Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 928930, 16 pages
doi:10.1155/2012/928930

Research Article

Preservation of Stability and Synchronization of a Class of Fractional-Order Systems

Armando Fabián Lugo-Peñaloza, José Job Flores-Godoy, and Guillermo Fernández-Anaya

Departamento de Física y Matemáticas, Universidad Iberoamericana, Prol. Paseo de la Reforma 880, Lomas de Santa Fe, 01210 México, DF, Mexico

Correspondence should be addressed to
Armando Fabián Lugo-Peñaloza, armando.lugo@correo.uia.mx

Received 18 April 2012; Revised 11 August 2012; Accepted 12 August 2012

Academic Editor: Ricardo Femat

We present sufficient conditions for the preservation of stability of fractional-order systems, and then we use this result to preserve the synchronization, in a master-slave scheme, of fractional-order systems. The systems treated herein are autonomous fractional differential linear and nonlinear systems with commensurate orders lying between 0 and 2, where the nonlinear ones can be described as a linear part plus a nonlinear part. These results are based on stability properties for equilibria of fractional-order autonomous systems and some similar properties for the preservation of stability in integer order systems. Some simulation examples are presented only to show the effectiveness of the analytic result.

1. Introduction

The applications of fractional calculus to science and engineering have been growing in the last few years [1]; this is due in part to the properties of these operators. The applications, specifically, that involve fractional-order chaotic systems or their synchronization had been one of the principal subjects of investigation; some of these works are [2–6]. There is also several works concerning chaotic systems or complex networks of integer order or their synchronization, for example, [7–10]. There are many different works on the synchronization of fractional autonomous systems that can be described as a linear plus a nonlinear part [11–14], in such works several schemes are proposed to ensure that the error dynamics satisfies the conditions from the celebrated theorem for autonomous commensurate differential systems with fractional order between 0 and 1 by [15]; this means that the error dynamics must hold a linear relation in order to achieve the synchronization. There is also a scheme proposed in [16], based on [17], where the dynamical system of the synchronization error

can be nonlinear, which when viewed from the analytical point can be important because it does not restrict the error dynamics to be only linear.

There is some other interesting theme, the preservation of stability and synchronization, which is the main issue in this work. This problem can be stated as follows: if we have an original autonomous nonlinear system that can be described as a linear plus a nonlinear part whose origin is stable, we want to investigate some kinds of modifications that can occur to the fractional order, the linear part, and the nonlinear part in such a way that the origin of the modified system is also stable. This subject is important because such modifications can be interpreted as perturbations on the system. Note that the modification of the linear part of the vector field associated with the fractional differential equation modifies some local properties of the vector field at the point of equilibrium, in particular local stability. In [18], the authors developed two results for the preservation of stability of integer-order nonlinear systems; one of such results gives conditions for the preservation of stability between systems of different orders of the state vector but does not give direct insight on the transformations, and the other result gives more insight but in return is a little more restrictive because as part of the hypothesis it asks for diagonalizability of the linear part of the system. In [19], the authors have reached conditions for the preservation of stability for integer-order systems in the presence of nonlinear modifications to the Jacobian matrix; such modifications can be applied on the characteristic polynomial or in form of a nonlinear polynomial matrix evaluation.

The main objective of this work is to state under which conditions a certain family of transformations applied to the fractional order, the linear part, and the nonlinear part of an autonomous fractional differential system with commensurate order will preserve stability of the origin. It is important to point out that this analytical result is of relevance for its relation with robustness not for the use of an advanced controller in the stabilization or the synchronization. As far as the authors know, this problem has not been addressed for the case of fractional-order systems.

In Section 2, we present the definitions and some results on the stability of autonomous commensurate fractional-order systems. In Section 3, the main results are stated in form of propositions and corollaries. Based on these propositions, in Section 4, we present a methodology to illustrate how these results can be used and this is complemented by the application of this methodology in two examples of simulation presented in Section 5. Finally in Section 6, we present the obtained conclusions.

2. Preliminary Results

There are several definitions of a fractional derivative of order $\alpha \in \mathbb{R}^+$ [20–22]. We will use the Caputo fractional operator because the meaning of the initial conditions for systems described using this operator is the same as for integer-order systems.

Definition 2.1 (Caputo fractional derivative). The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ of a function x is defined as (see [20])

$$x^{(\alpha)} = {}_{t_0}D_t^{\alpha} x = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{d^m x(\tau)}{d\tau^m} (t-\tau)^{m-\alpha-1} d\tau, \tag{2.1}$$

where $m-1 \leq \alpha < m$, $d^m x(\tau)/d\tau^m$ is the mth derivative of x in the usual sense, $m \in \mathbb{N}$, and Γ is the gamma function. (Throughout the paper, we use indistinctly $x^{(\alpha)} \equiv x^{(\alpha)}(t)$, $x \equiv x(t)$.)

We recall some previous results on the stability of autonomous commensurate fractional-order systems that are related to our study.

2.1. Autonomous Commensurate Fractional-Order Linear Systems Stability

Given an autonomous fractional-order system with state space representation

$$\begin{aligned} x^{(\alpha)} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{2.2}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, the state vector $x \in R^n$, the input vector $u \in R^m$, and the output vector $y \in R^p$.

Definition 2.2 (see [15]). The fractional-order autonomous system (2.2)

$$x^{(\alpha)} = Ax, \quad \text{with } x(0) = x_0, \tag{2.3}$$

is said to be

(i) stable if and only if forall $\epsilon > 0\ \exists\ \delta = \delta(\epsilon) > 0$, such that given $\|x_0\| < \delta$ then $\|x(t)\| < \epsilon$ for all $t \geq 0$;

(ii) asymptotically stable if and only if it is stable and $\lim_{t\to\infty} \|x(t)\| = 0$.

Firstly, we will introduce some results on fractional-order systems stability. First for $0 < \alpha < 1$, we have the celebrated Theorem [15] that gives us necessary and sufficient conditions for the asymptotic stability of the origin of a type of autonomous linear fractional-order systems; such conditions involve the argument of the eigenvalues of the system matrix.

Theorem 2.3. *The autonomous system*

$$x^{(\alpha)} = Ax, \quad \text{with } x(t_0) = x_0,\ 0 < \alpha < 1, \tag{2.4}$$

is asymptotically stable if and only if $|\arg(\operatorname{spec}(A))| > \alpha(\pi/2)$, *where* $\operatorname{spec}(A)$ *is the set of all the eigenvalues of* A. *Also, the state vector* x *decays towards* 0 *and meets the following condition:* $\|x\| < Nt^{-\alpha}$, $t > 0$, $\alpha > 0$.

And for $1 < \alpha < 2$, we have a similar result [23].

Theorem 2.4. *The autonomous fractional differential system*

$$x^{(\alpha)} = Ax, \quad t > t_0, \tag{2.5}$$

with initial conditions $x^{(k)}(t_0) = x_k (k = 0, 1)$, *with the Caputo derivative and where* $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ *is asymptotically stable if and only if* $|\arg(\operatorname{spec}(A))| > \alpha(\pi/2)$. *In this case, the components of the state decay towards 0 like* $t^{-\alpha-1}$. *Moreover, the system* (2.5) *is stable if and only if either it is*

asymptotically stable, or those eigenvalues which satisfy $|\arg(\mathrm{spec}(A))| = \alpha(\pi/2)$ *have the same algebraic and geometric multiplicities.*

2.2. Commensurate Fractional-Order Nonlinear Systems Stability

Given a commensurate fractional-order system with the Caputo fractional operator

$$x^{(\alpha)} = f(t, x) \tag{2.6}$$

with initial condition $x(t_0) = x_0$, $\alpha \in (0,1)$, $f : [t_0, \infty) \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_0, \infty) \times \Omega$, and $\Omega \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$.

The equilibrium point of (2.6) is defined as follows [24].

Definition 2.5. The constant x_e is an equilibrium point of the fractional-order system (2.6) if and only if $f(t, x_e) = 0$.

Without loss of generality, let the equilibrium point be $x = x_e = 0$. In this definition, we are considering that the result of the derivative of a constant is zero because we are using only the Caputo fractional operator.

Definition 2.6 (the Lyapunov stability). The equilibrium point $x = 0$ of the system (2.6) is said to be

(1) stable, if for all $\epsilon > 0$ $\exists \delta > 0$ such that if $\|x_0\| < \delta$ then $\|x\| < \epsilon$, for all $t \geq 0$. Otherwise the equilibrium point is called unstable;

(2) asymptotically stable, if it is stable and in addition the following equality holds:

$$\lim_{t \to \infty} \|x\| = 0. \tag{2.7}$$

As a starting point for the construction of our own results, we can use the following result for the stability of the origin of commensurate fractional-order systems with $0 < \alpha < 1$ [17].

Theorem 2.7. *Consider the n-dimensional nonlinear fractional-order dynamic system*

$$x^{(\alpha)} = Ax + g(x), \tag{2.8}$$

with a constant linear regular matrix A, a nonlinear function $g(x)$ of the states x, and $0 < \alpha < 1$. *If*

(1) *the zero solution of $x^{(\alpha)} = Ax$ is asymptotically stable and $\alpha\rho(A) > 1$;*

(2) $g(0) = 0$ and $\lim_{\|x\| \to 0} \|(g(x)\|/\|x\|) = 0$, *where $\rho(A)$ is the spectral radius of A,*

then $x = 0$, $0 \leq t_0 \leq t$ is a stable solution of the system (2.8).

The following result is valid for the asymptotic stability of systems with $1 < \alpha < 2$. Consider the n-dimensional nonlinear fractional-order dynamic system with the Caputo derivative

$$x^{(\alpha)} = Ax + g(t,x), \quad t > t_0, \tag{2.9}$$

under the initial conditions

$$x^{(\alpha-k)}(t)\Big|_{t=t_0} = x_{k-1} \quad (k = 1,2), \tag{2.10}$$

where $x \in \mathbb{R}$, matrix $A \in \mathbb{R}^{n\times n}$, and $1 < \alpha < 2$, $g(t,x) : [t_0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function in which $g(t,0) = 0$; moreover, $g(t,x)$ holds the Lipschitz condition with respect to x.

Theorem 2.8. *If the matrix A such that $|\arg(\operatorname{spec}(A))| \neq 0$, $|\arg(\operatorname{spec}(A))| > \alpha(\pi/2)$, $\alpha + 1/\|A\| < 2$, and suppose that the function $g(t,x)$ satisfies uniformly*

$$\lim_{x\to\infty} \frac{\|g(t,x)\|}{\|x\|} = 0, \quad t \in [t_0,\infty), \tag{2.11}$$

then the zero solution of (2.9) *is asymptotically stable.*

The proof of this theorem for the Caputo derivative follows from the proof of Theorem 3.3 in [23] and the application of Lemma 2.7 in [23] and Gronwall-Bellman inequality.

3. Preservation of Stability

So once given all these stability results, we need to give a definition for the preservation of stability in fractional-order systems in order to be in the possibility to state the conditions in form of a proposition.

Definition 3.1. Given an asymptotically stable autonomous commensurate fractional-order linear system of the kind

$$x^{(\alpha)} = Ax, \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$, $0 < \alpha < 2$ and $A = PJ_AP^{-1}$. If one has a transformation $\psi : R^+ \times R^{n\times n} \to R^+ \times R^{n\times n}$, namely, $\psi(\alpha, A) = (\alpha\beta, MA)$, such that the new system

$$x^{(\alpha\beta)} = MAx, \quad \text{with } 0 < \beta \leq 1 \tag{3.2}$$

is also asymptotically stable, where $MA = PJ_MJ_AP^{-1}$, $M \in \mathbb{R}^{n\times n}$, for some matrix $M = PJ_MP^{-1}$, where J_M and J_A are Jordan matrices, then one says that ψ is an asymptotically stability preserving transformation for commensurate fractional-order autonomous linear systems.

We should notice that for the matrices M and A we are using the same matrix P. It is also worth to mention that, given the Jordan matrix J_A that corresponds to the matrix A, the Jordan matrix J_M, that represents the modifications, must have the same order and type of Jordan blocks that J_A. The reason behind this fact is that in several applications we will need that $MA \in \mathbb{R}^{n\times n}$. But it should also be noticed that another canonical form, instead of Jordan form, could result more convenient in the construction of state feedback controllers.

Now in a similar way we state the definition for this concept in commensurate fractional-order nonlinear systems.

Definition 3.2. Given a commensurate fractional-order nonlinear system of the kind

$$x^{(\alpha)} = Ax + g(t, x), \tag{3.3}$$

where $A \in \mathbb{R}^{n\times n}$, $A = PJ_AP^{-1}$, $x \in \mathbb{R}^n$, $0 < \alpha < 2$, $g : [t_0, \infty) \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_0, \infty) \times \Omega$, and $\Omega \subset \mathbb{R}^n$ is a domain that contains the origin and the origin itself is a stable solution of the system. If one has a transformation $\Psi : R^+ \times R^{n\times n} \times C^k(R^n,\ R^n) \to R^+ \times R^{n\times n} \times C^k(R^n,\ R^n)$, namely, $\Psi(\alpha, A, g(\cdot)) = (\alpha\beta, MA, cg(\cdot))$, in such a way that in the new system

$$x^{(\alpha\beta)} = MAx + cg(t, x), \tag{3.4}$$

the origin is also a stable solution, where $c \in R$, $MA = PJ_MJ_AP^{-1}$, $M \in \mathbb{R}^{n\times n}$, for some matrix $M = PJ_MP^{-1}$, where J_M and J_A are Jordan matrices, then one calls to that transformation a stability preserving transformation for commensurate fractional-order nonlinear systems.

Remark 3.3. In Definition 3.2, for the case where $0 < \alpha < 1$, the nonlinear part is considered as autonomous, that is, for the system (3.3), we have $x^{(\alpha)} = Ax + g(x)$, and for the modified system (3.4), we have $x^{(\alpha\beta)} = MAx + cg(x)$.

Now based on the Theorems 2.3, 2.4, 2.7, and 2.8, and the results from [18] for the preservation of stability for integer-order systems, the following criterion for the preservation of stability in autonomous commensurate fractional-order systems can be stated as follows.

Proposition 3.4. *Consider an autonomous commensurate fractional-order nonlinear system of the form*

$$x^{(\alpha)} = Ax + g(x) \tag{3.5}$$

with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $g : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, D is a neighborhood of the origin for $0 < \alpha < 1$. Let $A \in \mathbb{R}^{n\times n}$ with the argument of its kth eigenvalue denoted by $\theta_k = \arg(\lambda_k(A))$. Given a transformation $\Psi(\alpha, A, g(\cdot)) = (\alpha\beta, MA, cg(\cdot))$ such that the new system is

$$x^{(\alpha\beta)} = MAx + cg(x), \tag{3.6}$$

where $c \in R$, $M \in \mathbb{R}^{n\times n}$, $0 < \beta \leq 1$, $\phi_k = \arg(\lambda_k(M))$ *is the argument of the kth eigenvalue of* M, $A = PJ_AP^{-1}$, $M = PJ_MP^{-1}$. *Also let* $\phi_{a_k} = -\theta_k + \alpha(\pi/2)$, $\phi_{b_k} = -\theta_k - \alpha\pi/2$, $\phi_{\max} = \max_k\{\phi_{a_k}\}$, $\phi_{\min} = \min_k\{\phi_{b_k}\}$, *if*

$$\phi_{\min} > \phi_k > \phi_{\max} \tag{3.7}$$

for each $k = 1, 2, \ldots, n$, *and if the system* $x^{(\alpha)} = Ax$ *is asymptotically stable,* $\alpha\rho(A) > 1$, $g(0) = 0$, $\lim_{\|x\|\to 0}\|g(x)\|/\|x\| = 0$, *and* $\rho(MA) \geq \rho(A)$, *then one claims that such transformation is a stability preserving transformation for fractional-order systems of the kind of* (3.5).

Proof . Summarizing the initial hypothesis, the original system (3.5) holds the conditions from Theorem 2.7, so we have $|\theta_k| > \alpha(\pi/2)$ for $k = 1, 2, \ldots, n$.

By the hypothesis $\rho(MA) \geq \rho(A)$, $\rho(A) > 1$, and $0 < \beta \leq 1$, we have that $\alpha\beta\rho(MA) > 1$, and we have asked for $g(x)$ to hold $g(0) = 0$ and $\lim_{\|x\|\to 0}\|g(x)\|/\|x\| = 0$. As a result we need the asymptotic stability of the system $x^{(\alpha\beta)} = MAx$ to hold the conditions of Theorem 2.7 for the new system (3.6).

By the properties of the complex numbers, and based on the fact that J_M and J_A are Jordan matrices with the same structure and that $MA = PJ_MJ_AP^{-1}$, in order to assure that the system $x^{(\alpha\beta)} = MAx$ is asymptotically stable, we need for $|\arg(\operatorname{spec}(MA))| > \alpha\beta(\pi/2)$ to hold, so first we want for

$$\left|\phi_k + \theta_k\right| > \alpha\frac{\pi}{2}, \quad k \in \{1, 2, \ldots, n\}, \tag{3.8}$$

to hold. The last part of the hypothesis states that the inequality (3.7) holds. From the right part of (3.7), we know that given that each ϕ_k is greater than $\phi_{\max}$, we have that $\phi_k > -\theta_k + \alpha(\pi/2)$. And similarly from the left part we have that $-\theta_k - \alpha(\pi/2) > \phi_k$ for any ϕ_k. Then these two parts together give us precisely that (3.8) holds, and taking from the hypothesis that $0 < \beta \leq 1$, we have that $\alpha\beta \leq \alpha$ and therefore $|\phi_k + \theta_k| > \alpha\beta(\pi/2)$, and thus the modified system holds all the conditions for the linear part from Theorem 3. From the demonstration of Theorem 3 given in [17], we can observe that $cg(x)$ also holds the corresponding conditions; therefore we claim that Ψ is a stability preserving transformation for the fractional-order autonomous systems of the form of (3.5). □

Now we have a similar result for systems with fractional orders lying between 1 and 2.

Proposition 3.5. *Consider a partially autonomous commensurate fractional-order nonlinear system of the form*

$$x^{(\alpha)} = Ax + g(t, x) \tag{3.9}$$

with $1 < \alpha < 2$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $g : [t_0, \infty) \times D \subset \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function,* D *is a neighborhood of the origin, and* $g(x)$ *holds the Lipschitz condition with respect to* x.

Let $A \in \mathbb{R}^{n\times n}$ *with the argument of its kth eigenvalue denoted by* $\theta_k = \arg(\lambda_k(A))$. *Given a transformation* $\Psi(\alpha, A, g(t, \cdot)) = (\alpha\beta, MA, cg(t, \cdot))$ *such that the new system is*

$$x^{(\alpha\beta)} = MAx + cg(t, x), \tag{3.10}$$

where $c \in R$, $1/\alpha < \beta < 2/\alpha$, $M \in \mathbb{R}^{n\times n}$, $\phi_k = \arg(\lambda_k(M))$ *is the argument of the kth eigenvalue of* M, $A = PJ_AP^{-1}$, $M = PJ_MP^{-1}$. *Also let* $\phi_{a_k} = -\theta_k + \alpha\pi/2$, $\phi_{b_k} = -\theta_k - \alpha\pi/2$, $\phi_{\max} = \max_k\{\phi_{a_k}\}$, $\phi_{\min} = \min_k\{\phi_{b_k}\}$, *if*

$$\phi_{\min} > \phi_k > \phi_{\max} \tag{3.11}$$

for each $k = 1, 2, \ldots, n$, *and if* $|\arg(\operatorname{spec}(A))| \neq 0$, $|arg(\operatorname{spec}(A))| > \alpha(\pi/2)$, $\alpha + 1/\|A\| < 2$, $\beta \leq 1$, $\beta < \alpha(2 - 1/\|MA\|)$, $\lim_{x\to 0}\|g(t,x)\|/\|x\| = 0$ *is uniformly satisfied for* $t \in [t_0, \infty)$, $\|MA\| \geq \|A\|$, *and* $|\arg(\operatorname{spec}(M))| \neq 0$, *then one claims that such transformation is a stability preserving transformation for fractional-order systems of the kind of* (3.9).

Proof. The first part of the hypothesis is that the original system (3.9) holds the conditions of Theorem 2.8; therefore, we have that $|\theta_k| > \alpha(\pi/2)$, for $k = 1, 2, \ldots, n$, and that its origin is an asymptotically stable solution.

Another part of the hypothesis is that $g(t,x)$ holds the conditions from Theorem 2.8, and from the proof of Theorem 2.8, we observe that such conditions also hold for $cg(t,x)$.

We have also asked for $\|MA\| \geq \|A\|$, $\alpha + 1/\|A\| < 2$, and $\beta < \alpha(2 - 1/\|MA\|)$ to hold, and thus we have that $\alpha\beta + 1/\|MA\| < 2$. As a result, we only need $|\arg(\operatorname{spec}(MA))| > \alpha\beta(\pi/2)$ to hold the conditions from Theorem 2.8 for the new system (3.10).

By very similar arguments as in the proof of Proposition 3.4, we have that (3.8) holds and we have asked for $\beta \leq 1$ to hold; therefore conditions from Theorem 2.8 are satisfied and we claim that Ψ is a stability preserving transformation for the fractional-order autonomous systems of the form of (3.5). □

The following corollaries are a direct consequence of Proposition 1 and Theorem 3.3 in [23].

Corollary 3.6. *Consider a linear autonomous commensurate fractional-order system of the form*

$$x^{(\alpha)} = Ax, \quad \text{with } 0 < \alpha < 2, \tag{3.12}$$

that holds the conditions from Theorem 2.4 or Theorem 2.3 accordingly, if one has a transformation $\Psi(\alpha, M) = (\alpha\beta, MA)$ *such that the new system is*

$$x^{(\alpha\beta)} = MAx, \tag{3.13}$$

with all the variables and matrices as defined before, and if the inequality

$$\phi_{\min} > \phi_k > \phi_{\max} \tag{3.14}$$

holds, with ϕ_k, $\phi_{\min}$, *and* $\phi_{\max}$ *as defined before, then one claims that* Ψ *is an asymptotic stability preserving transformation for the autonomous commensurate fractional-order linear systems of the form of* (3.12).

Remark 3.7. If we take the case of $\alpha = 1$ for Proposition 3.4, or Proposition 3.5 we will have similar conditions for the preservation of stability of the origin for an integer-order nonlinear

system $\dot{x} = Ax + g(x)$. And if we take the case of $\alpha = 1$ for Corollary 3.6, we will have the conditions for the preservation of asymptotic stability of an integer-order linear system $\dot{x} = Ax$, without the modification over the order, of course.

Remark 3.8. Notice that the conditions from [18, 19] are different from the ones that we obtain choosing $\alpha = 1$ in Proposition 3.4 and cannot be obtained as a particular case of Proposition 3.4.

4. Preservation of Synchronization

Given the conditions from Proposition 3.4, we want to illustrate how to use these criteria to ensure the preservation of stabilization and synchronization.

4.1. Preservation of Stabilization of Autonomous Commensurate Fractional-Order Nonlinear Systems

First for the stabilization of a system, we know from previous works that for an autonomous fractional commensurate order system of the form $x^{(\alpha)} = Ax + g(x) + u$, where $g(x)$ holds, the conditions from Theorem 2.7, and with $A = PJ_AP^{-1}$, we can choose a control $u = -K_1x$, $K_1 \in R^{n\times n}$, with $K_1 \in \mathbb{R}^{n\times n}$ in such a way that for the system $x^{(\alpha)} = (A-K_1)x+g(x)$, the origin is a stable solution. But in this particular case we want that $A-K_1 = P(J_A-J_{K_1})P^{-1}$; therefore, we need to construct K_1 as $K_1 = PJ_{K_1}P^{-1}$; the Jordan form J_{K_1} also has the restriction that its Jordan blocks must be of the same order and type as the ones of J_A.

Now for the modified system $x^{(\alpha\beta)} = MAx + cg(x) + u$, we will use a similar control (i.e., with the same K_1) defined as $u = -MK_1x$, where $M := PJ_MP^{-1}$ holds the conditions from 1 and all the other matrices defined as before, in such a way that for the system $x^{(\alpha\beta)} = (MA - MK_1)x + cg(x)$, the origin is also a stable solution.

4.2. Preservation of Complete Practical Synchronization of Autonomous Commensurate Fractional-Order Nonlinear Systems

First we need to describe the synchronization scheme. Let us consider two fractional-order systems as the master and the slave system, respectively,

$$\begin{gathered} x_{\mathcal{M}}^{(\alpha)} = A_{\mathcal{M}}x_{\mathcal{M}} + g(x_{\mathcal{M}}), \qquad x_{\mathcal{S}}^{(\alpha)} = A_{\mathcal{S}}x_{\mathcal{S}} + g(x_{\mathcal{S}}) + w, \\ y_{\mathcal{M}} = h_{\mathcal{M}}(x_{\mathcal{M}}), \qquad y_{\mathcal{S}} = h_{\mathcal{S}}(x_{\mathcal{S}}), \end{gathered} \tag{4.1}$$

where $x_{\mathcal{M}} \in R^n$ is the state vector of the master system, $y_{\mathcal{M}} :\in R^p$ is the output of the master system, $x_{\mathcal{S}} \in R^n$ is the state vector of the slave system, $w \in R^n$ is the control input, and $y_{\mathcal{S}} \in R^p$ is the output of the slave system.

In this synchronization scheme, the master system represents the *target* dynamics, while the slave system represents the system to be controlled.

Let us consider that all the outputs are available, only to illustrate the effectiveness of the method by showing more states on the graphs of the synchronization error in the

examples, of course the order of the output can be less than the order of the state vector. This consideration will lead us to the synchronization error:

$$e = x_{\mathcal{M}} - x_{\mathcal{S}}, \tag{4.2}$$

we must find a function w such that $\|e(t)\|$ is bounded in a subset that contains the origin, because the result presented in [17] only gives conditions for the stability of the origin, not for asymptotic stability. Because of this fact, this approach is called complete *practical* synchronization.

We specifically choose the control

$$w = g(x_{\mathcal{M}}) - g(x_{\mathcal{S}}) - g(x_{\mathcal{M}} - x_{\mathcal{S}}) + A_{\mathcal{M}} x_{\mathcal{M}} - A_{\mathcal{S}} x_{\mathcal{S}} - A_{\mathcal{S}}(x_{\mathcal{M}} - x_{\mathcal{S}}) + K_2(x_{\mathcal{M}} - x_{\mathcal{S}}), \tag{4.3}$$

where $K_2 \in R^{n\times n}$, in such a way that the error dynamics is given by

$$e^{(\alpha)} = x_{\mathcal{M}}^{(\alpha)} - x_{\mathcal{S}}^{(\alpha)} = (A_{\mathcal{S}} - K_2)e + g(e). \tag{4.4}$$

As before, given that $A_{\mathcal{S}} = P_{A_{\mathcal{S}}} J_{A_{\mathcal{S}}} P_{A_{\mathcal{S}}}^{-1}$, we will have $A_{\mathcal{S}} - K_2 = P_{A_{\mathcal{S}}}(J_{A_{\mathcal{S}}} - J_{K_2})P_{A_{\mathcal{S}}}^{-1}$, and therefore we need to construct K_2 as $K_2 = P_{A_{\mathcal{S}}} J_{K_2} P_{A_{\mathcal{S}}}^{-1}$ in such a way that the origin of the dynamic system of the error is stable. Again the Jordan form J_K has the restriction that its Jordan blocks must be of the same order and type as the ones of $J_{A_{\mathcal{S}}}$.

Then, we want to illustrate what kind of transformations can be applied to the master and slave systems in such a way that the same K_2 still stabilizes the origin of the modified synchronization error system. And to do this we define the modification matrices $M_{\mathcal{M}} := P_{A_{\mathcal{M}}} J_M P_{A_{\mathcal{M}}}^{-1}$ and $M_{\mathcal{S}} := P_{A_{\mathcal{S}}} J_M P_{A_{\mathcal{S}}}^{-1}$ that hold the conditions from Proposition 3.4. With these modifications applied to each of the systems in the following way:

$$x_{\mathcal{M}}^{(\alpha)} = M_{\mathcal{M}} A_{\mathcal{M}} x_{\mathcal{M}} + g(x_{\mathcal{M}}) \qquad x_{\mathcal{S}}^{(\alpha)} = M_{\mathcal{S}} A_{\mathcal{S}} x_{\mathcal{S}} + g(x_{\mathcal{S}}) + w, \tag{4.5}$$

with the synchronization error defined as in (4.2), and the control defined as

$$w = g(x_{\mathcal{M}}) - g(x_{\mathcal{S}}) - g(e) + M_{\mathcal{M}} A_{\mathcal{M}} x_{\mathcal{M}} - M_{\mathcal{S}} A_{\mathcal{S}} x_{\mathcal{S}} - M_{\mathcal{S}} A_{\mathcal{S}} x_{\mathcal{S}} + M_{\mathcal{S}} K_2 e \tag{4.6}$$

(the only change is that instead of the term $K_2 e$ now is $M_{\mathcal{S}} K_2 e$), one has that the autonomous commensurate fractional order dynamical system of the error is

$$e^{(\alpha)} = M_{\mathcal{S}}(A_{\mathcal{S}} - K_2)e + g(e) = (M_{\mathcal{S}} A_{\mathcal{S}} - M_{\mathcal{S}} K_2)e + g(e). \tag{4.7}$$

Given that one has constructed the matrix M in such a way that it holds the conditions from Proposition 3.4 it is straightforward to prove that the origin of the new dynamic system of the error is also stable. Note that the modification of the linear part of the vector field associated to the fractional differential equation, modifies the manifold of synchronization, but not the stability.

5. Examples

5.1. Preservation of Stabilization for Fractional-Order Lorenz Systems

Let us take the Lorenz system with commensurate fractional-order $\alpha = 0.97$ that can be written as [5]:

$$x^{(\alpha)} = Ax + g(x) + u = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\overline{\beta} \end{bmatrix} x + \begin{bmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{bmatrix} - K_1 x \tag{5.1}$$

with $x = [x_1 \;\; x_2 \;\; x_3]^T$, $\sigma = 10$, $\rho = 28$, $\overline{\beta} = 8/3$, initial conditions $x(0) = [-9 \;\; -5 \;\; 14]^T$. This system is chaotic as it is claimed in [25].

The objective is to stabilize the system and then apply a modification that holds the conditions of Proposition 3.4, to illustrate the validity of the analytical results.

In order to do this the next, we choose $u = -K_1 x$. With

$$J_{K_1} = \begin{bmatrix} -17 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \qquad K_1 = \begin{bmatrix} -5.1552 & 9.2338 & 0 \\ 25.8545 & 3.1552 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \tag{5.2}$$

the eigenvalues of the new matrix $A - K_1$ are $\lambda_1 \approx -5.827$, $\lambda_2 \approx -3.172$, and $\lambda_3 \approx -6.666$, with this and the fact that $g(x)$ holds the conditions from Theorem 2.7 (as it has been demonstrated in [17]), we know that the origin of the controlled system is a stable solution.

Now we are interested in verifying what will happen if we propose a modification such as $\alpha\beta = 0.95$, $\beta \approx 0.97938$, $c = 0.8$, and

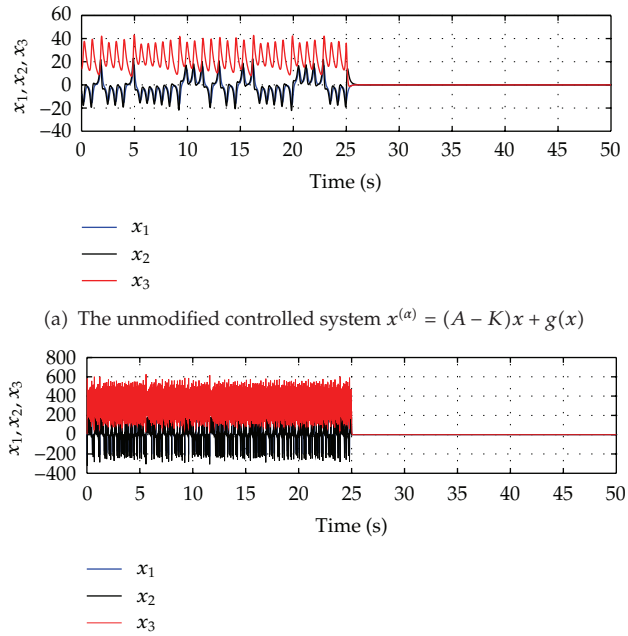
$$J_M = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 10.4 \end{bmatrix}, \qquad M = \begin{bmatrix} 10.2595 & -0.5771 & 0 \\ -1.6159 & 9.7403 & 0 \\ 0 & 0 & 10.4 \end{bmatrix}, \tag{5.3}$$

it can be easily verified that M holds the conditions of Proposition 3.4. The eigenvalues of the modified system $M(A - K_1)$ are $\lambda_1 \approx -64.105$, $\lambda_2 \approx -28.550$, and $\lambda_3 \approx -69.333$. Given that this eigenvalues hold the conditions of Theorem 2.7 and that $0.8g(x)$ also holds the rest of the conditions, the origin of the modified controlled system is also a stable solution.

In Figures 1 and 4, the simulation step and time were 0.004 and 50 s, respectively. In the first 25 s, the system was the original one ($u = 0$), and for the last 25 s the control u was activated, for the unmodified and modified systems (Figure 2).

5.2. Preservation of Complete Practical Synchronization for Fractional-Order Chen Systems

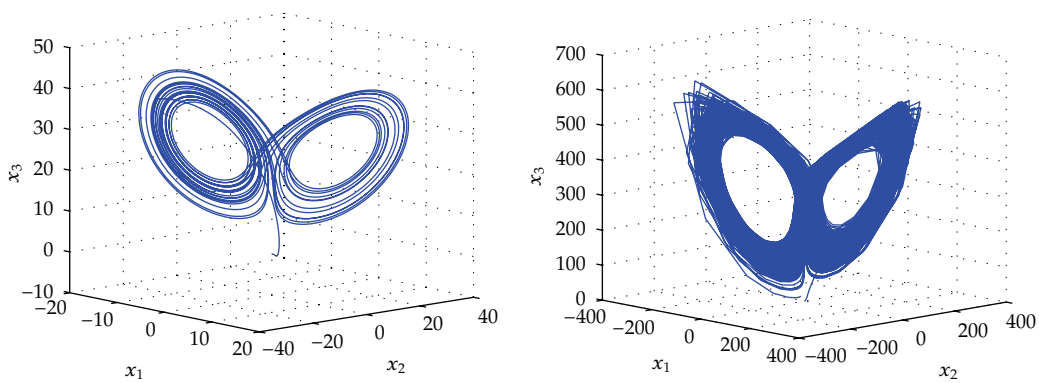
In this example specifically, we will make the synchronization of two fractional-order Chen systems with identical parameters and orders but different initial conditions, and because of these consideration, we will have that $A_{\mathcal{M}} = A_{\mathcal{S}}$, and therefore we drop the indices for the matrices A and M.



(a) The unmodified controlled system $x^{(\alpha)} = (A - K)x + g(x)$

(b) The modified controlled system $x^{(\alpha)} = M(A - K)x + g(x)$

Figure 1: Graphs of the states versus time of the controlled systems.



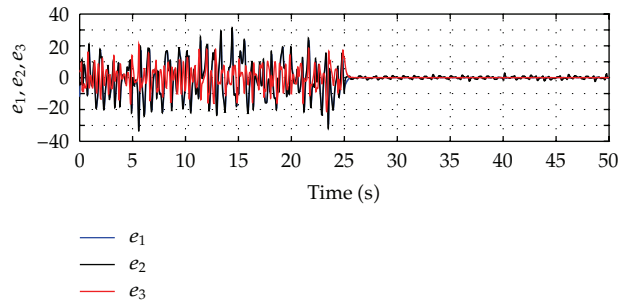
(a) The system $x^{(\alpha)} = (A - K)x + g(x)$

(b) The system $x^{(\alpha)} = M(A - K)x + g(x)$

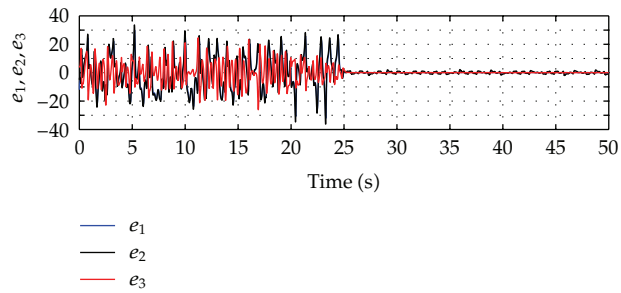
Figure 2: Phase plane of the unmodified and modified systems.

We use the structure of the Chen system with fractional-order α as presented in [3]. For the slave system, we have

$$x_{\mathcal{S}}^{(\alpha)} = Ax_{\mathcal{S}} + g(x_{\mathcal{S}}) = \begin{bmatrix} -a & a & 0 \\ d-a & d & 0 \\ 0 & 0 & -b \end{bmatrix} x_{\mathcal{S}} + \begin{bmatrix} 0 \\ -x_{S_1}x_{S_3} \\ x_{S_1}x_{S_2} \end{bmatrix} + w \tag{5.4}$$



(a) Synchronization error of the systems without modification



(b) Synchronization error of the modified systems

Figure 3: Errors graph with the control law applied at $t = 25$ s.

with w as defined in (4.6), $x_{\mathcal{S}} = [x_{S_1} \quad x_{S_2} \quad x_{S_3}]^T$, $a = 35$, $b = 3$, $d = 28$, and $\alpha = 0.975$, initial conditions $x_{\mathcal{S}}(0) = [3 \quad 0 \quad 10]^T$.

Now, for the master system, we take

$$x_{\mathcal{M}}^{(\alpha)} = Ax_{\mathcal{M}} + g(x_{\mathcal{M}}) = \begin{bmatrix} -a & a & 0 \\ d-a & d & 0 \\ 0 & 0 & -b \end{bmatrix} x_{\mathcal{M}} + \begin{bmatrix} 0 \\ -x_{M_1}x_{M_3} \\ x_{M_1}x_{M_2} \end{bmatrix} \tag{5.5}$$

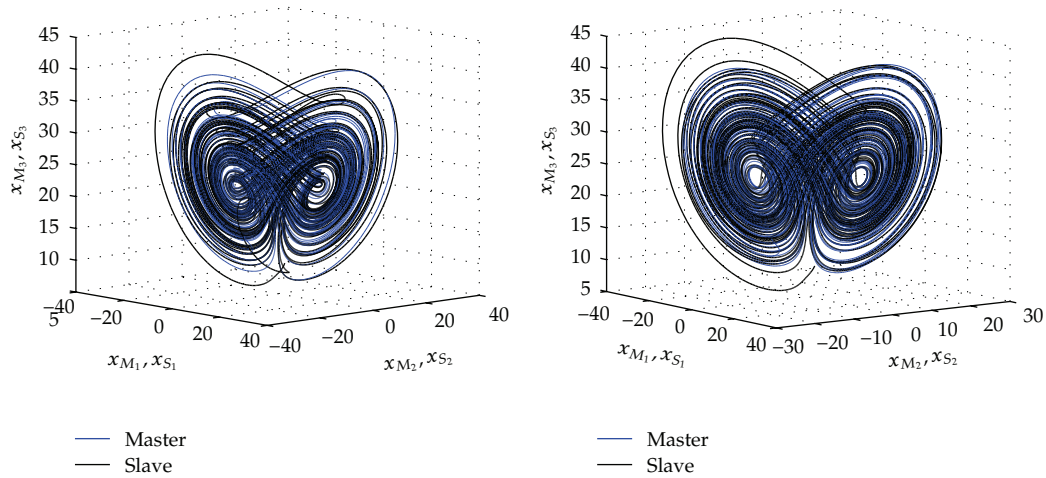
with $x_{\mathcal{M}} = [x_{M_1} \quad x_{M_2} \quad x_{M_3}]^T$, $a = 35$, $b = 3$, $d = 28$, $\alpha = 0.975$, and initial conditions $x_{\mathcal{M}}(0) = [-9 \quad -5 \quad 14]^T$.

Both master and slave systems are chaotic, as far as the conditions from [25]. For the control law (4.3), we have

$$J_{K_2} = \begin{bmatrix} -26 & 0 & 0 \\ 0 & 28 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -30.1130 & 34.5700 & 0 \\ -6.9140 & 32.1130 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \tag{5.6}$$

and for the modification we take $c = 0.9$, $\alpha\beta = 0.96$, $\beta \approx 0.9846$, and

$$J_M = \begin{bmatrix} 1.075 & 0 & 0 \\ 0 & 0.955 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}, \quad M = \begin{bmatrix} 1.0841 & -0.0768 & 0 \\ 0.0154 & 0.9459 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}. \tag{5.7}$$



(a) The unmodified synchronization (b) The modified synchronization

Figure 4: Phase plane of the unmodified and modified synchronizations; the master systems are in blue, and the slave systems in black; the control law is applied at $t = 50$ s.

With K_2 as proposed, it is assured that the synchronization error system without modification holds the conditions of Theorem 2.7. It can be easily verified that M and $0.9g(x)$ hold the conditions from Proposition 3.4, in such a way that we can assure that this is an example of preservation of synchronization.

Again the simulation step and time were 0.0028 and 50 s, respectively. In the first 25 s the system was the original one ($w = 0$), and for the last 25 s, the control w was activated, for the unmodified and modified systems.

In Figure 3, we can observe how the application of the control law, with the same K_2, stabilizes the origin of the unmodified and the modified synchronization error systems.

In Figure 4, we can observe the way the slave system follows to the master system for the unmodified and the modified systems with the same K_2. All the simulations were made using the algorithms presented in [26].

From several simulations, we have observed that under large variations in the parameters these transformations do not preserve chaos. This limits the possible variations in the transformations because, as it is well known, the chaos in dynamical systems is very sensitive to variations in the parameters.

6. Conclusions

As far as the authors know, this is the first time that the preservation of autonomous commensurate fractional order systems stability is made considering transformations that affect the fractional order, the linear part, and the nonlinear part of the vector field of the differential equation.

Furthermore, we have explained how these results can be used to ensure the preservation of stabilization and the preservation of synchronization of autonomous commensurate fractional-order systems and, through the presented examples, we have also showed the effectiveness of the results.

It is also worth to mention that there are some other results on the stability of fractional-order autonomous systems [25, 27, 28] that can be used in a similar way to Proposition 3.4 to state the conditions for the preservation of asymptotic stability of the solutions, for $0 < \alpha < 1$.

References

[1] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing, River Edge, NJ, USA, 2000.

[2] J. Lü and G. Chen, "Generating multiscroll chaotic attractors: theories, methods and applications," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 16, no. 4, pp. 775–858, 2006.

[3] C. Li and G. Chen, "Chaos in the fractional order Chen system and its control," *Chaos, Solitons and Fractals*, vol. 22, no. 3, pp. 549–554, 2004.

[4] C. Li and G. Chen, "Chaos and hyperchaos in the fractional-order Rössler equations," *Physica A*, vol. 341, no. 1–4, pp. 55–61, 2004.

[5] I. Grigorenko and E. Grigorenko, "Chaotic dynamics of the fractional Lorenz system," *Physical Review Letters*, vol. 91, no. 3, Article ID 034101, 4 pages, 2003.

[6] T. T. Hartley, C. F. Lorenzo, and H. K. Qammer, "Chaos in a fractional order Chua's system," *IEEE Transactions on Circuits and Systems I*, vol. 42, no. 8, pp. 485–490, 1995.

[7] J. Lü, S. Yu, H. Leung, and G. Chen, "Experimental verification of multidirectional multiscroll chaotic attractors," *IEEE Transactions on Circuits and Systems I*, vol. 53, no. 1, pp. 149–165, 2006.

[8] J. Lü and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Transactions on Automatic Control*, vol. 50, no. 6, pp. 841–846, 2005.

[9] J. Zhou, J.-A. Lu, and J. Lü, "Adaptive synchronization of an uncertain complex dynamical network," *IEEE Transactions on Automatic Control*, vol. 51, no. 4, pp. 652–656, 2006.

[10] J. Lü, X. Yu, G. Chen, and D. Cheng, "Characterizing the synchronizability of small-world dynamical networks," *IEEE Transactions on Circuits and Systems I*, vol. 51, no. 4, pp. 787–796, 2004.

[11] Y. Tang and J.-A. Fang, "Synchronization of N-coupled fractional-order chaotic systems with ring connection," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 2, pp. 401–412, 2010.

[12] G. Peng, Y. Jiang, and F. Chen, "Generalized projective synchronization of fractional order chaotic systems," *Physica A*, vol. 387, no. 14, pp. 3738–3746, 2008.

[13] X. Y. Wang and J. M. Song, "Synchronization of the fractional order hyperchaos Lorenz systems with activation feedback control," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 8, pp. 3351–3357, 2009.

[14] J. G. Lu, "Chaotic dynamics of the fractional-order Lü system and its synchronization," *Physics Letters, Section A*, vol. 354, no. 4, pp. 305–311, 2006.

[15] D. Matignon, "Stability results for fractional differential equations with applications to control processing," in *Proceedings of the Computational Engineering in System Application*, pp. 963–968, 1996.

[16] R. Martínez-Martínez, H. Jardón-Kojakhmetov, J. A. León, and G. Fernández-Anaya, *Estabilización de Redes Complejas Fraccionarias de Sistemas de Lorenz y Sistemas de Chen*, AMCA, 2009.

[17] X.-J. Wen, Z.-M. Wu, and J.-G. Lu, "Stability analysis of a class of nonlinear fractional-order systems," *IEEE Transsactions on Circuits and Systems-II*, vol. 55, pp. 1178–1182, 2008.

[18] G. Fernández-Anaya, J. J. Flores-Godoy, R. Femat, and J. J. Álvarez-Ramírez, "Preservation of stability and synchronization in nonlinear systems," *Physics Letters, Section A*, vol. 371, no. 3, pp. 205–212, 2007.

[19] D. Becker-Bessudo, G. Fernandez-Anaya, and J. J. Flores-Godoy, "Preserving synchronization under matrix product modifications," *Physica A*, vol. 387, no. 26, pp. 6631–6645, 2008.

[20] I. Podlubny, *Fractional Differential Equations*, vol. 198, Academic Press, San Diego, Calif, USA, 9th edition, 1999.

[21] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.

[22] I. Podlubny, "Geometric and physical interpretation of fractional integration and fractional differentiation," *Fractional Calculus & Applied Analysis*, vol. 5, no. 4, pp. 367–386, 2002.

[23] F. Zhang and C. Li, "Stability analysis of fractional differential systems with order lying in $(1, 2)$," *Advances in Difference Equations*, vol. 2011, Article ID 213485, 17 pages, 2011.

[24] Y. Li, Y. Chen, and I. Podlubny, "Mittag-Leffler stability of fractional order nonlinear dynamic systems," *Automatica*, vol. 45, no. 8, pp. 1965–1969, 2009.
[25] M. S. Tavazoei and M. Haeri, "Chaotic attractors in incommensurate fractional order systems," *Physica D. Nonlinear Phenomena*, vol. 237, no. 20, pp. 2628–2637, 2008.
[26] I. Petras, Fractional Order Chaotic Systems, 2010, http://www.mathworks.com/matlabcentral/fileexchange/27336-fractional-order-chaotic-systems.
[27] W. Deng, C. Li, and J. Lü, "Stability analysis of linear fractional differential system with multiple time delays," *Nonlinear Dynamics*, vol. 48, no. 4, pp. 409–416, 2007.
[28] M. S. Tavazoei and M. Haeri, "A necessary condition for double scroll attractor existence in fractional-order systems," *Physics Letters, Section A*, vol. 367, no. 1-2, pp. 102–113, 2007.