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Research Article

Golden Ratio Phenomenon of Random Data Obeying von Karman Spectrum

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von Karman originally deduced his spectrum of wind speed fluctuation based on the Stokes-Navier equation. Taking into account, the practical issues of measurement and/or computation errors, we suggest that the spectrum can be described from the point of view of the golden ratio. We call it the golden ratio phenomenon of the von Karman spectrum. To depict that phenomenon, we derive the von Karman spectrum based on fractional differential equations, which bridges the golden ratio to the von Karman spectrum and consequently provides a new outlook of random data following the von Karman spectrum in turbulence. In addition, we express the fractal dimension, which is a measure of local self-similarity, using the golden ratio, of random data governed by the von Karman spectrum.

1. Instruction

The golden ratio, denoted by φ, is an irrational number given by $\varphi = (1 + \sqrt{5})/2$ [1]. The paper by Ackermann [2] may likely be the earliest literature on the golden ratio in a mathematics journal in English in 1895, but it attracted and has attracted the interest of scientists and engineers in various fields of sciences and engineering, ranging from chemistry to computer science; see, for example, [1], Benassi [3], Putz [4], Orita et al. [5], Perez [6], Hassaballah et al. [7], Kellerhals [8], Henein et al. [9], Hurtley [10], Coldea et al. [11], Affleck [12], Jones et al. [13], Kaygn et al. [14], Cervantes et al. [15], Chebotarev [16], Benavoli et al. [17], Manikantan et al. [18], Assimakis et al. [19], Good [20], Davis and Jahnke [21], Totland [22], Moufarrège [23], Boeyens [24], Iñiguez et al. [25], Andrews and Zhang [26], Hofri and Rosberg [27], Itai and Rosberg [28], Cassandras and Julka [29], and Tanackov et al. [30], just to mention a few.

In the field of random functions, more precisely, turbulence in fluid mechanics, a kind of power spectra density (PSD) function introduced by von Karman [31], known as the von Karman spectra (VKS), has been widely used in the diverse fields, ranging from turbulence to acoustic wave propagation in random media; see, for example, Goedecke et al. [32] and the references therein. Among the von Karman spectra, the spectrum (VKSW for short) expressed in (1) is particularly useful in the field of wind engineering for the modeling of wind speed fluctuation; see, for example, [33–41]. That PSD is in the form

$$S_{\text{von}}(f) = \frac{4u_f^2 b_v w}{f(1 + 70.8w^2)^{5/6}}, \quad w = \frac{f L_u^x}{U}, \tag{1}$$

where f is frequency (Hz), L_u^x is turbulence integral scale, U is mean speed, u_f is friction velocity (ms^{-1}), and b_v is friction velocity coefficient such that the variance of wind speed $\sigma_u^2 = b_v u_f^2$.

Note that (1) was conventionally deduced based on the Stokes-Navier equation ([31], Bauer and Zeibig [42], Tropea [43], Monin and Yaglom [44], Xiushu [45]). It does not originally relate to the concept of either the golden ratio or fractal dimension. As a matter of fact, reports regarding turbulence's fractal dimension derived directly based on the Stokes-Navier equation are rarely seen as Gaoan stated in [46, page 55], letting alone the golden ratio.

This paper aims at contributing the following three results. First, we will propose a rigorous but concise derivation of (1). Then, we will generalize (1) such that the generalization may be described from the point of view of the golden ratio. Finally, we will explain the golden ratio phenomenon of the VKSW from the point of view of fractal dimension or local self-similarity. As a result, we achieve the goal of bridging the golden ratio to the VKSW as well as self-similarity of random data, establishing a new outlook of data following the VKSW.

The rest of the paper is organized as follows. The preliminaries are briefed in Section 2. The results are given in Section 3, which is followed by conclusions.

2. Preliminaries

2.1. Golden Ratio. One of the conventional ways to deduce the golden ratio φ is to solve the difference equation that produces the Fibonacci sequences. The equation is given by (see, e.g., [1], Jamieson [46], Ranum [47], and Eggar [48])

$$F(n) = F(n-1) + F(n-2), \quad n \in \mathbf{N} \text{ (the set of natural numbers)}. \tag{2}$$

Denote by $Z_F(z)$ the z-transform of $F(n)$. Then, doing the z-transform on both sides of (2) yields

$$Z_F(z) = Z_F(z)\left(z^{-1} + z^{-2}\right). \tag{3}$$

The above expression can be rewritten as

$$z^2 - z - 1 = 0. \tag{4}$$

The solutions to (4) are expressed by

$$z_{1,2} = \begin{cases} \dfrac{1+\sqrt{5}}{2} \\ \dfrac{1-\sqrt{5}}{2}. \end{cases} \tag{5}$$

The golden ratio equals z_1; that is,

$$\varphi = z_1 = \frac{1+\sqrt{5}}{2} \approx 1.618. \tag{6}$$

In addition,

$$z_2 = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi} \approx -0.618. \tag{7}$$

2.2. Fractional Oscillators. There are three types of fractional oscillators. The conventional type, see, for example, Achar et al. [49, 50], is given by

$$\frac{d^{2-\varepsilon}x(t)}{dt^{2-\varepsilon}} + \omega_0^2 x(t) = e(t), \quad 0 < \varepsilon < 1. \tag{8}$$

The second type was introduced by Lim and Muniandy [51]. It is in the form

$$\left(\frac{d^2}{dt^2} + \lambda\right)^{\beta} x(t) = e(t), \quad \beta > 0. \tag{9}$$

The third type introduced by Lim and Teo [52] is expressed by

$$\left({}_aD_t^{\alpha} + \lambda\right)^{\beta} x_{\alpha,\beta}(t) = e(t), \quad 0 < \alpha < 1, \ \beta > 0. \tag{10}$$

The symbol ${}_aD_t^{\alpha}$ is a fractional differential operator; see, for example, Eab and Lim [53, 54], Lim et al. [55], Klafter et al. [56], Machado et al. [57], and Cattani [58]. In what follows, we use (10) in the general sense.

We now consider the fractional differential equation with the coefficient A in the form

$$A\left(\frac{d}{dt} + \lambda\right)^{\beta} y_{\text{fOU}}(t) = \eta(t), \quad \beta > 0, \tag{11}$$

where $\eta(t)$ is a white noise. Then, the solution to (11) is the fractional Ornstein-Uhlenbeck (OU) process [59], referring to Coffey et al. [60] for the meaning of OU process.

3. Results

Denote that $g_{\text{fOU}}(t)$ is the impulse response function of (11). Then, it is the solution to the following equation with zero initial conditions

$$A\left(\frac{d}{dt} + \lambda\right)^{\beta} g_{\text{fOU}}(t) = \delta(t), \tag{12}$$

where $\delta(t)$ is the Dirac-δ function. Doing the Fourier transforms on both sides of the above equation yields

$$G_{\text{fOU}}(f) = \frac{A}{(\lambda - j2\pi f)^{\beta}}, \tag{13}$$

where $G_{\text{fOU}}(f)$ is the Fourier transform of $g_{\text{fOU}}(t)$.

Let $S_{y_{\text{fOU}}}(f)$ be the PSD of $y_{\text{fOU}}(t)$. Then, $S_{y_{\text{fOU}}}(f)$ is given by

$$S_{y_{\text{fOU}}}(f) = \left|G_{\text{fOU}}(f)\right|^2 = \frac{A^2}{\left[\lambda^2 + (2\pi f)^2\right]^{\beta}}. \tag{14}$$

Thus, we have the theorem below.

Theorem 1. *Let $X_{vk}(t)$ be the random function that obeys* (1). *Then, $X_{vk}(t)$ is governed by the fractional differential equation that is in the form*

$$\sqrt{A_{vk}}\left(\frac{d}{dt} + B_{vk}\right)^{5/6} X_{vk}(t) = \eta(t). \tag{15}$$

Its solution in frequency domain is given by

$$S_{vk}(f) = \frac{A_{vk}}{\left[(B_{vk})^2 + (2\pi f)^2\right]^{5/6}}. \tag{16}$$

Proof. Replacing A, λ, and β in (11) with $\sqrt{A_{\text{vk}}}$, B_{vk}, and 5/6, respectively, yields (15). Substituting A, λ, and β in (14) with $\sqrt{A_{\text{vk}}}$, B_{vk}, and 5/6, respectively, produces (16). Thus, Theorem 1 results. □

From Theorem 1, we obtain the following corollary.

Corollary 2 (modified VKSW). *Let $X_{vk\varphi}(t)$ be the random function that is governed by the fractional differential equation given by*

$$\sqrt{A_{vk}}\left(\frac{d}{dt}+B_{vk}\right)^{\varphi/2}X_{vk\varphi}(t)=\eta(t). \tag{17}$$

Then, its solution in frequency domain is in the form

$$S_{vk\varphi}(f)=\frac{A_{vk}}{\left[(B_{vk})^2+(2\pi f)^2\right]^{\varphi/2}}. \tag{18}$$

The proof is straightforward and omitted consequently.

From Theorem 1 and Corollary 2, we immediately have the remark below.

Remark 3. The VKSW may be approximately expressed by the golden ratio.

As a matter of fact,

$$\frac{\varphi}{2}\approx\frac{5}{6}. \tag{19}$$

Thus, (16) approximately equals (18). This may not be a simple approximation but substantially develops the implication of the VKSW from the point of view of the golden ratio.

Note that there are errors in measuring real random data [61–63] and computation errors [64–66]. Thus, from a view of practice, the power of the VKSW may not exactly be the value of 5/6 in most cases in engineering. Rather, it may be in the form

$$\left(\frac{5}{6}\right)+e, \tag{20}$$

where e is error. Thus, by using the golden ratio, (18) is quite reasonable to characterize random functions that obey the VKSW.

For the purpose of exhibiting the results in time domain, we denote by F and F^{-1} the operator of the Fourier transform and its inverse, respectively. Then, we get the theorem below.

Theorem 4. *The inverse Fourier transform of $1/[1+(2\pi f)^2]^{\varphi/2}$ is given by*

$$F^{-1}\left\{\frac{1}{\left[1+(2\pi f)^2\right]^{\varphi/2}}\right\}=\frac{2\sqrt{\pi}}{\Gamma(\varphi/2)}\left(\frac{|\tau|}{2}\right)^{(\varphi-1)/2}K_{(\varphi-1)/2}(|\tau|), \tag{21}$$

where $K_\nu(z)$ is the modified Bessel function of second kind or the MacDonald function and τ is the time lag.

Proof. Because $(\varphi-1)/2>1/2$, according to the computation formula in Gelfand and Vilenkin [67, page 188, in Section 2, Chapter 2], (21) holds. □

Recall that the Fourier transform of $|t|^\alpha$ is expressed by [68]

$$F\left(|t|^\alpha\right)=-2\sin\left(\frac{\alpha\pi}{2}\right)\Gamma(\alpha+1)|\omega|^{-\alpha-1}, \tag{22}$$

where $\alpha\neq 1,3,\ldots$.

Note that

$$S_{\mathrm{vk}\varphi}(f)\sim\frac{1}{f^{\varphi/2}}\quad\text{for } f\longrightarrow\infty. \tag{23}$$

Then, denoting $r_{\mathrm{vk}\varphi}(\tau)$ is the inverse Fourier transform of $S_{\mathrm{vk}\varphi}(f)$, one has

$$r_{\mathrm{vk}\varphi}(\tau)\sim|\tau|^{(\varphi-1)/2}\quad\text{for }\tau\longrightarrow 0. \tag{24}$$

The fractal dimension of a process can be determined by its autocorrelation function (ACF) for $\tau\to 0$ [69]. Thus,

$$r_{\mathrm{vk}}(0)-r_{\mathrm{vk}}(\tau)\sim|\tau|^{(\varphi-1)/2}\quad\text{for }\tau\longrightarrow 0. \tag{25}$$

Therefore, with the probability one [69], the fractal dimension of the modified von Karman process based on the golden ratio is given by

$$D_{\mathrm{vk}\varphi}=\left(2-\frac{\varphi-1}{4}\right)=\frac{7-\varphi}{4}. \tag{26}$$

Approximately, it is expressed by

$$D_{\mathrm{vk}\varphi}\approx 1.346. \tag{27}$$

Note that fractal dimension is a measure of local self-similarity, irregularity, or roughness [70]. High value of fractal dimension of a sample path implies high irregularity of that path. Thus, (26) means that the modified von Karman process with the golden ratio has considerable local irregularity.

We would like to call out the work described above as the golden ratio phenomenon of the von Karman process. From the point of view of our work in data science or big data, this research may not be enough. The future work will investigate possible golden ratio phenomena in other topics of data such as those discussed in [71–82], exploring laws associating with the golden ratio in the universe.

4. Conclusions

We have given the derivation of the von Karman spectrum based on the fractional differential equation (10). The results suggest that the process obeying VKSW is in the class of fractional OU processes. Moreover, we have explained the reasons why the VKSW may be described from the point of view of the golden ratio. The fractal dimension of random data obeying the VKSW by using the golden ratio has also been discussed.

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