies that $J_\alpha(x_n)+J_\alpha(x_m) \geq 2I+(\alpha/2)\|x_n + x_m\|_{H^p}^2$. Since the left-hand side converges to $2I$ as n, m tend to infinity. This shows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence and thus convergent. Let $\lim_{n\to\infty} x_n = x^\alpha$, noting that $x^\alpha \in X$. From the continuity of $J_\alpha(x)$, we conclude that $J_\alpha(x_n) \to J_\alpha(x^\alpha)$, that is, $J_\alpha(x^\alpha) = I$. This proves the existence of a minimum of $J_\alpha(x)$.

Step 2. The equivalence of the quasinormal equation with the minimization problem for $J_\alpha(x)$ is shown. According to the definition $J_\alpha(x)$ and (2.1), we can obtain the following formula:

$$\begin{aligned}
&J_\alpha(x) - J_\alpha(x^\alpha) \\
&\quad = \|Ax - y\|_{L^2}^2 + \alpha\|x\|_{H^p}^2 - \|Ax^\alpha - y\|_{L^2}^2 - \alpha\|x^\alpha\|_{H^p}^2 \\
&\quad = \langle Ax - y, Ax - y\rangle_{L^2} - \langle Ax^\alpha - y, Ax^\alpha - y\rangle_{L^2} + \alpha\langle x, x\rangle_{H^p} - \alpha\langle x^\alpha, x^\alpha\rangle_{H^p} \\
&\quad = \|A(x - x^\alpha)\|_{L^2}^2 + 2\,\mathrm{Re}\left\langle x - x^\alpha, A^*(Ax^\alpha - y) + \alpha T^2 x^\alpha\right\rangle_{L^2} + \alpha\|T(x - x^\alpha)\|_{L^2}^2,
\end{aligned} \tag{2.4}$$

for all $x \in X$. If x^α satisfies $A^*Ax^\alpha + \alpha T^2 x^\alpha = A^*y$, then $J_\alpha(x) - J_\alpha(x^\alpha) = \|A(x - x^\alpha)\|_{L^2}^2 + \alpha\|T(x - x^\alpha)\|_{L^2}^2 \geq 0$, that is, x^α minimizes $J_\alpha(x)$.

Conversely, if x^α minimizes $J_\alpha(x)$, then we substitute $x - x^\alpha = t\xi$ for any $t > 0$ and $\xi \in X$, and then we can arrive at

$$2t\,\mathrm{Re}\left\langle \xi, A^*(Ax^\alpha - y) + \alpha T^2 x^\alpha\right\rangle_{L^2} + t^2\left(\|A\xi\|_{L^2}^2 + \alpha\|T\xi\|_{L^2}^2\right) \geq 0. \tag{2.5}$$

Dividing both sides of the above inequality by $t > 0$ and taking $t \to 0$, we get

$$\operatorname{Re}\left\langle \xi, A^*(Ax^\alpha - y) + \alpha T^2 x^\alpha \right\rangle_{L^2} \geq 0, \tag{2.6}$$

for all $\xi \in X$. This implies that $A^*(Ax^\alpha - y) + \alpha T^2 x^\alpha = 0$. It follows that x^α solves the quasinormal equation. From this, the equivalence of the quasinormal equation with the minimization problem for $J_\alpha(x)$ is shown exactly.

Step 3. We show that the operator $A^*A + \alpha T^2$ is one-one for every $\alpha > 0$. Let $(A^*A + \alpha T^2)x = 0$. Multiplication by x yields $\langle Ax, Ax\rangle_{L^2} + \alpha\langle Tx, Tx\rangle_{L^2} = 0$, that is, $x = 0$. □

3. A Tikhonov-Type Regularization Method

In this section, we first present a Tikhonov-type regularization method to obtain the approximate solution of (1.2) and then consider an a priori strategy and a posteriori choice rule to find the regularization parameter. Under each choice of the regularization parameter, the corresponding estimate can be obtained.

Since (1.2) is an ill-posed problem, we give its regularized solution $f^{\alpha,\delta}(x)$ which minimizes the Tiknonov functional

$$J_\alpha(f(x)) := \left\| Kf - g^\delta \right\|_{L^2}^2 + \alpha \|f\|_{H^p}^2, \tag{3.1}$$

where the operator K is defined as in (1.10), and $\alpha > 0$ is a regularization parameter.

According to Lemma 2.1, this minimum $f^{\alpha,\delta}(x)$ is the unique solution of the quasinormal equation $K^*Kf^{\alpha,\delta}(x) + \alpha T^2 f^{\alpha,\delta}(x) = K^* g^\delta(x)$, that is, $f^{\alpha,\delta}(x) = (K^*K + \alpha T^2)^{-1} K^* g^\delta(x)$. Because K is a linear self-adjoint operator, that is, $K^* = K$, we have the equivalent form of $f^{\alpha,\delta}(x)$ as

$$f^{\alpha,\delta}(x) = \left(K^2 + \alpha T^2\right)^{-1} K g^\delta(x). \tag{3.2}$$

Further, the function $f^{\alpha,\delta}(x)$ can be reduced to

$$f^{\alpha,\delta}(x) = -\sum_{n=1}^{\infty} \frac{(n^2 + k^2)/\left(1 - e^{-\sqrt{n^2+k^2}}\right)}{1 + \alpha\left((1+n^2)^p \left((n^2+k^2)/\left(1 - e^{-\sqrt{n^2+k^2}}\right)\right)\right)^2} g_n^\delta X_n(x). \tag{3.3}$$

Now we are ready to formulate the main results of this paper. Before proceeding, the following lemmas are needed.

Lemma 3.1. *For any $n \in N^+$, $k > 0$, it holds $n^2 + k^2 \leq (n^2 + k^2)/(1 - e^{-\sqrt{n^2+k^2}}) \leq 1 + n^2 + k^2$.*

Proof. The proof is elementary and is omitted. □

Table 1: Relative errors $\operatorname{reff}(f(x))$ with $\varepsilon = 0.01$, $k = 2$, $p = 2$, and $N = 10$ for different M.

M	10	50	100	200	400	800	1600
$\operatorname{reff}(f(x))$	0.0471	0.0141	0.0116	0.0036	0.0039	0.0020	0.0018

Lemma 3.2. *For $k > 0$, $p \geq 0$, $h \in L^2(0,\pi)$ and the operator K defined in* (1.10), *one has*

$$\left\|K^{-1}h\right\|_{L^2} \leq \left(1+k^2\right)^{p/(p+2)} \|h\|_{L^2}^{p/(p+2)} \left\|K^{-1}h\right\|_{H^p}^{2/(p+2)}. \tag{3.4}$$

Proof. By the Hölder inequality and Lemma 3.1, we have

$$\begin{aligned}
\left\|K^{-1}h\right\|_{L^2}^2 &= \sum_{n=1}^{\infty}\left(\frac{n^2+k^2}{1-e^{-\sqrt{n^2+k^2}}}|h_n|\right)^2 \\
&= \sum_{n=1}^{\infty}\left(\frac{n^2+k^2}{1-e^{-\sqrt{n^2+k^2}}}\right)^2 |h_n|^{4/(p+2)}|h_n|^{2p/(p+2)} \\
&\leq \left(\sum_{n=1}^{\infty}\left(\left(\frac{n^2+k^2}{1-e^{-\sqrt{n^2+k^2}}}\right)^2 |h_n|^{4/(p+2)}\right)^{(p+2)/2}\right)^{2/(p+2)} \\
&\quad \times \left(\sum_{n=1}^{\infty}\left(|h_n|^{2p/(p+2)}\right)^{(p+2)/p}\right)^{p/(p+2)} \\
&= \left(\sum_{n=1}^{\infty}\left(\frac{n^2+k^2}{1-e^{-\sqrt{n^2+k^2}}}\right)^{p+2}|h_n|^2\right)^{2/(p+2)} \|h\|_{L^2}^{2p/(p+2)} \\
&= \left(\sum_{n=1}^{\infty}\left(\frac{n^2+k^2}{1-e^{-\sqrt{n^2+k^2}}}\right)^{p}\left(\frac{n^2+k^2}{1-e^{-\sqrt{n^2+k^2}}}|h_n|\right)^2\right)^{2/(p+2)} \|h\|_{L^2}^{2p/(p+2)} \\
&\leq \left(\sum_{n=1}^{\infty}\left(1+n^2\right)^{p}\left(\frac{1+n^2+k^2}{1+n^2}\right)^{p}\left(\frac{n^2+k^2}{1-e^{-\sqrt{n^2+k^2}}}|h_n|\right)^2\right)^{2/(p+2)} \|h\|_{L^2}^{2p/(p+2)} \\
&\leq \left(1+k^2\right)^{2p/(p+2)} \|h\|_{L^2}^{2p/(p+2)} \left\|K^{-1}h\right\|_{H^p}^{4/(p+2)}.
\end{aligned} \tag{3.5}$$

The proof is completed. □

In the following we give the corresponding convergence results for an a priori choice rule and an a posteriori choice rule.

Table 2: Relative errors $\operatorname{reff}(f(x))$ with $\varepsilon = 0.01$, $k = 2$, $p = 2$ and $M = 100$ for different N.

N	1	5	10	20	40	80	100
$\operatorname{reff}(f(x))$	$3.8681e-004$	0.0055	0.0040	0.0081	0.0067	0.0054	0.0046

Table 3: Relative errors $\operatorname{reff}(f(x))$ with $k = 2$, $M = 100$, and $N = 10$, for different p and ε.

$\operatorname{reff}(f(x))$	$p = 0$	$p = 1/2$	$p = 1$	$p = 2$	$p = 4$	$p = 8$	$p = 10$
$\varepsilon = 0.1$	0.1660	0.1345	0.0564	0.0241	0.0097	0.0076	0.0223
$\varepsilon = 0.01$	0.0283	0.0308	0.0221	0.0040	0.0063	0.0033	0.0021
$\varepsilon = 0.001$	0.0026	0.0048	0.0030	0.0018	0.0011	$1.4765e-004$	$1.3137e-004$

3.1. An A Priori Choice Rule

Choose the regularization parameter α_1 as

$$\alpha_1 = \left(\frac{\delta}{E}\right)^2. \tag{3.6}$$

The next theorem shows that the choice (3.6) is valid under suitable assumptions.

Theorem 3.3. *Let $f^{\alpha_1,\delta}(x)$ be the minimizer of $J_{\alpha_1}(f(x))$ defined by* (3.1) *and $f(x)$ be the exact solution of* (1.2)*, and let assumptions (A), (B), and (C) hold. If α_1 is chosen by* (3.6)*, then $f^{\alpha_1,\delta}(x)$ is convergent to the exact solution $f(x)$ as the noise level δ tends to zero. Furthermore, one has the following estimate:*

$$\left\|f^{\alpha_1,\delta} - f\right\|_{L^2} \le \left(\sqrt{2}+1\right)\left(1+k^2\right)^{p/(p+2)} \delta^{p/(p+2)} E^{2/(p+2)}. \tag{3.7}$$

Proof. Since $f^{\alpha_1,\delta}(x)$ is the minimizer of $J_{\alpha_1}(f(x))$ defined by (3.1), we can obtain

$$\begin{aligned} \left\|f^{\alpha_1,\delta}\right\|_{H^p}^2 &\le \frac{1}{\alpha_1} J_{\alpha_1}\left(f^{\alpha_1,\delta}(x)\right) \le \frac{1}{\alpha_1} J_{\alpha_1}(f(x)) \le 2E^2, \\ \left\|Kf^{\alpha_1,\delta} - g^\delta\right\|_{L^2}^2 &\le J_{\alpha_1}\left(f^{\alpha_1,\delta}(x)\right) \le J_{\alpha_1}(f(x)) \le 2\delta^2. \end{aligned} \tag{3.8}$$

Furthermore, we get

$$\begin{aligned} \left\|f^{\alpha_1,\delta} - f\right\|_{H^p} &\le \left\|f^{\alpha_1,\delta}\right\|_{H^p} + \|f\|_{H^p} \le \left(\sqrt{2}+1\right)E, \\ \left\|Kf^{\alpha_1,\delta} - g\right\|_{L^2} &\le \left\|Kf^{\alpha_1,\delta} - g^\delta\right\|_{L^2} + \left\|g^\delta - g\right\|_{L^2} \le \left(\sqrt{2}+1\right)\delta. \end{aligned} \tag{3.9}$$

Table 4: Relative errors $\operatorname{reff}(f(x))$ with $p = 2$, $M = 100$, and $N = 10$, for different k and ε.

$\operatorname{reff}(f(x))$	$k = 1$	$k = 2$	$k = 4$	$k = 8$
$\varepsilon = 0.1$	0.0412	0.0548	0.0304	0.0245
$\varepsilon = 0.01$	0.0133	0.0174	0.0064	0.0036
$\varepsilon = 0.001$	0.0032	0.0031	$8.2509e-004$	$5.2171e-004$

Table 5: Relative errors $\operatorname{reff}(f(x))$ with $\varepsilon = 0.01$, $k = 2$, $p = 2$, and $N = 10$ for different M.

M	10	50	100	200	400	800	1600
$\operatorname{reff}(f(x))$	0.0364	0.0239	0.0194	0.0093	0.0074	0.0060	0.0045

By Lemma 3.2, we have

$$\begin{aligned}\left\|f^{\alpha_1,\delta}-f\right\|_{L^2} &= \left\|K^{-1}\left(Kf^{\alpha_1,\delta}-g\right)\right\|_{L^2}\\ &\le \left(1+k^2\right)^{p/(p+2)}\left\|Kf^{\alpha_1,\delta}-g\right\|_{L^2}^{p/(p+2)}\left\|f^{\alpha_1,\delta}-f\right\|_{H^p}^{2/(p+2)}\\ &\le \left(\sqrt{2}+1\right)\left(1+k^2\right)^{p/(p+2)}\delta^{p/(p+2)}E^{2/(p+2)}.\end{aligned} \tag{3.10}$$

The proof is completed. □

3.2. An A Posteriori Choice Rule

Choose the regularization parameter α_2 as the solution of the equation

$$\left\|Kf^{\alpha_2,\delta}-g^{\delta}\right\|_{L^2} = \tau\delta, \tag{3.11}$$

where the operator K is defined by (1.10) and $\tau > 1$.

In the following theorem, an a posteriori rule based on the discrepancy principle [27] is considered in the convergence estimate.

Theorem 3.4. *Let $f^{\alpha_2,\delta}(x)$ be the minimizer of $J_{\alpha_2}(f(x))$ defined by* (3.1) *and $f(x)$ be the exact solution of* (1.2), *and let assumptions (A), (B), and (C) hold. If α_2 is chosen as the solution of* (3.11), *then $f^{\alpha_2,\delta}(x)$ is convergent to the exact solution $f(x)$ as the noise level δ tends to zero. Furthermore, one has the following estimate:*

$$\left\|f^{\alpha_2,\delta}-f\right\|_{L^2} \le 2\left(1+k^2\right)^{p/(p+2)}\delta^{p/(p+2)}E^{2/(p+2)}. \tag{3.12}$$

Proof. Since $f^{\alpha_2,\delta}(x)$ is the minimizer of $J_{\alpha_2}(f(x))$ defined by (3.1), we can obtain

$$\left\|Kf^{\alpha_2,\delta}-g^{\delta}\right\|_{L^2}^2+\alpha_2\left\|f^{\alpha_2,\delta}\right\|_{H^p}^2 = J_{\alpha_2}\left(f^{\alpha_2,\delta}(x)\right) \le J_{\alpha_2}(f(x)) = \left\|g-g^{\delta}\right\|_{L^2}^2+\alpha_2\|f\|_{H^p}^2. \tag{3.13}$$

Table 6: Relative errors $\operatorname{reff}(f(x))$ with $\varepsilon = 0.01$, $k = 2$, $p = 2$, and $M = 200$ for different N.

N	1	5	10	20	40	80	100
$\operatorname{reff}(f(x))$	1.0000	0.0041	0.0071	0.0086	0.0098	0.0094	0.0090

Table 7: Relative errors $\operatorname{reff}(f(x))$ with $k = 2$, $M = 200$, and $N = 10$, for different p and ε.

$\operatorname{reff}(f(x))$	$p = 0$	$p = 1/2$	$p = 1$	$p = 2$	$p = 4$	$p = 8$	$p = 10$
$\varepsilon = 0.1$	0.1160	0.1849	0.1986	0.1230	0.0179	0.0170	0.0337
$\varepsilon = 0.01$	0.0292	0.0128	0.0227	0.0026	0.0031	0.0040	$5.9243e-004$
$\varepsilon = 0.001$	0.0017	0.0027	0.0010	0.0025	$1.3769e-004$	0.0028	$1.1640e-004$

Consequently, it has

$$\left\|f^{\alpha_2,\delta}\right\|_{H^p}^2 \le \|f\|_{H^p}^2 + \frac{1}{\alpha_2}\left(\left\|g - g^\delta\right\|_{L^2}^2 - \tau^2\delta^2\right) \le \|f\|_{H^p}^2 + \frac{1}{\alpha_2}\left(1-\tau^2\right)\delta^2 < \|f\|_{H^p}^2 \le E^2. \tag{3.14}$$

This leads to $\|f^{\alpha_2,\delta} - f\|_{H^p} \le \|f^{\alpha_2,\delta}\|_{H^p} + \|f\|_{H^p} \le 2E$. It follows from Lemma 3.2 that the assertion of this theorem is true. □

4. Numerical Tests

In this section, we present an example to illustrate the effectiveness and stability of our proposed method. The numerical results verify the validity of the theoretical results for the two cases of the a priori and a posteriori parameter choice rules.

Substituting (1.8) into (1.9), and then using trapezoid's rule to discretize (1.9) can result in the following discrete form:

$$-\frac{2}{\pi}\sum_{i=1}^{M+1}\sum_{n=1}^{N}\frac{1-e^{-\sqrt{n^2+k^2}}}{n^2+k^2}f(x_i)\sin(nx_i)\sin(nx_j)\frac{\pi}{M} = g(x_j), \tag{4.1}$$

where $x_i = (i-1)M/\pi$, $i = 1, 2, \ldots, M+1$, and $j = 1, 2, \ldots, M+1$.

We conduct two tests, and the tests are performed in the following way: first, from (1.6) and (1.11), we can select the source term $f(x) = -(1+k^2)\sin x$ and then $u(x,y) = (1-e^{-\sqrt{1+k^2}y})\sin x$. Consequently, the data function $g(x) = (1-e^{-\sqrt{1+k^2}})\sin x$, and

$$\|f\|_{H^p} = \left(\sum_{n=1}^{\infty}\left(1+n^2\right)^p\left|f_n\right|^2\right)^{1/2} = 2^{p/2}\left(1+k^2\right)\sqrt{\pi/2}. \tag{4.2}$$

We choose $E = 2^{p/2}(1+k^2)\sqrt{\pi/2}$. Next, we add a random distributed perturbation to each data function, giving the vector

$$g^\delta = g + \varepsilon\operatorname{randn}(\operatorname{size}(g)). \tag{4.3}$$

Table 8: Relative errors $\operatorname{reff}(f(x))$ with $p = 2$, $M = 200$, and $N = 10$, for different k and ε.

$\operatorname{reff}(f(x))$	$k = 1$	$k = 2$	$k = 4$	$k = 8$
$\varepsilon = 0.1$	0.0250	0.0190	0.0315	0.0516
$\varepsilon = 0.01$	0.0076	0.0074	0.0030	0.0536
$\varepsilon = 0.001$	0.0047	0.0013	$5.6243e-004$	0.0071

The function randn$(\cdot)$ generates arrays of random numbers whose elements are normally distributed with mean 0 and variance 1. Thus, the total noise level δ can be measured in the sense of root mean square error according to

$$\delta = \left\| g^{\delta} - g \right\|_2 := \left(\frac{1}{M+1} \sum_{j=1}^{M+1} \left(g^{\delta}(x_j) - g(x_j) \right)^2 \right)^{1/2}. \tag{4.4}$$

And g_n^{δ} can be obtained according to (1.12). Our error estimates use the relative error, which is given as follows:

$$\operatorname{reff}(f(x)) := \frac{\| f^{\alpha,\delta} - f \|_2}{\| f \|_2}, \tag{4.5}$$

where $\| \cdot \|_2$ is given by (4.4).

Test 1. In the case of the a priori choice rule, we, respectively, compute $\operatorname{reff}(f(x))$ with different parameters M, N, p, k, and ε. Tables 1 and 2 show that M and N have small influence on $\operatorname{reff}(f(x))$ when they become larger. So, we always take $M = 100$ and $N = 10$ in this test. Table 3 shows $\operatorname{reff}(f(x))$ for $p = 0, 1/2, 1, 2, 4, 8$, and 10 with the perturbation $\varepsilon = 0.1$, 0.01, and 0.001. Table 4 shows $\operatorname{reff}(f(x))$ for $k = 1, 2, 4$, and 8 with the perturbation $\varepsilon = 0.1$, 0.01, and 0.001. In conclusion, the regularized solution $f^{\alpha,\delta}$ well converges to the exact solution $f(x)$ when ε tends to zero.

Test 2. In the case of the a posteriori choice rule (3.11), by taking $\tau = 1.5$, we also give the corresponding results as described in Test 1. The results can be easily seen from Tables 5, 6, 7, and 8.

From Tests 1 and 2, we conclude that the proposed regularization method is effective and stable.

5. Conclusion

In this paper, we proposed a Tikhonov-type regularization method to deal with the inverse source identification for the modified Helmholtz equation and set up a theoretical frame to analyze the convergence of such method. For instance, we provided the quasinormal equation to obtain the regularized solution. Moreover, besides the a priori parameter choice rule we studied an a posteriori rule for choosing the regularization parameter. Finally, we presented a numerical example whose results seem to be in excellent agreement with the convergence estimates of the method.

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