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Research Article

Contact Problem for an Elastic Layer on an Elastic Half Plane Loaded by Means of Three Rigid Flat Punches

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The frictionless contact problem for an elastic layer resting on an elastic half plane is considered. The problem is solved by using the theory of elasticity and integral transformation technique. The compressive loads P and Q (per unit thickness in z direction) are applied to the layer through three rigid flat punches. The elastic layer is also subjected to uniform vertical body force due to effect of gravity. The contact along the interface between elastic layer and half plane is continuous, if the value of the load factor, λ, is less than a critical value, λ_{cr}. In this case, initial separation loads, λ_{cr} and initial separation points, x_{cr} are determined. Also the required distance between the punches to avoid any separation between the punches and the elastic layer is studied and the limit distance between punches that ends interaction of punches is investigated for various dimensionless quantities. However, if tensile tractions are not allowed on the interface, for $\lambda > \lambda_{cr}$ the layer separates from the interface along a certain finite region. Numerical results for distance determining the separation area, vertical displacement in the separation zone, contact stress distribution along the interface between elastic layer and half plane are given for this discontinuous contact case.

1. Introduction

Contact between deformable bodies abounds in industry and everyday life. Because of the industrial importance of the physical processes that take place during contact, a considerable effort has been made in their modeling, analysis, and numerical simulations.

The range of application in contact mechanics starts with problems like foundations in civil engineering, where the lift off the foundation from soil due to eccentric forces acting on a building, railway ballasts, foundation grillages, continuous foundation beams, runaways, liquid tanks resting on the ground, and grain silos is considered. Furthermore, foundation including piles as supporting members or the driving of piles into the soil is of interest. Also classical bearing problem of steel constructions and the connection of structural members by bolts or screws are areas in which contact analysis enters the design process in civil engineering [1].

A complete analysis of the interaction problem for elastic bodies generally requires the determination of stresses and strain within the individual bodies in contact, together with information regarding the distribution of displacements and stresses at the contact regions.

The contact problem for an elastic layer has attracted considerable attention in the past because of its possible application to a variety of structures of practical interest. The layer usually rests on a foundation which may be either elastic or rigid and the body force due to gravity may be neglected [2–27] or effect of gravity may be taken into account [28–35]. Studies regarding the frictionless contact along the interface may be found in [2–15]. Also contact region may exhibit frictional characteristics, giving rise to normal and shear traction at the contact surface [16–27].

While initial contact is determined by the geometric features of the bodies, the extent of the contact generally changes not only by the particular loads applied to the bodies but also with the elastic constants of the materials. Due to the bending of the layer under local compressive loads, in the absence of gravity effects, the contact area would decrease to a finite size which is independent of the magnitude of the applied load [2–7]. If the effect of gravity was taken into account, the normal stress along the layer subspace interface will be compressive and the contact is maintained through

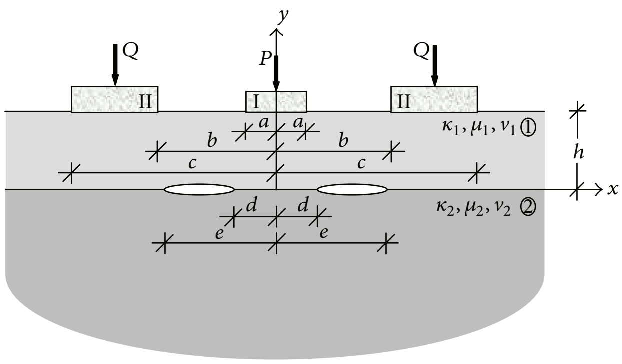


Figure 1: Elastic layer resting on an elastic half plane loaded by means of three rigid flat punches.

the frictionless interface unless the compressive applied load exceeds a certain critical value. When the magnitude of the compressive external load exceeds a certain value, a separation will take place between the layer and the foundation. The length of separation region along the interface is unknown, and the problem becomes a discontinuous contact problem [28–35].

Interaction between an elastic medium and a rigid punch forms another group of contact problems. Rigid punches may be structural elements such as foundations, beams, and plates of finite or infinite extent resting on idealized linearly deformable elastic media. Here, the shape of the contact region may be known a priori and remains constant, or contact region may be changed due to the shape of punch profile [8–27]. The problem of flat-ended rigid punch has important applications in soil mechanics, particularly in estimating the safety of foundations. The application of the three punches for an elastic layer resting on an elastic half plane in soil mechanics is obvious; for example, the punches can be taken as foundations placed on layered soil. When the foundations are placed on soil, there is a possibility of pressure isobars of adjacent foundations overlapping each other. The soil is highly stressed in the zones of overlapping, or the difference of settlement between two adjacent foundations, commonly referred to as differential settlement may cause damage to the structure. It is possible to avoid overlapping of pressures or differential settlement by installing the foundations at considerable distance apart from each other.

In this study, contact problem of the three punches for an elastic layer resting on an elastic half plane is considered according to the theory of elasticity with integral transformation technique. The compressive loads P and Q (per unit thickness in z direction) are applied to the layer through three rigid flat punches. The width of midmost punch can be different from the other two punches and thickness of the layer is constant, h. The layer is subjected to homogeneous vertical body force due to gravity, $\rho_1 g$. All surfaces are frictionless. The layer remains in contact with the elastic half plane where the magnitude of the load factor λ is less than a critical value, λ_{cr} ($\lambda = P/\rho_1 g h^2$). If $\lambda > \lambda_{cr}$, the contact is discontinuous and a separation takes place between the layer and the half plane. A numerical integration procedure is performed for the solution of the problems, and different parameters are researched for various dimensionless quantities for both continuous and discontinuous contact cases. Finally, numerical results are analyzed and conclusions are drawn.

2. General Expressions for Stresses and Displacements

Consider a frictionless elastic layer of thickness h lying on an elastic half plane. The geometry and coordinate system are shown in Figure 1. The governing equations are

$$\mu_k \nabla^2 u_k + \frac{2\mu_k}{(\kappa_k - 1)} \frac{\partial}{\partial x} \left(\frac{\partial u_k}{\partial x} + \frac{\partial v_k}{\partial y} \right) = 0, \quad (1a)$$

$$\mu_k \nabla^2 v_k + \frac{2\mu_k}{(\kappa_k - 1)} \frac{\partial}{\partial y} \left(\frac{\partial u_k}{\partial x} + \frac{\partial v_k}{\partial y} \right) = \rho_k g, \quad (1b)$$

$$(k = 1, 2)$$

where $\rho_k g$ is the intensity of the body force acting vertically in which ρ_k and g are mass density and gravity acceleration. u_k and v_k are the x and y components of the displacement vector, μ_k and κ_k represent shear modulus and elastic constant of the layer and the half plane, respectively. $\kappa_k = (3 - \gamma_k)/(1 + \gamma_k)$ for plane stress and $\kappa_k = (3 - 4\gamma_k)$ for plane strain. γ_k is the Poisson ratio ($k = 1, 2$). Subscript 1 indicates the elastic layer and subscript 2 indicates the elastic half plane.

u_p and v_p represent the displacements for the case in which gravity forces are considered. u_h and v_h are the displacements when the gravity forces are ignored, and total field of displacements may be expressed as

$$u = u_p + u_h, \quad (2a)$$

$$v = v_p + v_h. \quad (2b)$$

Observing that $x = 0$ is a plane of symmetry, it is sufficient to consider the problem in the region $0 \le x \le \infty$ only. Using the symmetry consideration, the following expressions may be written:

$$u_1(x, y) = -u_1(-x, y), \quad (3a)$$

$$v_1(x, y) = v_1(-x, y), \quad (3b)$$

$$u_1(x, y) = \frac{2}{\pi} \int_0^{\infty} \phi_1(\alpha, y) \sin(\alpha x)\, d\alpha, \quad (3c)$$

$$v_1(x, y) = \frac{2}{\pi} \int_0^{\infty} \Psi_1(\alpha, y) \cos(\alpha x)\, d\alpha, \quad (3d)$$

where ϕ_1 and Ψ_1 functions are inverse Fourier transforms of u_1 and v_1 respectively. Taking necessary derivatives of (3c) and (3d), substituting them into (1a) and (1b), and solving the second-order differential equations, the following equations may be obtained for displacements:

$$u_{1_h}(x,y) = \frac{2}{\pi}\int_0^\infty [(A+By)e^{-\alpha y} + (C+Dy)e^{\alpha y}]\sin(\alpha x)\,d\alpha, \tag{4a}$$

$$v_{1_h}(x,y) = \frac{2}{\pi}\int_0^\infty \left[\left[A+\left(\frac{\kappa_1}{\alpha}\kappa + y\right)B\right]e^{-\alpha y} + \left[-C+\left(\frac{\kappa_1}{\alpha} - y\right)D\right]e^{\alpha y}\right]\cos(\alpha x)\,d\alpha. \tag{4b}$$

Using Hooke's law and (4a) and (4b), stress components which do not include the gravity force may be expressed as follows:

$$\sigma_{x_{1_h}}(x,y) = \frac{4\mu_1}{\pi}\int_0^\infty \left[\left[\alpha(A+By) - \left(\frac{3-\kappa_1}{2}\right)B\right]e^{-\alpha y} + \left[\alpha(C+Dy) + \left(\frac{3-\kappa_i}{2}\right)D\right]e^{\alpha y}\right] \times \cos(\alpha x)\,d\alpha, \tag{5a}$$

$$\sigma_{y_{1_h}}(x,y) = \frac{4\mu_1}{\pi}\int_0^\infty \left[-\left[\alpha(A+By) + \left(\frac{\kappa_1+1}{2}\right)B\right]e^{-\alpha y} + \left[-\alpha(C+Dy) + \left(\frac{\kappa_1+1}{2}\right)D\right]e^{\alpha y}\right] \times \cos(\alpha x)\,d\alpha, \tag{5b}$$

$$\tau_{xy_{1_h}}(x,y) = \frac{4\mu_1}{\pi}\int_0^\infty \left[-\left[\alpha(A+By) + \left(\frac{\kappa_i-1}{2}\right)B\right]e^{-\alpha y} + \left[\alpha(C+Dy) - \left(\frac{\kappa_i-1}{2}\right)D\right]e^{\alpha y}\right] \times \sin(\alpha x)\,d\alpha. \tag{5c}$$

For the case in which gravity force exist, particular part of the displacement components corresponding to $\rho_1 g$, the following expressions are obtained, that is, special solution of the Navier equations for a layer with a height h:

$$u_{1_p} = \frac{3-\kappa_1}{8\mu_1}\frac{\rho_1 g h}{2}x, \tag{6a}$$

$$v_{1_p} = \frac{\rho_1 g y}{2\mu_1}\left[\frac{(\kappa_1-1)}{(\kappa_1+1)}(y-h) - \frac{(\kappa_1+1)}{8}h\right], \tag{6b}$$

$$\sigma_{y_{1_p}} = \rho_1 g(y-h), \tag{6c}$$

$$\sigma_{x_{1_p}} = \rho_1 g\left(y-\frac{h}{2}\right)\frac{(1+\kappa_2)}{(1+\kappa_2)+2\mu_2(1+\kappa_1)}, \tag{6d}$$

$$\tau_{xy_{1_p}} = 0. \tag{6e}$$

Considering the orthogonal axes shown in Figure 1, displacements will be zero for $y = -\infty$, and if μ_2, ν_2 are the elastic constants of the half plane, then the homogenous field of displacements and stresses of the elastic half plane may be obtained as

$$u_{2_h}(x,y) = \frac{2}{\pi}\int_0^\infty [(E+Fy)e^{\alpha y}]\sin(\alpha x)\,d\alpha, \tag{7a}$$

$$v_{2_h}(x,y) = \frac{2}{\pi}\int_0^\infty \left[\left[-E+\left(\frac{\kappa_2}{\alpha} - y\right)F\right]e^{\alpha y}\right]\cos(\alpha x)\,d\alpha, \tag{7b}$$

$$\sigma_{x_{2_h}}(x,y) = \frac{4\mu_2}{\pi}\int_0^\infty \left[\left[\alpha(E+Fy) + \left(\frac{3-\kappa_2}{2}\right)F\right]e^{\alpha y}\right] \times \cos(\alpha x)\,d\alpha, \tag{7c}$$

$$\sigma_{y_{2_h}}(x,y) = \frac{4\mu_2}{\pi}\int_0^\infty \left[\left[-\alpha(E+Fy) + \left(\frac{\kappa_2+1}{2}\right)F\right]e^{\alpha y}\right]\cos(\alpha x)\,d\alpha, \tag{7d}$$

$$\tau_{xy_{2_h}}(x,y) = \frac{4\mu_2}{\pi}\int_0^\infty \left[\left[\alpha(E+Fy) - \left(\frac{\kappa_2-1}{2}\right)F\right]e^{\alpha y}\right] \times \sin(\alpha x)\,d\alpha, \tag{7e}$$

subscript 2 indicates the elastic half plane. Note that the body force acting in the foundation is neglected since it does not disturb the contact pressure distribution. *A, B, C, D, E,* and *F* are the unknown constants, which will be determined from the boundary and continuity conditions at $y = 0$ and $y = h$.

3. Case of Continuous Contact

An elastic layer with a height of h resting on an elastic half plane, shown in Figure 1, is analyzed for unit thickness in z direction. Widths of punches at both sides are similar, $(c-b)/h$, and each of these punches transmits a concentrated load of Q to the elastic layer. The width of the midmost punch is different, $2a/h$ and it is subjected to a concentrated load, P. All surfaces are frictionless. Particularly, the initial separation load (λ_{cr}) and point (x_{cr}) where the layer separated from the elastic half plane and the variation of the stress distribution between elastic layer and elastic half plane is examined depending on material properties, width of punches, and magnitude of the external loads, P and Q. Due to the different settlement of punches, a separation takes place between punch I and elastic layer, if punches are close enough. Therefore, the critical distance between the punches indicating the initiation of separation between the punch I and the elastic layer is researched and also limit distance between the punches where the interaction of punches ends is investigated.

If load factor (λ) is sufficiently small, then the contact along the layer-subspace, $y = 0, 0 < x < \infty$, will be continuous, and $A, B, C, D, E,$ and F must be determined from the following boundary and continuity conditions:

$$\sigma_{y_1}(x,h) = \begin{cases} -P(x), & 0 < x < a \\ -Q(x), & b < x < c \\ 0, & a < x < b, \quad c < x < \infty, \end{cases} \tag{8a}$$

$$\tau_{xy_1}(x,h) = 0, \quad 0 < x < \infty, \tag{8b}$$

$$\tau_{xy_1}(x,0) = 0, \quad 0 < x < \infty, \tag{8c}$$

$$\tau_{xy_2}(x,0) = 0, \quad 0 < x < \infty, \tag{8d}$$

$$\sigma_{y_1}(x,0) = \sigma_{y_2}(x,0), \quad 0 < x < \infty, \tag{8e}$$

$$\frac{\partial}{\partial x}\left[v_2(x,0) - v_1(x,0)\right] = 0, \quad 0 < x < \infty, \tag{8f}$$

$$\frac{\partial}{\partial x}\left[v_1(x,h)\right] = 0, \quad 0 < x < a, \tag{8g}$$

$$\frac{\partial}{\partial x}\left[v_1(x,h)\right] = 0, \quad b < x < c, \tag{8h}$$

in which subscripts 1 and 2 indicate relation to the elastic layer and the elastic half plane, respectively. $P(x)$ is the unknown contact pressure under punch I and $Q(x)$ is the unknown contact pressure under punch II, which have not been determined yet. If a separation occurs between the elastic layer and elastic half plane, this will give rise to a discontinuous contact position and the following results for former solution will no longer be valid and new solution will be attained for the latter case.

Equilibrium conditions of the problem may be expressed as

$$\int_0^a P(x)\,dx = \frac{P}{2}, \tag{9a}$$

$$\int_b^c Q(x)\,dx = Q. \tag{9b}$$

Displacement and stress expressions (4a), (4b), (5a)–(5c), (6a)–(6e), and (7a)–(7e) are substituted into boundary conditions (8a)–(8f), and unknown constants $A, B, C, D, E,$ and F are determined in terms of unknown functions $P(x)$ and $Q(x)$. By making use of (8g) and (8h), after some simple manipulations, one may obtain the following singular integral equations for $P(x)$ and $Q(x)$ [36, 37]:

$$\begin{aligned} &-\frac{1}{\pi\mu_1}\int_0^a \left[k_1(x,t) + \frac{(1+\kappa_1)}{4}\left(\frac{1}{t+x} + \frac{1}{t-x}\right)\right] P(t)\,dt \\ &\quad - \frac{1}{\pi\mu_1}\int_b^c \left[k_1(x,t) + \frac{(1+\kappa_1)}{4}\left(\frac{1}{t+x} + \frac{1}{t-x}\right)\right] \\ &\quad \times Q(t)\,dt = 0, \quad 0 < x < a, \end{aligned} \tag{10a}$$

$$\begin{aligned} &-\frac{1}{\pi\mu_1}\int_0^a \left[k_1(x,t) + \frac{(1+\kappa_1)}{4}\left(\frac{1}{t+x} + \frac{1}{t-x}\right)\right] P(t)\,dt \\ &\quad - \frac{1}{\pi\mu_1}\int_b^c \left[k_1(x,t) + \frac{(1+\kappa_1)}{4}\left(\frac{1}{t+x} + \frac{1}{t-x}\right)\right] \\ &\quad \times Q(t)\,dt = 0, \quad b < x < c, \end{aligned} \tag{10b}$$

where

$$\begin{aligned} k_1(x,t) = \int_0^\infty \Bigg\{ &\Bigg\{ \frac{\alpha^3}{8(1+\kappa_1)} \\ &\times \Bigg[\left[(1+\kappa_2) + \frac{\mu_2}{\mu_1(1+\kappa_1)}\right] \\ &+ e^{-2\alpha h}\left[4\alpha h(1+\kappa_2) - \frac{2\mu_2}{\mu_1(1+\kappa_1)}\right] \\ &+ e^{-4\alpha h}\left[-1-\kappa_2 + \frac{\mu_2}{\mu_1(1+\kappa_1)}\right]\Bigg]\Bigg\}(\Delta)^{-1} \\ &- \frac{(1+\kappa_1)}{4}\Bigg\} \\ &\times \{\sin\alpha(t+x) - \sin\alpha(t-x)\}\,d\alpha, \end{aligned} \tag{11}$$

in which

$$\begin{aligned} \Delta = \alpha^3 \Bigg(2\Bigg\{&\left(-1-\kappa_2 - \frac{\mu_2}{\mu_1(1+\kappa_1)}\right) \\ &+ e^{-4\alpha h}\left[-1-\kappa_2 + \frac{\mu_2}{\mu_1(1+\kappa_1)}\right] \\ &+ e^{-2\alpha h}\Bigg[(1+\kappa_2)\left(2+4\alpha^2h^2\right) \\ &+ \frac{4\alpha h\mu_2}{\mu_1(-1-\kappa_1)}\Bigg]\Bigg\}\Bigg)^{-1}. \end{aligned} \tag{12}$$

If evaluated values of $A, B, C,$ and D in terms of $P(x)$ and $Q(x)$ are substituted into (5b), the expression of the contact stress between elastic layer and half plane may be obtained as

$$\begin{aligned} \sigma_{y_1}(x,0) = &-\rho_1 g h - \frac{1}{\pi}\int_0^a k_2(x,t)\,P(t)\,dt \\ &- \frac{1}{\pi}\int_b^c k_2(x,t)\,Q(t)\,dt, \quad 0 < x < \infty, \end{aligned} \tag{13}$$

in which ρ_1 and g are mass density and gravity acceleration, respectively, where

$$k_2(x,t) = \int_0^{\infty}\left\{\frac{\alpha^3\mu_2}{\mu_1(1+\kappa_1)} \times\left[e^{-3\alpha h}(-1+\alpha h)+e^{-\alpha h}(1+\alpha h)\right]\right\}(\Delta)^{-1} \times\{\cos\alpha(t+x)+\cos\alpha(t-x)\}d\alpha. \tag{14}$$

To simplify the numerical analysis, the following dimensionless quantities are introduced:

$$x_1 = ar_1, \tag{15a}$$

$$t_1 = as_1, \tag{15b}$$

$$x_2 = \frac{c-b}{2}r_2 + \frac{c+b}{2}, \tag{15c}$$

$$t_2 = \frac{c-b}{2}s_2 + \frac{c+b}{2}, \tag{15d}$$

$$g_1(s_1) = \frac{P(as_1)}{P/h}, \tag{15e}$$

$$g_2(s_2) = \frac{Q(((c-b)/2)s_2+(c+b)/2)}{P/h}, \tag{15f}$$

$$\alpha = wh, \tag{15g}$$

$$\lambda = \frac{P}{\rho_1 g h^2}. \tag{15h}$$

Substituting from (15a)–(15h), (9a), (9b) and (10a), (10b) may be expressed as

$$\int_0^1 g_1(s_1)\frac{a}{h}ds_1 = \frac{1}{2}, \tag{16a}$$

$$\int_{-1}^1 g_2(s_2)\frac{c-b}{2h}ds_2 = \frac{Q}{P}, \tag{16b}$$

$$\begin{aligned}&-\frac{1}{\pi}\int_0^1 g_1(s_1)\frac{a}{h}ds_1\left[m_1(r_1,s_1)+\frac{(1+\kappa_1)}{4}\times\left(\frac{1}{a(s_1+r_1)}-\frac{1}{a(s_1-r_1)}\right)\right]\\&-\frac{1}{\pi}\int_{-1}^1 g_2(s_2)\frac{c-b}{2h}ds_2\\&\times\left[m_2(r_1,s_2)+\frac{(1+\kappa_1)}{4}\times\left(\frac{1}{(((c-b)/2)s_2+(c+b)/2)+ar_1}-\frac{1}{(((c-b)/2)s_2+(c+b)/2)-ar_1}\right)\right]=0,\\&\qquad 0<r_1<1,\end{aligned} \tag{16c}$$

$$\begin{aligned}&-\frac{1}{\pi}\int_0^1 g_1(s_1)\frac{a}{h}ds_1\\&\times\left[m_3(r_2,s_1)+\frac{(1+\kappa_1)}{4}\times\left(\frac{1}{as_1+(((c-b)/2)r_2+(c+b)/2)}-\frac{1}{as_1-(((c-b)/2)r_2+(c+b)/2)}\right)\right]\\&-\frac{1}{\pi}\int_{-1}^1 g_2(s_2)\frac{c-b}{2}ds_2\\&\times\left[m_4(r_2,s_2)+\frac{(1+\kappa_1)}{4}\times\left(\frac{1}{((c-b)/2)(s_2+r_2)+(c+b)}-\frac{1}{((c-b)/2)(s_2-r_2)}\right)\right]=0,\\&\qquad -1<r_2<1,\end{aligned} \tag{16d}$$

$$\frac{\sigma_{y_1}(x,0)}{P/h} = -\frac{1}{\lambda}-\frac{1}{\pi}\int_0^1 m_5(r_1,s_1)g_1(s_1)\frac{a}{h}ds_1 -\frac{1}{\pi}\int_{-1}^1 m_6(r_2,s_2)g_2(s_2)\frac{c-b}{2h}ds_2, \tag{16e}$$

where

$$m_1(r_1,s_1) = k_1(x_1,t_1), \tag{17a}$$

$$m_2(r_1,s_2) = k_1(x_1,t_2), \tag{17b}$$

$$m_3(r_2,s_1) = k_1(x_2,t_1), \tag{17c}$$

$$m_4(r_2,s_2) = k_1(x_2,t_2), \tag{17d}$$

$$m_5(r_1,s_1) = k_2(x_1,t_1), \tag{17e}$$

$$m_6(r_2,s_2) = k_2(x_2,t_2). \tag{17f}$$

The index of the integral equations in (16a)–(16e) is +1; so the functions $g_1(s_1)$ and $g_2(s_2)$ may be expressed in the following forms:

$$g_1(s_1) = G_1(s_1)\left(1-s_1^2\right)^{-1/2}, \quad (0<s_1<1), \tag{18a}$$

$$g_2(s_2) = G_2(s_2)\left(1-s_2^2\right)^{-1/2}, \quad (-1<s_2<1), \tag{18b}$$

where $G_1(s_1)$ is bounded in $[0,1]$ and $G_2(s_2)$ is bounded in $[-1,1]$. Then using appropriate Gauss-Chebyshev integration formula given in [36], (16a)–(16d) are replaced by the following algebraic equations:

$$\sum_{i=1}^{n}\pi W_i\frac{a}{h}G_1\left(s_{1_i}\right) = \frac{1}{2}, \tag{19a}$$

$$\sum_{i=1}^{n}\pi W_i \frac{c-b}{2h} G_2\left(s_{2_i}\right) = \frac{Q}{P}, \tag{19b}$$

$$\begin{aligned}
&-\sum_{i=1}^{n} W_i G_1\left(s_{1_i}\right)\frac{a}{h}\left[m_1\left(r_{1_j}, s_{1_i}\right) + \frac{(1+\kappa_1)}{4}\right.\\
&\qquad\left.\times\left(\frac{1}{a\left(s_{1_i}+r_{1_j}\right)} - \frac{1}{a\left(s_{1_i}-r_{1_j}\right)}\right)\right]\\
&-\sum_{i=1}^{n} W_i G_2\left(s_{2_i}\right)\frac{c-b}{2h}\\
&\times\left[m_2\left(r_{1_j}, s_{2_i}\right) + \frac{(1+\kappa_1)}{4}\right.\\
&\quad\times\left(\frac{1}{\left(((c-b)/2)\, s_{2_i} + (c+b)/2\right) + ar_{1_j}}\right.\\
&\quad\left.\left.-\frac{1}{\left(((c-b)/2)\, s_{2_i} + (c+b)/2\right) - ar_{1_j}}\right)\right] = 0,\\
&\qquad\qquad\left(j = 1, \ldots, n-1\right),
\end{aligned} \tag{19c}$$

$$\begin{aligned}
&-\sum_{i=1}^{n} W_i G_1\left(s_{1_i}\right)\frac{a}{h}\\
&\times\left[m_3\left(r_{2_j}, s_{1_i}\right) + \frac{(1+\kappa_1)}{4}\right.\\
&\quad\times\left(\frac{1}{as_{1_i} + \left(((c-b)/2)\, r_{2_j} + (c+b)/2\right)}\right.\\
&\quad\left.\left.-\frac{1}{as_{1_i} - \left(((c-b)/2)\, r_{2_j} + (c+b)/2\right)}\right)\right]\\
&-\sum_{i=1}^{n} W_i G_2\left(s_{2_i}\right)\frac{c-b}{2h}\\
&\times\left[m_4\left(r_{2_j}, s_{2_i}\right) + \frac{(1+\kappa_1)}{4}\right.\\
&\quad\times\left(\frac{1}{((c-b)/2)\left(s_{2_i}+r_{2_j}\right) + (c+b)}\right.\\
&\quad\left.\left.-\frac{1}{((c-b)/2)\left(s_{2_i}-r_{2_j}\right)}\right)\right] = 0,\\
&\qquad\qquad\left(j = 1, \ldots, n-1\right),
\end{aligned} \tag{19d}$$

$$W_1 = W_n = \frac{1}{2n-2}, \qquad W_i = \frac{1}{n-1}, \quad (i = 2, \ldots, n-1), \tag{19e}$$

$$s_{1_i} = \cos\left(\frac{i-1}{2n-1}\pi\right), \qquad s_{2_i} = \cos\left(\frac{i-1}{n-1}\pi\right), \quad (i = 1, \ldots, n), \tag{19f}$$

$$r_{1_j} = \cos\left(\frac{2j-1}{4n-2}\pi\right), \qquad r_{2_j} = \cos\left(\frac{2j-1}{2n-2}\pi\right), \quad \left(j = 1, \ldots, n-1\right). \tag{19g}$$

The unknowns $G_1(s_{1_i})$ and $G_2(s_{2_i})$ $(i = 1, \ldots, n)$ are determined from the system (19a)–(19d). By using (18a), (18b), substituting the results into (16e), and using Gauss-Chebyshev integration formula, the contact stress $\sigma_{y_1}(x,0)h/P$ is evaluated. It should be observed that the integral equations (16c) and (16d) are valid provided the contact stress $\sigma_{y_1}(x,0)h/P$ is compressive everywhere; that is, $0 < x < \infty$. The critical load factor, λ_{cr} and the corresponding location of interface separation, x_{cr} can be determined through the use of the following condition for various dimensionless quantities:

$$\frac{\sigma_{y_1}(x,0)}{P/h} = 0. \tag{20}$$

4. Case of Discontinuous Contact

Since the interface cannot carry tensile tractions for $\lambda > \lambda_{cr}$, there will be separation between the elastic layer and the elastic half plane in the neighborhood of $x = x_{cr}$ on the contact plane $y = 0$, as shown in Figure 1. Assuming that the separation region is described by $d < x < e$, $y = 0$, where d and e are unknowns and functions of λ, boundary and continuity conditions for the discontinuous contact case are defined as follows:

$$\sigma_{y_1}(x,h) = \begin{cases} -P(x), & 0 < x < a \\ -Q(x), & b < x < c \\ 0, & a < x < b,\ c < x < \infty, \end{cases} \tag{21a}$$

$$\tau_{xy_1}(x,h) = 0, \quad 0 < x < \infty, \tag{21b}$$

$$\tau_{xy_1}(x,0) = 0, \quad 0 < x < \infty, \tag{21c}$$

$$\tau_{xy_2}(x,0) = 0, \quad 0 < x < \infty, \tag{21d}$$

$$\sigma_{y_1}(x,0) = \sigma_{y_2}(x,0), \quad 0 < x < \infty, \tag{21e}$$

$$\frac{\partial}{\partial x}\left[v_2(x,0) - v_1(x,0)\right] = \begin{cases} \varphi(x), & d < x < e \\ 0, & 0 < x < d,\ e < x < \infty, \end{cases} \tag{21f}$$

$$\sigma_{y_1}(x,0) = \sigma_{y_2}(x,0) = 0, \quad d < x < e, \tag{21g}$$

$$\frac{\partial}{\partial x}\left[v_1(x,h)\right]=0, \quad 0<x<a, \tag{21h}$$

$$\frac{\partial}{\partial x}\left[v_1(x,h)\right]=0, \quad c<x<b. \tag{21i}$$

After utilizing the boundary and continuity conditions defined in (21a)–(21f), new values for the constants A, B, C, D, E, and F which appear in (4a), (4b), (5a)–(5c), and (7a)–(7e) may be obtained in terms of new unknown functions $P(x)$, $Q(x)$, and $\varphi(x)$. Unknown functions are then determined from the conditions (21h)-(21i) which have not yet been satisfied. These conditions give the following system of singular integral equations:

$$\begin{aligned}
&-\frac{1}{\pi\mu_1}\int_0^a\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\left(\frac{1}{t+x}-\frac{1}{t-x}\right)\right]P(t)\,dt\\
&-\frac{1}{\pi\mu_1}\int_b^c\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\left(\frac{1}{t+x}-\frac{1}{t-x}\right)\right]Q(t)\,dt\\
&+\frac{1}{\pi}\int_d^e k_2(x,t)\varphi(t)\,dt=0, \quad 0<x<a,
\end{aligned} \tag{22a}$$

$$\begin{aligned}
&-\frac{1}{\pi\mu_1}\int_0^a\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\left(\frac{1}{t+x}-\frac{1}{t-x}\right)\right]P(t)\,dt\\
&-\frac{1}{\pi\mu_1}\int_b^c\left[k_1(x,t)+\frac{(1+\kappa_1)}{4}\left(\frac{1}{t+x}-\frac{1}{t-x}\right)\right]Q(t)\,dt\\
&+\frac{1}{\pi}\int_d^e k_2(x,t)\varphi(t)\,dt=0, \quad b<x<c,
\end{aligned} \tag{22b}$$

$$\begin{aligned}
&\frac{1}{\pi}\int_0^a k_2(x,t)P(t)\,dt+\frac{1}{\pi}\int_b^c k_2(x,t)Q(t)\,dt\\
&-\frac{\mu_1}{\pi}\int_d^e\left[k_3(x,t)-\frac{4\mu_2/\mu_1}{(1+\kappa_2)+\mu_2/\mu_1(1+\kappa_1)}\right.\\
&\qquad\left.\times\left(\frac{1}{t+x}+\frac{1}{t-x}\right)\right]\varphi(t)\,dt-\rho_1 gh=0,\\
&\qquad d<x<e,
\end{aligned} \tag{22c}$$

where kernels $k_1(x,t)$ and $k_2(x,t)$ are given by (11) and (14) and

$$\begin{aligned}
k_3(x,t)=&\int_0^\infty\left\{\left\{-\frac{2\alpha^3\mu_2}{\mu_1}\left\{e^{-2\alpha h}\left(2+4\alpha^2h^2\right)-e^{-4\alpha h}-1\right\}\right\}\right.\\
&\left.\times(\Delta)^{-1}+\frac{4\mu_2/\mu_1}{(1+\kappa_2)+\mu_2/\mu_1(1+\kappa_1)}\right\}\\
&\times\{\sin\alpha(t+x)+\sin\alpha(t-x)\}\,d\alpha,
\end{aligned} \tag{23}$$

in which Δ is given by (12).

The index of integral equations (22a) and (22b) is +1. On the other hand, the index of the singular integral equation (22c) is –1 due to the physical requirement of smooth contact at the end points d and e [37]. Thus, in solving the problem the two conditions which would account for the unknowns d and e are the consistency condition of integral equation (22c) and the single-valuedness condition

$$\int_d^e\varphi(x)\,dx=0. \tag{24}$$

Defining the following dimensionless quantities

$$x_3=\frac{e-d}{2}r_3+\frac{e+d}{2}, \qquad t_3=\frac{e-d}{2}s_3+\frac{e+d}{2}, \tag{25a}$$

$$g_3(s_3)=\frac{\mu_1\varphi\left(((e-d)/2)s_3+(e+d)/2\right)}{P/h}; \tag{25b}$$

by making use of (15a)–(15h), the integral equations (22a)–(22c) may be expressed as follows:

$$\begin{aligned}
&-\frac{1}{\pi}\int_0^1 g_1(s_1)\frac{a}{h}ds_1\left[m_1^*(r_1,s_1)+\frac{(1+\kappa_1)}{4}\right.\\
&\qquad\left.\times\left(\frac{1}{a(s_1+r_1)}-\frac{1}{a(s_1-r_1)}\right)\right]\\
&-\frac{1}{\pi}\int_{-1}^1 g_2(s_2)\frac{c-b}{2h}ds_2\\
&\times\left[m_2^*(r_1,s_2)+\frac{(1+\kappa_1)}{4}\right.\\
&\times\left(\frac{1}{(((c-b)/2)s_2+(c+b)/2)+ar_1}\right.\\
&\left.\left.-\frac{1}{(((c-b)/2)s_2+(c+b)/2)-ar_1}\right)\right]\\
&+\frac{1}{\pi}\int_{-1}^1 g_3(s_3)m_3^*(r_1,s_3)\frac{e-d}{2h}ds_3=0,\\
&\qquad 0<r_1<1,
\end{aligned} \tag{26a}$$

$$\begin{aligned}
&-\frac{1}{\pi}\int_{-1}^1 g_1(s_1)\frac{a}{h}ds_1\\
&\times\left[m_4^*(r_2,s_1)+\frac{(1+\kappa_1)}{4}\right.\\
&\times\left(\frac{1}{as_1+(((c-b)/2)r_2+(c+b)/2)}\right.\\
&\left.\left.-\frac{1}{as_1-(((c-b)/2)r_2+(c+b)/2)}\right)\right]\\
&-\frac{1}{\pi}\int_{-1}^1 g_2(s_2)\frac{c-b}{2h}ds_2\left[m_5^*(r_2,s_2)+\frac{(1+\kappa_1)}{4}\right.\\
&\times\left(\frac{1}{((c-b)/2)(s_2+r_2)+(c+b)}\right.\\
&\left.\left.-\frac{1}{((c-b)/2)(s_2-r_2)}\right)\right]\\
&+\frac{1}{\pi}\int_{-1}^1 g_3(s_3)m_6^*(r_2,s_3)\frac{e-d}{2h}ds_3=0,\\
&\qquad -1<r_2<1,
\end{aligned} \tag{26b}$$

$$-\frac{1}{\pi}\int_{-1}^{1} g_1(s_1)\, m_7^*(r_3, s_1)\,\frac{a}{h} ds_1$$
$$-\frac{1}{\pi}\int_{-1}^{1} g_2(s_2)\, m_8^*(r_3, s_2)\,\frac{c-b}{2h} ds_2$$
$$-\frac{1}{\pi}\int_{-1}^{1} g_3(s_3)\,\frac{e-d}{2h} ds_3$$
$$\times\Bigg[m_9^*(r_3, s_3) - \frac{4\mu_2/\mu_1}{(1+\kappa_2)+\mu_2/\mu_1(1+\kappa_1)}$$
$$\times\left(\frac{1}{((e-d)/2)(s_3+r_3)+(e+d)}\right.$$
$$\left.+\frac{1}{((e-d)/2)(s_3-r_3)}\right)\Bigg] - \frac{1}{\lambda} = 0,$$
$$-1 < r_3 < 1, \tag{26c}$$

where

$$m_1^*(r_1, s_1) = k_1(x_1, t_1), \tag{27a}$$
$$m_2^*(r_1, s_2) = k_1(x_1, t_2), \tag{27b}$$
$$m_3^*(r_1, s_3) = k_2(x_1, t_3), \tag{27c}$$
$$m_4^*(r_2, s_1) = k_1(x_2, t_1), \tag{27d}$$
$$m_5^*(r_2, s_2) = k_1(x_2, t_2), \tag{27e}$$
$$m_6^*(r_2, s_3) = k_2(x_2, t_3), \tag{27f}$$
$$m_7^*(r_3, s_1) = k_2(x_3, t_1), \tag{27g}$$
$$m_8^*(r_3, s_2) = k_2(x_3, t_2), \tag{27h}$$
$$m_9^*(r_3, s_3) = k_3(x_3, t_3). \tag{27i}$$

Similar to (16a), (16b), additional condition (24) may be expressed as

$$\int_{-1}^{1} g_3(s_3)\, ds_3 = 0. \tag{28}$$

To solve the system of integral equations, it is found to be more convenient to assume that (26c) as well as (26a) and (26b) has an index +1 [29]; consequently, the function $g_i(s_i)$ $(i = 1, \ldots, 3)$ may be expressed in the form

$$g_1(s_1) = G_1(s_1)\left(1-s_1^2\right)^{-1/2}, \quad 0 < s_1 < 1, \tag{29a}$$

$$g_i(s_i) = G_i(s_i)\left(1-s_i^2\right)^{-1/2}, \quad -1 < s_i < 1, \quad (i = 2, 3), \tag{29b}$$

where $G_i(s_i)$ is a bounded function. In order to insure smooth contact at the end points of the separation region, we then impose the following conditions on $G_3(s_3)$:

$$G_3(-1) = 0, \qquad G_3(1) = 0. \tag{30}$$

Equations (26a)–(26c), (16a), (16b), and (28) can easily be reduced to the following system of linear algebraic equations by employing the appropriate Gauss-Chebyshev integration formula [36]:

$$-\sum_{i=1}^{n} W_i G_1\left(s_{1_i}\right)\frac{a}{h}\left[m_1^*\left(r_{1_j}, s_{1_i}\right) + \frac{(1+\kappa_1)}{4}\times\frac{1}{a\left(s_{1_i}+r_{1_j}\right)} - \frac{1}{a\left(s_{1_i}-r_{1_j}\right)}\right]$$
$$-\sum_{i=1}^{n} W_i G_2\left(s_{2_i}\right)\frac{c-b}{2h}\Bigg[m_2^*\left(r_{1_j}, s_{2_i}\right) + \frac{(1+\kappa_1)}{4}$$
$$\times\left(\frac{1}{\left(((c-b)/2)\, s_{2_i} + (c+b)/2\right) + ar_{1_j}} - \frac{1}{\left(((c-b)/2)\, s_{2_i} + (c+b)/2\right) - ar_{1_j}}\right)\Bigg]$$
$$+\sum_{i=2}^{n-1} W_i G_3\left(s_{3_i}\right)\frac{e-d}{2h} m_3^*\left(r_{1_j}, s_{3_i}\right) = 0,$$
$$(j = 1, \ldots, n-1), \tag{31a}$$

$$-\sum_{i=1}^{n} W_i G_1\left(s_{1_i}\right)\frac{a}{h}\Bigg[m_4^*\left(r_{2_j}, s_{1_i}\right) + \frac{(1+\kappa_1)}{4}$$
$$\times\left(\frac{1}{as_{1_i} + \left(((c-b)/2)\, r_{2_j} + (c+b)/2\right)} - \frac{1}{as_{1_i} - \left(((c-b)/2)\, r_{2_j} + (c+b)/2\right)}\right)\Bigg]$$
$$-\sum_{i=1}^{n} W_i G_2\left(s_{2_i}\right)\frac{c-b}{2h}\Bigg[m_5^*\left(r_{2_j}, s_{2_i}\right) + \frac{(1+\kappa_1)}{4}$$
$$\times\left(\frac{1}{((c-b)/2)\left(s_{2_i}-r_{2_j}\right)+(c+b)} \pm \frac{1}{((c-b)/2)\left(s_{2_i}-r_{2_j}\right)}\right)\Bigg]$$
$$+\sum_{i=2}^{n-1} W_i G_3\left(s_{3_i}\right)\frac{e-d}{2h} m_6^*\left(r_{2_j}, s_{3_i}\right) = 0$$
$$(j = 1, \ldots, n-1), \tag{31b}$$

$$-\sum_{i=1}^{n} W_i G_1\left(s_{1_i}\right)\frac{a}{h} m_7^*\left(r_{3_j}, s_{1_i}\right)$$
$$-\sum_{i=1}^{n} W_i G_2\left(s_{2_i}\right)\frac{c-b}{2h} m_8^*\left(r_{3_j}, s_{2_i}\right)$$
$$-\sum_{i=2}^{n-1} W_i G_3\left(s_{3_i}\right)\frac{e-d}{2h}$$
$$\times\left[m_9^*\left(r_{3_j}, s_{3_i}\right)-\frac{4\mu_2/\mu_1}{(1+\kappa_2)+\mu_2/\mu_1(1+\kappa_1)}\right. \tag{31c}$$
$$\times\left(\frac{1}{((e-d)/2)\left(s_{3_i}-r_{3_j}\right)+(e+d)}\right.$$
$$\left.\left.+\frac{1}{((e-d)/2)\left(s_{3_i}-r_{3_j}\right)}\right)\right]-\frac{1}{\lambda}=0,$$
$$(j=1,\ldots,n-1),$$

$$\sum_{i=1}^{n}\pi W_i\frac{a}{h}G_1\left(s_{1_i}\right)=\frac{1}{2}, \tag{31d}$$

$$\sum_{i=1}^{n}\pi W_i\frac{c-b}{2h}G_2\left(s_{2_i}\right)=\frac{Q}{P}, \tag{31e}$$

$$\sum_{i=2}^{n-1}\pi W_i G_3\left(s_{3_i}\right)=0, \tag{31f}$$

where W_i, s_i, and r_j are given by (19e)–(19g) ($r_2 = r_3$, $s_2 = s_3$). It was shown in [36] that the consistency condition is automatically satisfied if the Gauss-Chebyshev integration formula is used for solving integral equations. Thus, (31a)–(31d) and (31e)-(31f) give $3n$ equation for $3n$ unknowns $G_1(s_i)$, $G_2(s_i)$, $G_3(s_j)$ ($i = 1,\ldots,n$), ($j = 2,\ldots,n-1$), d and e. The equation system is nonlinear in d and e; so an interpolation scheme is required for the solution. Selected values of d and e are substituted into (31a)–(31d), and $G_1(s)$, $G_2(s)$, and $G_3(s)$ are obtained which must satisfy (31e), (31f) at the same time for known $\lambda > \lambda_{cr}$. If (31e), (31f) are not satisfied, then solution must be repeated with new values of d and e until the (31e), (31f) are satisfied at the same time.

It should be noted that (22c) gives the $\sigma_{y_1}(x,0)h/P$ outside as well as inside the separation region (e, f). Thus, once the functions $G_1(s)$, $G_2(s)$, and $G_3(s)$ and the constants d and e are determined, contact stress $\sigma_{y_1}(x,0)h/P$ may be easily evaluated. The displacement component $v^*(x,0)$ in the separation region (d, e), referring to (21f) and (25b), may be obtained from

$$v^*(x,0)=v_2(x,0)-v_1(x,0)=\int_d^x g_3(t)\,dt, \qquad d<x<e \tag{32a}$$

Table 1: Variation of minimum value of distance between two punches $(b-a)/h$ to avoid separation under first punch with a/h and $(c-b)/h$ for various values of Q/P ($\mu_2/\mu_1 = 6.48$).

$(c-b)/h$	a/h	$(b-a)/h$ 2P	4P	6P	8P	10P	12P
0.25	0.5	0.2773	0.5465	0.7263	0.8564	0.9569	1.0383
	0.75	0.4645	0.7676	0.9485	1.075	1.1716	1.250
	1.0	0.603	0.9107	1.089	1.2138	1.3105	1.3893
0.5	0.5	0.185	0.4403	0.617	0.7466	0.8474	0.9291
	0.75	0.3617	0.6597	0.8405	0.9675	1.065	1.144
	1.0	0.4983	0.8035	0.983	1.1086	1.206	1.2856
0.75	0.5	0.108	0.3432	0.516	0.645	0.746	0.8285
	0.75	0.2715	0.5613	0.742	0.8699	0.9685	1.0485
	1.0	0.4045	0.7073	0.8875	1.015	1.1134	1.4445
1.0	0.5	0.0357	0.2504	0.4195	0.548	0.6497	0.7332
	0.75	0.191	0.4694	0.6502	0.7797	0.8798	0.9613
	1.0	0.320	0.6188	0.8005	0.930	1.030	1.1123

or

$$\frac{\mu_1}{P/h}v^*(x,0)=\frac{e-d}{2h}\int_{-1}^{r_3}G_3(s_3)\,ds_3, \qquad -1<r_3<1, \tag{32b}$$

where

$$x=\frac{e-d}{2}r_3+\frac{e+d}{2}. \tag{32c}$$

Also using appropriate Gauss-Chebyshev integration formula and taking +1 as the index of (32b), the following expression may be written for the vertical displacement in the separation region:

$$\frac{\mu_1}{P/h}v^*(x,0)=\frac{e-d}{2h}\sum_{i=2}^{k-1}W_i G_3\left(s_{3_i}\right), \qquad (k=2,\ldots,n-1), \tag{33}$$

where W_i and s_i are given by (19e)–(19g).

5. Results and Discussion

Some of the calculated results obtained from the solution of the continuous and discontinuous contact problems for various dimensionless quantities such as μ_2/μ_1, a/h, $(c-b)/h$, Q/P, λ, and $(b-a)/h$ are presented in Figures 2, 3, 4, 5, and 6 and Tables 1, 2, 3, 4, and 5. First separation point between elastic layer and elastic half plane is determined and first separation region is investigated. Contact pressure $\sigma_{y_1}(x,0)h/P$ is presented. Depending on load factor λ, possibilities of other separation regions between elastic layer and elastic half plane are determined. Also possibility of separation between first punch and elastic layer is researched. Besides, the distance $(b-a)/h$, that ends the interaction of punches is examined. It is assumed that $Q/h \geq P/h$.

TABLE 2: Variation of distance between two punches $(b-a)/h$ that ends interaction of punches with Q/P ($\mu_2/\mu_1 = 2.75$, $a/h = 0.25$, and $(c-b)/h = 0.5$).

Q	$(b-a)/h$	Punch I		Punch II	
		$\lambda_{\mathrm{cr}_{\mathrm{right}}}$	$(x_{\mathrm{cr}_{\mathrm{right}}}-a)/h$	$\lambda_{\mathrm{cr}_{\mathrm{left}}}=\lambda_{\mathrm{cr}_{\mathrm{right}}}$	$(b-x_{\mathrm{cr}_{\mathrm{left}}})/h=(x_{\mathrm{cr}_{\mathrm{right}}}-c)/h$
P	10.5585	96.6748	2.091	96.6175	2.092
$2P$	9.5874	96.0544	2.091	48.3612	2.092
$4P$	8.6032	94.2026	2.094	24.1974	2.092
$6P$	8.1313	91.6959	2.100	16.1360	2.092
$8P$	7.8582	88.6769	2.111	12.1037	2.091
$10P$	7.6742	85.2495	2.128	9.6838	2.091
$12P$	7.5378	81.4924	2.153	8.0704	2.091
$14P$	7.4308	81.4924	2.192	6.9178	2.090

TABLE 3: Variation of distance between two punches $(b-a)/h$ that ends interaction of punches with μ_2/μ_1 ($Q = 6P$, $a/h = 0.25$, and $(c-b)/h = 0.5$).

μ_2/μ_1	$(b-a)/h$	Punch I		Punch II	
		$\lambda_{\mathrm{cr}_{\mathrm{right}}}$	$(x_{\mathrm{cr}_{\mathrm{right}}}-a)/h$	$\lambda_{\mathrm{cr}_{\mathrm{left}}}=\lambda_{\mathrm{cr}_{\mathrm{right}}}$	$(b-x_{\mathrm{cr}_{\mathrm{left}}})/h=(x_{\mathrm{cr}_{\mathrm{right}}}-c)/h$
0.15	14.8404	164.5971	5.309	36.6977	4.795
0.36	12.1858	156.6651	3.678	30.6651	3.596
0.61	10.8897	143.5372	3.066	26.8566	3.027
1.65	9.2688	114.5372	2.418	20.5185	2.402
2.75	8.1313	91.6959	2.100	16.1360	2.092
6.48	6.4981	65.5235	1.812	11.4136	1.793

Figure 2 shows limit distance between punches that initiates separation under first punch for various values of material constant, μ_2/μ_1 with Q/P. The distance $(b-a)/h$ increases, maintaining continuous contact between first punch and elastic layer with Q/P. Besides, for bigger values of μ_2/μ_1, limit distance that initiates separation under first punch decreases. In such a case, elastic half plane gets stiffer and it becomes easy to separate first punch from the elastic layer.

Variation of critical distance between punches with a/h and $(c-b)/h$ for various values of Q/P is presented in Table 1. For fixed values of second punch width, $(c-b)/h$, an increase in first punch width requires longer distance between punches to avoid separation under first punch. On the contrary, for fixed values of first punch width, a/h, an increase in second punch width decreases $(b-a)/h$ and punches can be placed closer to each other without separation.

Interaction between punches ends for a definite value of $(b-a)/h$. Tables 2–4 show the critical value of the distance that ends interaction between punches with dimensionless quantities μ_2/μ_1, a/h, $(c-b)/h$, and Q/P. In such a case, there is no need to consider punches together. Also these tables show the values of the load factor that cause separation between elastic layer and elastic half plane, λ_{cr}. For $\lambda=\lambda_{\mathrm{cr}}$, $\sigma_{y_1}(x,0)h/P$ is zero. Contact between punches and elastic layer is continuous.

Table 3 shows the critical distance between punches that ends interaction of punches with elastic constant, μ_2/μ_1. For small values of μ_2/μ_1; that is, it is easy to bend elastic layer, interaction between punches ends in a longer distance. Initial separation point x_{cr} between elastic layer and elastic half plane from the origin $x = 0$ decreases with an increase in μ_2/μ_1. Critical load factor also decreases in this situation.

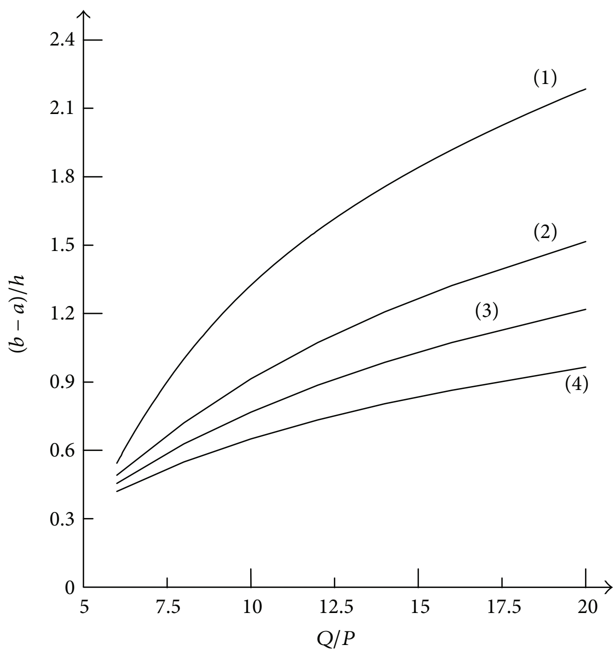
Distance $(b-a)/h$ that ends interaction between punches increases with a decrease in second punch width while first punch width is fixed. If both first and second widths are increased, interaction of punches ends in a shorter distance. This situation is presented in Table 4. In this case, critical load factor, λ_{cr} increases but initial separation point, x_{cr} decreases with increment in punch widths. Separation also occurs at the right-hand side of the second punch.

For fixed values of $a/h = 0.5$, $(c-b)/h = 1$, $Q = 2P$, and $\mu_2/\mu_1 = 1.65$, variations of critical load factor, λ_{cr} and initial separation point, x_{cr} are given in Table 5. For small values of $(b-a)/h$, initial separation point between elastic layer and elastic half plane appears at the right-hand side of second punch. If $(b-a)/h$ increases, in this case separation takes place between two punches. Keeping on increasing, the distance $(b-a)/h$ ends the interaction of punches. For $(b-a)/h = 7.9446$, there is no need to consider punches together.

In Figures 3–5, the normalized contact stress distribution $\sigma_{y_1}(x,0)h/P$ at the interface of elastic layer and elastic half plane is given for the problems described in Section 3, and Section 4. Different scales have been used for continuous and discontinuous contact cases in order to include the entire pressure distribution and to give sufficient details in compact forms.

Table 4: Variation of distance between two punches $(b-a)/h$ that ends interaction of punches with a/h and $(c-b)/h$ ($Q = 4P$, $\mu_2/\mu_1 = 0.36$).

a/h	$(c-b)/h$	$(b-a)/h$	Punch I		Punch II	
			$\lambda_{\mathrm{cr}_{\mathrm{right}}}$	$(x_{\mathrm{cr}_{\mathrm{right}}} - a)/h$	$\lambda_{\mathrm{cr}_{\mathrm{left}}} = \lambda_{\mathrm{cr}_{\mathrm{right}}}$	$(b - x_{\mathrm{cr}_{\mathrm{left}}})/h = (x_{\mathrm{cr}_{\mathrm{right}}} - c)/h$
0.25	1.5	10.8345	150.2590	3.775	54.1240	3.3816
0.25	1.0	11.8775	164.4214	3.655	49.0259	3.4495
0.25	0.5	12.7008	169.3000	3.628	45.9626	3.5968
0.50	1.0	11.6754	173.4645	3.523	49.0235	3.4494
0.75	1.5	10.4960	176.6011	3.421	54.1289	3.3810



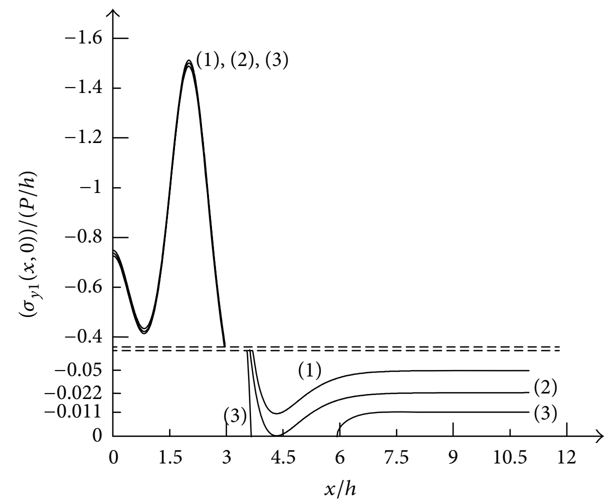
(1) $\mu_2/\mu_1 = 0.61$
(2) $\mu_2/\mu_1 = 1.65$
(3) $\mu_2/\mu_1 = 2.75$
(4) $\mu_2/\mu_1 = 6.48$

Figure 2: Variation of minimum value of distance between two punches $(b-a)/h$ to avoid separation under first punch with Q/P for various values of μ_2/μ_1 ($a/h = 0.5$, $(c-b)/h = 1$).

Table 5: Variation of load factor values with distance between two punches $(b-a)/h$ ($Q = 2P$, $\mu_2/\mu_1 = 1.65$, $a/h = 0.5$, and $(c-b)/h = 1$).

$(b-a)/h$	Punch I		Punch II			
	$\lambda_{\mathrm{cr}_{\mathrm{right}}}$	$x_{\mathrm{cr}_{\mathrm{right}}}$	$\lambda_{\mathrm{cr}_{\mathrm{left}}}$	$x_{\mathrm{cr}_{\mathrm{left}}}$	$\lambda_{\mathrm{cr}_{\mathrm{right}}}$	$x_{\mathrm{cr}_{\mathrm{right}}}$
0.5					**30.6814**	**4.294**
1					**32.4279**	**4.806**
3					**34.0783**	**6.808**
5	**28.4331**	**3.167**	**28.4331**	**3.167**	34.2756	8.803
6	**32.0727**	**4.184**	**32.0727**	**4.184**	34.3055	9.801
7	**33.8531**	**5.211**	**33.8531**	**5.211**	34.3175	10.800
7.9446	123.1714	2.835	**34.3182**	**6.148**	**34.3182**	**11.743**

Variation of the contact stress distribution $\sigma_{y_1}(x,0)h/P$ with load factor $\lambda = P/\rho_1 g h^2$ for fixed values of $Q = 2P$, $\mu_2/\mu_1 = 2.75$, $a/h = 0.25$, $(b-a)/h = 1.5$, and $(c-b)/h = 0.5$ is presented in Figure 3, $\lambda < \lambda_{\mathrm{cr}}$ and $\lambda > \lambda_{\mathrm{cr}}$ show contact stress distribution for the continuous and discontinuous contact cases, respectively. It can be seen from the figure that another separation region is possible if distance between punches $(b-a)/h$ or load factor λ is increased. Contact pressure has peaks around the edges of the rigid punches.



(1) $\lambda = 90 > \lambda_{\mathrm{cr}}$ ($d/h = 3.66$, $e/h = 5.926$)
(2) $\lambda = \lambda_{\mathrm{cr}} = 45.894$ ($x_{\mathrm{cr}} = 4.327$)
(3) $\lambda = 20 < \lambda_{\mathrm{cr}}$

Figure 3: Contact stress distribution between elastic layer and elastic half plane for the cases of continuous ($\lambda < \lambda_{\mathrm{cr}}$) and discontinuous contact ($\lambda > \lambda_{\mathrm{cr}}$) ($Q = 2P$, $\mu_2/\mu_1 = 2.75$, $a/h = 0.25$, $(b-a)/h = 1.5$, and $(c-b)/h = 0.5$).

In Figure 4, the variation of the normalized contact stress $\sigma_{y_1}(x,0)h/P$ with Q/P is given for the discontinuous contact case ($\lambda = 75 > \lambda_{\mathrm{cr}}$). Figure 4 shows that both contact stress and separation zone $(e-d)/h$ increase with an increase in Q/P. Variation of the normalized contact stress $\sigma_{y_1}(x,0)h/P$ for the discontinuous contact case shows three different regions between elastic layer and elastic half plane. These are continuous contact region, separation zone, and also continuous contact region, where the effects of external load Q/h and P/h decrease and disappear infinitely.

Variation of contact stress $\sigma_{y_1}(x,0)h/P$ with $\mu_2/\mu_1 = 1.65$ is shown in Figure 5. Separation zone increases as elastic half plane gets stiffer than the elastic layer. In this case, peak value of the contact stress also increases.

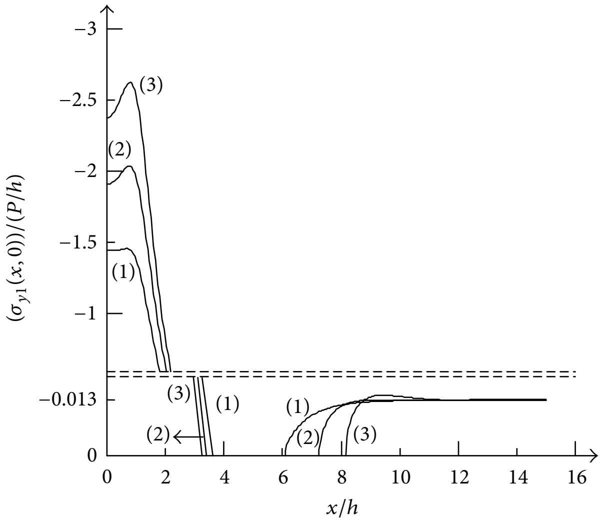


Figure 4: Contact stress distribution between elastic layer and elastic half plane for the case of discontinuous contact ($\mu_2/\mu_1 = 0.36$, $a/h = 0.125$, $(b-a)/h = 0.5$, $(c-b)/h = 0.5$, and $\lambda = 75 > \lambda_{cr}$).

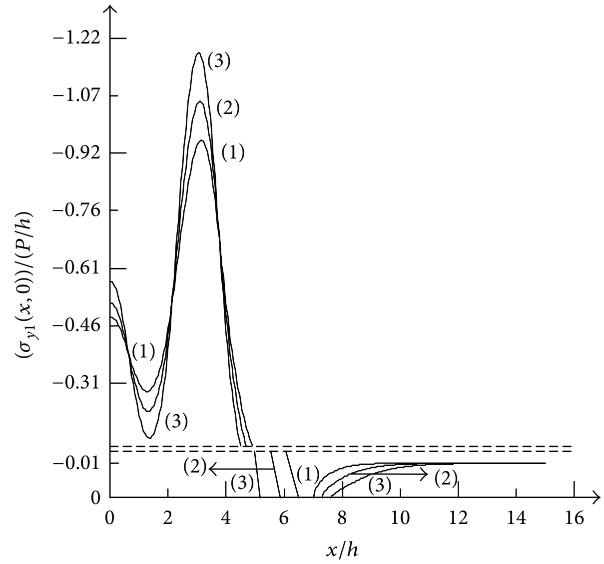


Figure 5: Contact stress distribution between elastic layer and elastic half plane for the case of discontinuous contact ($Q = 2P$, $a/h = 0.5$, $(b-a)/h = 2$, $(c-b)/h = 1$, and $\lambda = 100 > \lambda_{cr}$).

Results calculated from (33) giving the displacement $v^*(x,0)$ in the separation region $d < x < e$ as a function of x are shown in Figure 6. Also results are compared with those of [35]. It is seen that separation region $d < x < e$ and displacement values increase with an increase in Q/P ratio as this is the case in [35]. Separation zone is nearly the same, but displacement values are smaller than those of [35] for fixed values of $\mu_2/\mu_1 = 6.48$, $a/h = 0.75$, $(b-a)/h = 1$, and $(c-b)/h = 1.5$.

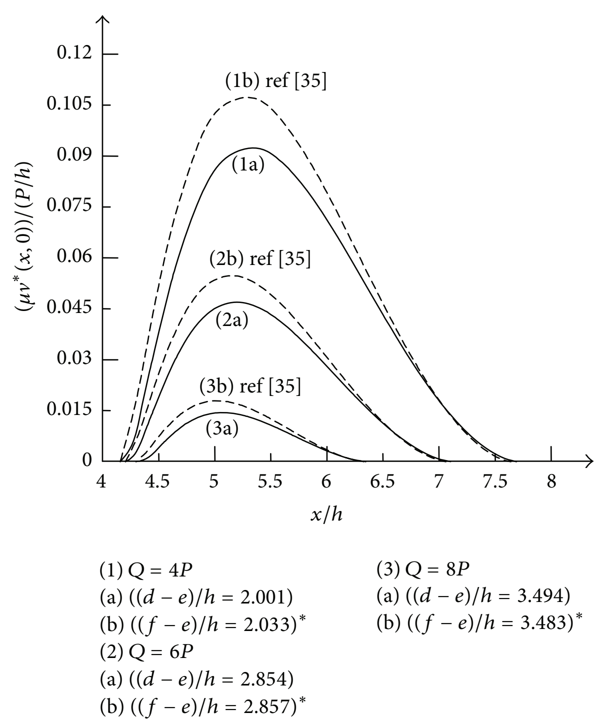


Figure 6: Separation displacement $v^*(x,0)$ between elastic layer and elastic half space as a function of x for various values of second punch load Q ($\mu_2/\mu_1 = 6.48$, $a/h = 0.75$, $(b-a)/h = 1$, $(c-b)/h = 1.5$, and $\lambda = 50 > \lambda_{cr}$), ($*$: [35]).

6. Conclusion

In this paper, continuous and discontinuous contact problems for an elastic layer resting on an elastic half plane loaded by means of three rigid flat punches are considered. Numerical procedures developed in this study can be used to find approximate solutions to problems of engineering interest.

Numerical results show that the punch widths, elastic constants, external loads, distance between punches play a very important role in the formation of the continuous and discontinuous contact areas, initial separation point and load, separation displacement, limit distance that ends the interaction of punches, critical distance that causes separation under first punch, and the contact pressure distribution.

From the study, the following conclusions may be drawn:

(i) it is easy to separate first punch from elastic layer if Q/P or a/h increases. This results also decrease in μ_2/μ_1 or $(c-b)/h$.

(ii) Increment in Q/P, a/h or $(c-b)/h$ ends interaction of punches in a shorter distance. Also an increase in μ_2/μ_1 causes the same result.

(iii) First separation between elastic layer and elastic half plane occurs between punches or in the region $c < x < \infty$ depending on λ. If $(b-a)/h$ is big enough, interaction of punches disappears.

(iv) Size of separation region is not affected much with Q/P, but separation displacement is increased with an increase in Q/P.

Nomenclature

P: Compressive load per unit thickness in z direction on punch I, (N/m)
Q: Compressive load per unit thickness in z direction on punch II and punch III, (N/m)
h: Thickness of the layer, (m)
λ: Load factor
u: x-component of the displacement, (m)
v: y-component of the displacement, (m)
ρ_k: Mass density, (kg/m^3)
g: Gravity acceleration, (m/sn2)
μ: Shear modulus, (Pa)
κ: Elastic constant
γ: Poisson's ratio
$(c-b)$: Width of punch II and punch III, (m)
a: Half width of punch I, (m)
$P(x)$: Contact pressure under punch I, (Pa)
$Q(x)$: Contact pressure under punch II and punch III, (Pa)
$(e-d)$: Separation length, (m).

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