# Nonlinear Dynamic Analysis and Optimization of Closed-Form Planetary Gear System 

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#### Abstract

A nonlinear purely rotational dynamic model of a multistage closed-form planetary gear set formed by two simple planetary stages is proposed in this study. The model includes time-varying mesh stiffness, excitation fluctuation and gear backlash nonlinearities. The nonlinear differential equations of motion are solved numerically using variable step-size Runge-Kutta. In order to obtain function expression of optimization objective, the nonlinear differential equations of motion are solved analytically using harmonic balance method (HBM). Based on the analytical solution of dynamic equations, the optimization mathematical model which aims at minimizing the vibration displacement of the low-speed carrier and the total mass of the gear transmission system is established. The optimization toolbox in MATLAB program is adopted to obtain the optimal solution. A case is studied to demonstrate the effectiveness of the dynamic model and the optimization method. The results show that the dynamic properties of the closed-form planetary gear transmission system have been improved and the total mass of the gear set has been decreased significantly.


## 1. Introduction

Planetary gear sets have been widely used in engineering including automotive transmissions, aviation transmissions, and crane gearboxes as well as other marine and industrial power transmission systems. Planetary gear trains have many advantages over fixed-center counter-shaft gear systems. The flow of power via multiple-gear meshes increases the power density and helps to reduce the overall size of the transmission train. The ability of multistage planetary sets in providing multiple speed reduction ratios has been the main reason for their extensive use in automatic transmission applications. Closed-form planetary trains are obtained from a number of single-stage differential planetary gear sets and one quasi-planetary stage whose central members are connected according to a given power flow configuration. Input, output, and fixed member assignments are made to certain central members to achieve a given gear ratio.

Because of complexity of structure, most of the earlier published studies on the planetary gear systems were
confined to single-stage planetary. In addition, these early models were of linear time-invariant type, so that the eigen solutions and model summation techniques were used to predict the natural modes and the forced response [1-3]. Kahraman [4] employed a purely rotational dynamic model for all possible power flow configurations of complex compound planetary gear sets. He classified the natural model in two categories: asymmetric planet modes and axi-symmetric overall modes. Sun and Hu [5] investigated the frequency response of nonlinear planetary transmission system with multiple clearances using single-term harmonic balance method and focusing only on a single power flow configuration, in which the ring gear was fixed. Al-shyyab and Kahraman [6] developed a rotational single-stage nonlinear dynamic model of a simple planetary gear set and provided a semianalytical forced response solution using multiterm HBM and showed that these HBM solutions in well agreement with direct numerical integration solution. Also, a recent study by Al-shyyab [7] investigated a compound planetary gear set formed by any number of simple planetary stages, and each


Figure 1: (a) A closed-form planetary gear system; (b) A discrete model of closed-form planetary gear system.
planetary stage has a distinct fundamental mesh frequency and any number of planets spaced in any angular positions using multiterm HBM.

Those studies cited previously are mainly aimed at the modeling of planetary gear trains, the analysis of dynamic response as well as the analysis of parameters stability, and so on. Whereas, the studies on nonlinear dynamic optimization design of planetary transmission system with multiple clearances are still very limited. Zeng Bao [8] and Guan Wei [9] investigated multistage helical gear trains and singlestage planetary gear train taking the dynamic properties as objective functions, respectively. But, in their studies the design variables of optimization were limited to these parameters: the number of gears, the pitch-cycle helicalangle, and the modification coefficients. This study aims at providing numerical solutions and analytical solutions for the dynamic response of a closed-form planetary gear train having three planets spaced equally position angle using RungeKutta numerical integration and HBM, respectively. Based on the analytical solution, the optimization mathematical model that focused on minimizing the vibration acceleration of structure and the total mass of the gear transmission system is established. Some key design parameters such as the number of each gear, the module of each stage planetary, the transmission ratio of each stage planetary and the pressure angle are chosen as design varies. This study is available for the designing of closed-form planetary gear sets of both minimum weight and best dynamic characteristic for reference.

## 2. Dynamic Model of System

2.1. Model and Assumptions. The closed-form planetary gear train consists of a single-stage differential planetary (lowspeed stage) and a single-stage quasi-planetary (high-speed stage, carrier is fixed) in this study, as shown in Figure 1. Each stage is comprised of three central elements: the sun gear $\left(s_{1}, s_{2}\right)$, the ring gear $\left(r_{1}, r_{2}\right)$, and carrier $\left(c_{1}, c_{2}\right)$. Each stage


Figure 2: Lumped parameter dynamic model of single-stage planetary gear set.
planetary has $n$ planet gears. The parameter $n$ representing the number of planet gears is taken as 3 throughout this paper. The planets of each stage are free to rotate with respect to their common carrier. All the gears are mounted on their rigid shafts supported by rolling element bearings. The two rings are connected by torsional linear springs of stiffness $k_{r 1 r 2}$, as shown in Figure 2. Likewise, the other central elements $c_{1}, c_{2}$ and $s_{1}, s_{2}$ are constrained by torsional linear springs of stiffness $k_{c 1 s 2}, k_{c 2}$ and $k_{s 1}, k_{c 1 s 2}$, respectively. In order to establish the mathematical model of the transmission system, a number of simplified assumptions are introduced in the case of speed reduction in the closed-form planetary gear set, as shown in Figure 1.
(1) All of the gears in the set are assumed to be rigid and the flexibilities of each gear teeth at the gear mesh interface are modeled by an equivalent spring having time-varying stiffness acting along the mesh directions. These periodically time-varying mesh stiffnesses are subject to piece-linear functions representing gear backlashes.
(2) Because the bending stiffness of shafts in the set is very large, the deflection of these shafts can be neglected. Thus, the transverse displacements of gears are not considered.
(3) Because the damping mechanisms at the gear meshes and bearings of a planetary gear set are not easy to give a description of mathematical model, viscous gear mesh damping elements are introduced to represent energy dissipation of the transmission system.
2.2. Equivalent Displacements. In order to establish the equations of motion easily, all of torsional angular displacements are unified on the pressure line in terms of equivalent displacements. The equivalent transverse displacements in the mesh line direction caused by rotational displacements are written as follows:

$$
\begin{array}{r}
u_{c j}=r_{c j} \theta_{c j}, \quad u_{r j}=r_{r j} \theta_{r j}, \quad u_{s j}=r_{s j} \theta_{s j}, \quad u_{p j n}=r_{p j} \theta_{j n}, \\
j=1,2 ; n=1,2,3 \tag{1}
\end{array}
$$

where $\theta$ and $r$ (subscripts $c j, r j, s j, j n ; j=h, l ; n=1,2,3$ ) are angular displacements and base circle radius of the parts, respectively. While $r$ (subscripts $c 1, c 2$ ) is the equivalent base circle radius of the carriers defined as follows:

$$
\begin{align*}
& r_{s 1}+r_{p h}=r_{r 1}-r_{p h}=r_{c 1} \cos \alpha  \tag{2}\\
& r_{s 2}+r_{p l}=r_{r 2}-r_{p l}=r_{c 2} \cos \alpha
\end{align*}
$$

With the symbol $U$ is used to represent the relative displacements in the direction of pressure line, the relative displacements are obtained according to the meshing relation and the equivalent displacements as follows. The positive direction of the relative displacements is assumed to be the same direction of the compressive deformation. The relative displacements of the parts in the pressure line direction are written as follows:

$$
\begin{gather*}
\begin{array}{c}
U_{s 1 p h i}=r_{s 1}\left(\theta_{s 1}-\theta_{c 1}\right)-r_{p h}\left(\theta_{h i}+\theta_{c 1}\right) \\
=r_{s 1} \theta_{s 1}-r_{c 1} \theta_{c 1} \cos \alpha-r_{p h} \theta_{h i}, \\
U_{r 1 p h i}=r_{p h}\left(\theta_{h i}+\theta_{c 1}\right)-r_{r 1}\left(\theta_{r 1}+\theta_{c 1}\right) \\
=r_{p h} \theta_{h i}-r_{c 1} \theta_{c 1} \cos \alpha-r_{r 1} \theta_{r 1}, \\
U_{c 1 s 2}=r_{c 1}\left(\theta_{c 1}-\theta_{s 2}\right) \cos \alpha, \\
U_{c 2}=r_{c 2} \theta_{c 2} \cos \alpha,
\end{array} .
\end{gather*}
$$

$$
\begin{align*}
U_{s 2 p l i} & =r_{s 2}\left(\theta_{s 2}-\theta_{c 2}\right)-r_{p l}\left(\theta_{l i}+\theta_{c 2}\right)  \tag{3e}\\
& =r_{s 2} \theta_{s 2}-r_{c 2} \theta_{c 2} \cos \alpha-r_{p l} \theta_{l i} \\
U_{r 2 p l i} & =r_{p l}\left(\theta_{l i}+\theta_{c 2}\right)-r_{r 2}\left(\theta_{r 2}+\theta_{c 2}\right)  \tag{3f}\\
& =r_{p l} \theta_{l i}-r_{c 2} \theta_{c 2} \cos \alpha-r_{r 2} \theta_{r 2} \\
& U_{r 1 r 2}=r_{r 1}\left(\theta_{r 1}-\theta_{r 2}\right) \tag{3~g}
\end{align*}
$$

2.3. Equations of Motion. The closed-form planetary gear system consists of four different kinds of gear pairs, the external gear pair, that is, the sun gear/planet gear- $i$ pair (subscripts $s 1, p h i$ and $s 2$, pli), and the internal gear pair, that is, the ring gear/planet gear-i pair (subscripts $r 1, p h i$ and $r 2, p l i)$. The mesh of gear $j$ (si or $r i, i=1,2)$ with a planet $p i(i=h, l)$ is represented by a periodically timevarying stiffness element $k_{j p i}(t)$ subjected to a piecewise linear backlash function $g$ that includes a clearance of gap width $2 b_{j p i}$. Accordingly, the dynamic model of a closed-form gear set with $n$ planets includes $2 n$ clearances. In this closedform planetary gear system, damper is described by constant viscous damper coefficient $c_{j p i}$. This is a rather simplified mesh contact model; in reality these contacts are subjected to the hydro-elastic-dynamic regime of lubrication [10]. In this paper, it is supposed that all planets $p i(i=h, l)$ and their respective meshes with gear $j(s 1, s 2$ or $r 1, r 2)$ are identical so that $k_{j p i}(t), b_{j p i}$, and $c_{j p i}$ are the same for each $j p i$ mesh, except the phase angles of $k_{j p i}(t)$ which differ according to planet phasing conditions.

Thus, the nonlinear dynamic differential equations of motion of closed-form planetary gear set can be established using the Lagrange principle as follows:

$$
\begin{gather*}
M_{c 1} \ddot{u}_{c 1}+C_{c 1 s 2} \dot{U}_{c 1 s 2} \\
-\sum_{i=1}^{3}\left(C_{s 1 n} \dot{U}_{s 1 p h i}+C_{r 1 n} \dot{U}_{r 1 p h i}\right)+k_{c 1 s 2} U_{c 1 s 2}  \tag{4a}\\
-\sum_{i=1}^{3}\left[k_{s 1 n} U_{s 1 p h i}+k_{r 1 n} U_{r 1 p h i}\right]=0 \\
M_{r 1} \ddot{u}_{r 1}+C_{r 1 r 2} \dot{U}_{r 1 r 2} \\
-\sum_{i=1}^{3} C_{r 1 n} \dot{U}_{r 1 p h i}+k_{r 1 r 2} U_{r 1 r 2}  \tag{4b}\\
\quad-\sum_{i=1}^{3} k_{r 1 n} U_{r 1 p h i}=0 \\
M_{s 1} \ddot{u}_{s 1}+\sum_{i=1}^{3} C_{s 1 n} \dot{U}_{s 1 p h i}+\sum_{i=1}^{3} k_{s 1 n} U_{s 1 p h i}=\frac{T_{\text {in }}}{r_{s 1}}  \tag{4c}\\
M_{p h} \ddot{u}_{h n}-C_{s 1 n} \dot{U}_{s 1 p h n}+C_{r 1 n} \dot{U}_{r 1 p h n}  \tag{4d}\\
-k_{s 1 n} U_{s 1 p h n}+k_{r 1 n} U_{r 1 p h n}=0 \quad(n=1,2,3),
\end{gather*}
$$

$$
\begin{align*}
& M_{c 2} \ddot{u}_{c 2}+C_{c 2} \dot{U}_{c 2} \\
& -  \tag{4e}\\
& \sum_{i=1}^{3}\left(C_{s 2 n} \dot{U}_{s 2 p l i}+C_{r 2 n} \dot{U}_{r 2 p l i}\right)+k_{c 2} U_{c 2} \\
& - \\
& \sum_{i=1}^{3}\left[k_{s 2 n} U_{s 2 p l i}+k_{r 2 n} U_{r 2 p l i}\right]=0,  \tag{4f}\\
& \\
& M_{r 2} \ddot{u}_{r 2}-C_{r 1 r 2} \dot{U}_{r 1 r 2} \frac{r_{r 1}}{r_{r 2}} \\
& \quad-\sum_{i=1}^{3} C_{r 2 n} \dot{U}_{r 2 p l i}-k_{r 1 r 2} U_{r 1 r 2} \frac{r_{r 1}}{r_{r 2}}  \tag{4g}\\
& \quad-\sum_{i=1}^{3} k_{r 2 n} U_{r 2 p l i}=\frac{T_{\text {out }}}{r_{r 2}}, \\
& M_{s 2} \ddot{u}_{s 2}-C_{c 1 s 2} \dot{U}_{c 1 s 2} \frac{r_{c 1} \cos \alpha}{r_{s 2}} \\
& \quad+\sum_{i=1}^{3} C_{s 2 n} \dot{U}_{s 2 p l i}-k_{c 1 s 2} U_{c 1 s 2} \frac{r_{c 1} \cos \alpha}{r_{s 2}}  \tag{4h}\\
& \quad+\sum_{i=1}^{3} k_{s 2 n} U_{s 2 p l i}=0, \\
& M_{p l} \ddot{u}_{l n}-C_{s 2 n} \dot{U}_{s 2 p l n}+C_{r 2 n} \dot{U}_{r 2 p l n} \\
& -k_{s 2 n} U_{s 2 p l n}+k_{r 2 n} U_{r 2 p l n}=0 \quad(n=1,2,3),
\end{align*}
$$

where, $I_{c 1}=I_{c 1}^{\prime}+3 m_{p h} r_{c 1}^{2}$ is the equivalent mass moment of inertia of the carrier including planet gears in high-speed stage; $I_{c 1}^{\prime}$ is the inertia of the carrier in high-speed stage; $m_{p h}$ is the actual mass of planet-gear in high-speed stage; and $r_{c 1}$ is the distribution circle radius of the planet gears. $I_{c 1}=M_{c 1} r_{b c 1}^{2}, M_{c 1}$ is the equivalent mass of the carrier in high-speed stage with respect to the its equivalent base circle radius; $r_{b c 1}$ is equivalent base circle radius of the carrier in high-speed stage; $r_{b c 1}=r_{c 1} \cos \alpha ; I_{r 1}=M_{r 1} r_{r 1}^{2}$, and $M_{r 1}$ is the equivalent mass of the ring in low-speed stage with respect to its base circle radius; $I_{s 1}=M_{s 1} r_{s 1}^{2}$, and $M_{s 1}$ is the equivalent mass of the sun gear in low-speed stage with respect to its base circle radius; $I_{p h n}=M_{p h} r_{p h}^{2}$, and $M_{p h}$ is the equivalent mass of the planet-gear in high-speed stage with respect to its equivalent base circle radius; $I_{c 2}=I_{c 2}^{\prime}+3 m_{p l} r_{c 2}^{2}$, and $I_{c 2}^{\prime}$ is the moment of inertia of the carrier in low-speed stage.

Dynamic model of motion of transmission has considerable difficulties in its solution procedure as follows. (1) As a semidefinite system, its first-order natural frequency is zero corresponding to a rigid body motion. (2) The piecewiselinear function $g$ represents the gear backlash, and the number of variables is even different according to the external and internal gear pairs. (3) As both linear and nonlinear restoring forces exist in the equations, it is not possible to write out the governing equation in matrix form, while a general solution technique applicable to the systems of multiple degrees of freedom must be based on matrix form.

Therefore, (4a)-(4h) are simplified further by introducing a set of new variables:

$$
\begin{gather*}
U_{s 1 p h n}=u_{s 1}-u_{c 1}-u_{h n}  \tag{5a}\\
U_{r 1 p h n}=u_{h n}-u_{c 1}-u_{r 1},  \tag{5b}\\
U_{c 1 s 2}=u_{c 1}-\frac{r_{c 1} \cos \alpha}{r_{s 2}} u_{s 2}  \tag{5c}\\
U_{c 2}=u_{c 2}  \tag{5d}\\
U_{s 2 p h n}=u_{s 2}-u_{c 2}-u_{l n}  \tag{5e}\\
U_{r 2 p h n}=u_{l n}-u_{c 2}-u_{r 2}  \tag{5f}\\
U_{r 1 r 2}=u_{r 1}-\frac{r_{r 1}}{r_{r 2}} u_{r 2} . \tag{5~g}
\end{gather*}
$$

The new coordinate variables defined previously not only have intuitional physical meaning, but also eliminate the rigid body motion. Furthermore, the piecewise-linear backlash function $g$ can be written as a set of functions with a single variable according to (5a)-(5g).

In addition, nondimensional parameters of (4a)-(4h) can be obtained by a characteristic length $b$ and frequency $\omega_{n}=$ $\sqrt{k_{s 1 n} / M_{s 1}}$, such that

$$
\begin{equation*}
t=\omega_{n} \bar{t} \tag{6}
\end{equation*}
$$

Hence, a set of simplified equations of motions of the transmission system is obtained by substituting (5a)-(5g) and (6) into (4a)-(4h):

$$
\begin{align*}
& \ddot{U}_{s 1 p h n}+\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left[\frac{C_{s 1 n} \dot{U}_{s 1 p h i}}{M_{s 1 c 1}}+\frac{C_{r 1 n} \dot{U}_{r 1 p h i}}{M_{c 1}}\right] \\
&+\frac{1}{\omega_{n}^{2}} {\left[\frac{C_{s 1 n} \dot{U}_{s 1 p h n}-C_{r 1 n} \dot{U}_{r 1 p h n}}{M_{p h}}\right.} \\
&\left.-\frac{C_{c 1 s 2} \dot{U}_{c 1 s 2}}{M_{c 1}}\right] \\
&+\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left[\frac{k_{s 1 n} g\left(U_{s 1 p h i}\right)}{M_{s 1 c 1}}+\frac{k_{r 1 n} g\left(U_{r 1 p h i}\right)}{M_{c 1}}\right] \\
&+\frac{1}{\omega_{n}^{2}} {\left[\frac{k_{s 1 n} g\left(U_{s 1 p h n}\right)-k_{r 1 n} g\left(U_{r 1 p h n}\right)}{M_{p h}}\right.} \\
&=\left.-\frac{k_{c 1 s 2} g\left(U_{c 1 s 2}\right)}{M_{c 1}}\right] \\
&=\frac{T_{\text {in }}}{r_{s 1} M_{s 1} b \omega_{n}^{2}} \quad(n=1,2,3) \tag{7a}
\end{align*}
$$

$$
\begin{align*}
& \ddot{U}_{r 1 p h n}+\frac{1}{\omega_{n}^{2}}\left[\frac{C_{r 1 n} \dot{U}_{r 1 p h n}-C_{s 1 n} \dot{U}_{s 1 p h n}}{M_{p h}}\right. \\
& \left.-\frac{C_{c 1 s 2} \dot{U}_{c 1 s 2}}{M_{c 1}}-\frac{C_{r 1 r 2} \dot{U}_{r 1 r 2}}{M_{r 1}}\right] \\
& +\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left[\frac{C_{s 1 n} \dot{U}_{s 1 p h i}}{M_{c 1}}+\frac{C_{r 1 n} \dot{U}_{r 1 p h i}}{M_{c 1 r 1}}\right] \\
& +\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left[\frac{k_{s 1 n} g\left(U_{s 1 p h i}\right)}{M_{c 1}}+\frac{k_{r 1 n} g\left(U_{r 1 p h i}\right)}{M_{c 1 r 1}}\right]  \tag{7b}\\
& +\frac{1}{\omega_{n}^{2}}\left[\frac{k_{r 1 n} g\left(U_{r 1 p h n}\right)-k_{s 1 n} g\left(U_{s 1 p h n}\right)}{M_{p h}}\right. \\
& \left.-\frac{k_{c 1 s 2} g\left(U_{c 1 s 2}\right)}{M_{c 1}}-\frac{k_{r 1 r 2} g\left(U_{r 1 r 2}\right)}{M_{r 1}}\right]=0 \\
& (n=1,2,3) \text {, } \\
& \ddot{U}_{c 1 s 2}+\frac{C_{c 1 s 2}}{M_{c 1} \omega_{n}^{2}} \dot{U}_{c 1 s 2} \\
& -\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left[\frac{C_{s 1 n} \dot{U}_{s 1 p h i}+C_{r 1 n} \dot{U}_{r 1 p h i}}{M_{c 1}}\right. \\
& \left.+\frac{C_{s 2 n} \dot{U}_{s 2 p l i} r_{c 1} \cos \alpha}{M_{s 2} r_{s 2}}\right] \\
& +\frac{C_{c 1 s 2}}{M_{s 2} \omega_{n}^{2}}\left(\frac{r_{c 1} \cos \alpha}{r_{s 2}}\right)^{2} \dot{U}_{c 1 s 2}+\frac{k_{c 1 s 2} g\left(U_{c 1 s 2}\right)}{M_{c 1} \omega_{n}^{2}}  \tag{7e}\\
& +\frac{k_{c 1 s 2} g\left(U_{c 1 s 2}\right)}{M_{s 2} \omega_{n}^{2}}\left(\frac{r_{c 1} \cos \alpha}{r_{s 2}}\right)^{2} \\
& -\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left[\frac{k_{s 1 n} g\left(U_{s 1 p h i}\right)+k_{r 1 n} g\left(U_{r 1 p h i}\right)}{M_{c 1}}\right. \\
& \left.+\frac{k_{s 2 n} g\left(U_{s 2 p l i}\right) r_{c 1} \cos \alpha}{M_{s 2} r_{s 2}}\right]=0,  \tag{7c}\\
& \ddot{U}_{c 2}+\frac{C_{c 2} \dot{U}_{c 2}}{M_{c 2} \omega_{n}^{2}} \\
& -\sum_{i=1}^{3} \frac{C_{s 2 n} \dot{U}_{s 2 p l i}+C_{r 2 n} \dot{U}_{r 2 p l i}}{M_{c 2} \omega_{n}^{2}}+\frac{k_{c 2} g\left(U_{c 2}\right)}{M_{c 2} \omega_{n}^{2}}  \tag{7d}\\
& -\sum_{i=1}^{3} \frac{k_{s 2 n} g\left(U_{s 2 p l i}\right)+k_{r 2 n} g\left(U_{r 2 p l i}\right)}{M_{c 2} \omega_{n}^{2}}=0,
\end{align*}
$$

$$
\begin{aligned}
& \ddot{U}_{s 2 p l n}+\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left[\frac{C_{s 2 n} \dot{U}_{s 2 p l i}}{M_{s 2}}\right. \\
& \left.+\frac{C_{s 2 n} \dot{U}_{s 2 p l i}+C_{r 2 n} \dot{U}_{r 2 p l i}}{M_{c 2}}\right] \\
& +\frac{1}{\omega_{n}^{2}}\left[\frac{C_{s 2 n} \dot{U}_{s 2 p l n}-C_{r 2 n} \dot{U}_{r 2 p l n}}{M_{p l}}\right. \\
& \left.-\frac{C_{c 2} \dot{U}_{c 2}}{M_{c 2}}-\frac{C_{c 1 s 2}}{M_{s 2}} \frac{r_{c 1} \cos \alpha \dot{U}_{c 1 s 2}}{r_{s 2}}\right] \\
& +\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left[\frac{k_{s 2 n} g\left(U_{s 2 p l i}\right)}{M_{s 2}}\right. \\
& \left.+\frac{k_{s 2 n} g\left(U_{s 2 p l i}\right)+k_{r 2 n} g\left(U_{r 2 p l i}\right)}{M_{c 2}}\right] \\
& +\frac{1}{\omega_{n}^{2}}\left[\frac{k_{s 2 n} g\left(U_{s 2 p l n}\right)-k_{r 2 n} g\left(U_{r 2 p l n}\right)}{M_{p l}}\right. \\
& -\frac{k_{c 2} g\left(U_{c 2}\right)}{M_{c 2}}-\frac{k_{c 1 s 2}}{M_{s 2}} \\
& \left.\times \frac{g\left(U_{c 1 s 2}\right) r_{c 1} \cos \alpha}{r_{s 2}}\right]=0 \quad(n=1,2,3), \\
& \ddot{U}_{r 2 p l n}+\frac{1}{\omega_{n}^{2}}\left[\frac{C_{r 2 n} \dot{U}_{r 2 p l n}-C_{s 2 n} \dot{U}_{s 2 p l n}}{M_{p l}}\right. \\
& \left.-\frac{C_{c 2} \dot{U}_{c 2}}{M_{c 2}}+\frac{C_{r 1 r 2} r_{r 1} \dot{U}_{r 1 r 2}}{M_{r 2} r_{r 2}}\right] \\
& +\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left[\frac{C_{s 2 n} \dot{U}_{s 2 p l i}+C_{r 2 n} \dot{U}_{r 2 p l i}}{M_{c 2}}\right. \\
& \left.+\frac{C_{r 2 n} \dot{U}_{r 2 p l n}}{M_{r 2}}\right] \\
& +\frac{1}{\omega_{n}^{2}}\left[\frac{k_{r 2 n} g\left(U_{r 2 p l n}\right)-k_{s 2 n} g\left(U_{s 2 p l n}\right)}{M_{p l}}\right. \\
& -\frac{k_{c 2} g\left(U_{c 2}\right)}{M_{c 2}} \\
& \left.+\frac{k_{r 122} r_{r 1} g\left(U_{r 1 r 2}\right)}{M_{r 2} r_{r 2}}\right]
\end{aligned}
$$

$$
\begin{align*}
& +\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left[\frac{k_{s 2 n} g\left(U_{s 2 p l i}\right)+k_{r 2 n} g\left(U_{r 2 p l i}\right)}{M_{c 2}}\right. \\
& \left.+\frac{k_{r 2 n} g\left(U_{r 2 p l i}\right)}{M_{r 2}}\right] \\
& =\frac{-T_{\text {out }}}{M_{r 2} r_{r 2} b \omega_{n}^{2}} \quad(n=1,2,3) \text {, }  \tag{7f}\\
& \ddot{U}_{r 1 r 2}+\frac{C_{r 1 r 2} \dot{U}_{r 1 r 2}}{M_{r 1 r 2} \omega_{n}^{2}} \\
& +\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left(\frac{C_{r 2 n} r_{r 1}}{M_{r 2} r_{r 2}} \dot{U}_{r 2 p l i}-\frac{C_{r 1 n}}{M_{r 1}} \dot{U}_{r 1 p h i}\right) \\
& +\frac{k_{r 1 r 2} g\left(U_{r 1 r 2}\right)}{M_{r 1 r 2} \omega_{n}^{2}}  \tag{7g}\\
& +\frac{1}{\omega_{n}^{2}} \sum_{i=1}^{3}\left(\frac{k_{r 2 n} r_{r 1} g\left(U_{r 2 p l i}\right)}{M_{r 2} r_{r 2}}-\frac{k_{r 1 n} g\left(U_{r 1 p h i}\right)}{M_{r 1}}\right) \\
& =-\frac{T_{\text {out }} r_{r 1}}{M_{r 2} r_{r 2}^{2} b \omega_{n}^{2}},
\end{align*}
$$

where,

$$
\begin{align*}
& \frac{1}{M_{s 1 c 1}}=\frac{1}{M_{s 1}}+\frac{1}{M_{c 1}} \\
& \frac{1}{M_{c 1 r 1}}=\frac{1}{M_{c 1}}+\frac{1}{M_{r 1}}  \tag{8}\\
& \frac{1}{M_{r 1 r 2}}=\frac{1}{M_{r 1}}+\frac{r_{r 1}}{M_{r 2} r_{r 2}} .
\end{align*}
$$

Equations (7a)-(7g) can be written in matrix form as

$$
\begin{equation*}
\ddot{\mathbf{U}}+\mathbf{C} \dot{\mathbf{U}}+\mathbf{K} g(\mathbf{U})=\mathbf{T}, \tag{9}
\end{equation*}
$$

where, the piecewise-linear backlash function is defined as

$$
g\left[U_{i j}(t)\right]= \begin{cases}U_{i j}(t)-b_{i j}, & U_{i j}>b_{i j}  \tag{10}\\ 0, & -b_{i j} \leq U_{i j} \leq b_{i j} \\ U_{i j}(t)+b_{i j}, & U_{i j}<-b_{i j}\end{cases}
$$

## 3. Solution of Dynamic Equations

3.1. Dynamic Response Using the Numerical Integration. In this section, the mathematical model will be solved numerically by the variable step-size Runge-Kutta integration method firstly. The parameters of the closed-form planetary gear set shown in Figure 1 are given in Tables 1 and 2. In this work, the values of the gear mesh damping coefficients $C_{i j}$ are assumed to be constants as 0.01 [11].

By resolving the dynamic differential equations of motions, the dynamic responses (displacement and speed) of low-stage carrier are gained as shown in Figure 3.
3.2. The Acquisition of Analytical Solutions. In order to establish the optimization mathematical model of the dynamic behavior of the closed-form planetary gear set, it is necessary to obtain the analytical solutions of model using the harmonic balance method. For limiting the number of algebraic balance equations, only the fundamental frequency harmonic of the mesh stiffness functions are considered in this case. Similarly, external torque functions is also considered to be in the form of mesh stiffness functions. And then, attention is paid to the periodic vibrations of system under the harmonic excitation. The procedure of solving (7a)-(7g) using HBM [5] includes some aspects as follows.
(1) According to the assumption previously mentioned, the external excitations that represent fundamental frequency pulsations can be written in the form

$$
\begin{equation*}
\mathbf{T}=\mathbf{T}_{m i}+\mathbf{T}_{a i} \cos \left(\Omega t+\phi_{i}\right) \tag{11}
\end{equation*}
$$

where $\mathbf{T}_{m i}$ is the mean component of torque and $\mathbf{T}_{a i}$ is the amplitude of the alternating component of the fundamental frequency mesh force. $\phi_{i}$ is the phase angle.
(2) According to the harmonic excitations given in (11), the harmonic balance method solution to $(7 \mathrm{a})-(7 \mathrm{~g})$ is assumed in the same form

$$
\begin{equation*}
\mathbf{U}=\mathbf{U}_{m i}+\mathbf{U}_{a i} \cos \left(\Omega t+\phi_{i}\right) \tag{12}
\end{equation*}
$$

where $\mathbf{U}_{m i}$ and $\mathbf{U}_{a i}$ are the mean and alternating components of the steady state response, respectively, and $\phi_{i}$ is the phase angle.
(3) For relative mesh displacements, the piecewise-linear function in (4a)-(4h) can be written in a unified form:

$$
\begin{equation*}
g\left(U_{i}\right)=N_{m i} U_{m i}+N_{a i} U_{a i} \cos \left(\Omega t+\phi_{i}\right), \tag{13}
\end{equation*}
$$

where

$$
\begin{gather*}
N_{m i}=1+\frac{q_{a i}}{2 q_{m i}}\left[G\left(\mu_{+}\right)-G\left(\mu_{-}\right)\right],  \tag{14}\\
N_{a i}=1-\frac{1}{2}\left[H\left(\mu_{+}\right)-H\left(\mu_{-}\right)\right]
\end{gather*}
$$

The expressions of $G$ and $H$ are given in the Appendix.
(4) Considering the mean value and the fundamental harmonic value of periodically time-varying mesh stiffness in Figure 2, the elements of stiffness matrix $\mathbf{K}$ in (9) can be written as

$$
\begin{equation*}
k_{i j}=k_{m i j}+k_{a i j} \cos \left(\Omega \tau+\varphi_{i j}\right) \tag{15}
\end{equation*}
$$

So, the stiffness matrix $\mathbf{K}$ is written in terms of two separate matrices for mean stiffness and alternating stiffness as

$$
\begin{equation*}
\mathbf{K}=\mathbf{K}_{m}+\Delta \mathbf{K} \tag{16}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathbf{K}_{m}=\left[k_{m i j}\right]_{n \times n}  \tag{17}\\
\Delta \mathbf{K}=\left[k_{a i j} \cos \left(\Omega \tau+\varphi_{i j}\right)\right]_{n \times n}
\end{gather*}
$$

Table 1: Geometric parameters of the closed-form planetary gear set.

|  | Sun gear | High-speed stage <br> Planet gear | Ring gear | Sun gear | Low-speed stage <br> Planet gear | Ring gear |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Number of teeth | 16 | 38 | 92 | 16 | 24 | 62 |
| Module (mm) |  | 3 |  |  | 3 |  |
| Pressure angle (deg) |  | $21.5^{\circ}$ |  |  | $21.5^{\circ}$ |  |
| Modification coefficient | 0.3 | 0.12 | 0.54 | 0.475 | 0.471 | 0.309 |
| Face width $(\mathrm{mm})$ | 20 | 20 | 34.5 | 30 | 32 |  |

Table 2: Physical parameters of the closed-form planetary gear set.

|  | High-speed stage |  |  |  | Low-speed stage |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | Sun | Planet | Ring | Carrier | Sun | Planet | Ring | Carrier |
| Inertia ( $\mathrm{kg} \cdot \mathrm{mm}^{2}$ ) | 88.54 | 827 | 68883.72 | 6800.79 | 339.59 | 462.92 | 110213.96 | 9400.9 |
| Rotation radius (mm) | 11.07 | 32.78 | 113 | 57.25 | 23.28 | 31.12 | 113 | 28 |
| Mean mesh stiffness ( $\mathrm{N} / \mathrm{mm}$ ) | $K_{s 1 n}=K_{r 1 n}=2 e^{7}$ |  |  |  | $K_{s 2 n}=K_{r 2 n}=2 e^{7}$ |  |  |  |
| Torsional stiffness ( $\mathrm{N} / \mathrm{mm}$ ) | $K_{c 1 s 2}=2 e^{7}$ |  |  |  | $K_{r 122}=10 e^{7}, K_{c 2}=10 e^{8}$ |  |  |  |
| Total mass | 69 kg |  |  |  |  |  |  |  |

By substituting (9)-(14) into (7a)-(7g) and balancing the like harmonic terms, the algebraic equations of system can be gained

$$
\begin{gather*}
\mathbf{K}_{m} \mathbf{y}_{m}+\frac{\mathbf{K}_{1} \mathbf{y}_{3}}{2}+\frac{\mathbf{K}_{2} \mathbf{y}_{4}}{2}-\mathbf{T}_{m}=\{0\}_{n \times 1}, \\
\mathbf{K}_{m} \mathbf{y}_{3}+\mathbf{K}_{1} \mathbf{y}_{m}-\Omega^{2} \mathbf{y}_{1}-\Omega \mathbf{C y}_{2}-\mathbf{T}_{1}=\{0\}_{n \times 1},  \tag{18}\\
\mathbf{K}_{m} \mathbf{y}_{4}+\mathbf{K}_{2} \mathbf{y}_{m}-\Omega^{2} \mathbf{y}_{2}+\mathbf{C} \Omega \mathbf{y}_{1}-\mathbf{T}_{2}=\{0\}_{n \times 1},
\end{gather*}
$$

where $\mathbf{y}_{1}=\left\{q_{a i} \cos \varphi_{i}\right\}_{n \times 1}, \mathbf{y}_{2}=\left\{q_{a i} \sin \varphi_{i}\right\}_{n \times 1}, \mathbf{y}_{3}=$ $\left\{N_{a i} q_{a i} \cos \varphi_{i}\right\}_{n \times 1}, \mathbf{y}_{4}=\left\{N_{a i} q_{a i} \sin \varphi_{i}\right\}_{n \times 1}, \mathbf{y}_{m}=\left\{N_{m i} q_{m i}\right\}_{n \times 1}$, $\mathbf{T}_{1}=\left\{T_{a i} \cos \phi_{i}\right\}_{n \times 1}, \mathbf{T}_{2}=\left\{T_{a i} \sin \phi_{i}\right\}_{n \times 1}, \mathbf{K}_{1}=$ $\left\{k_{a i j} \cos \phi_{i j}\right\}_{n \times 1}$, and $\mathbf{K}_{2}=\left\{k_{a i j} \sin \phi_{i j}\right\}_{n \times 1}$.

The matrices $\mathbf{k}_{m}$ and $\mathbf{C}$ are given in the Appendix.

## 4. Optimization of the Transmission System

4.1. Variables and Objective Function of Optimization. A number of key design parameters have great influence on the dynamic characteristics of the gear transmissions, so they can be chosen as design variables, such as number of each gear, module, gear width, and pressure angle. To simplify optimization process, the module of each stage $m_{1}$, $m_{2}$, number of sun gears of each stage $z_{s 1}, z_{s 2}$, and pressure angle of each stage $\alpha_{1}, \alpha_{2}$ are chosen as optimization variables in this study. So, the recurrence variables vector for this optimization procedure can be written as

$$
\begin{equation*}
x=\left[m_{1}, m_{2}, z_{s 1}, z_{s 2}, \alpha_{1}, \alpha_{2}\right]^{T} . \tag{19}
\end{equation*}
$$

The dynamic characteristics standards of the gear transmission include maximum dynamic loads, dynamic load factor, stiffness, and displacement/velocity/acceleration of vibration, each of these standards can be chosen as optimization objective. Considering that the rotation center of the planets in low-stage is unfixed, and reducing the
total weight of the transmission system simultaneously, the displacement of rotational vibration of the low-stage carrier and total mass of the gear set are chosen as optimization objective. According to the basic idea of the multiobjective optimization, two optimization objective functions unified using the normalized weighting method [11] can be written in form

$$
\begin{equation*}
f=\lambda_{1} f_{1}+\lambda_{2} f_{2} \tag{20}
\end{equation*}
$$

where $\lambda_{1}$ and $\lambda_{2}$ are weighting coefficient, here, $\lambda_{1}=0.3$ and $\lambda_{2}=0.7$ [12]. $f_{1}$ and $f_{2}$ are the ampler of the rotational vibration displacement of the low-stage carrier and total mass of the gear set, respectively. Due to limited space, the expressions of $f_{1}$ and $f_{2}$ are not given in detail.

### 4.2. Constraints

(1) Distributing the number of teeth of each gear: concentric conditions, adjacent conditions, and assembly conditions;
(2) contact fatigue strength and bending fatigue strength constraints

$$
\begin{array}{r}
S_{H}(i) \geq S_{H \min }, \quad S_{F}(i) \geq S_{F \min }  \tag{21}\\
i=s_{1}, s_{2}, r_{1}, r_{2}, p h, p l,
\end{array}
$$

(3) contact ratio constraints

$$
\begin{equation*}
\varepsilon_{i j} \geq 1.5, \quad i=s_{1}, s_{2}, r_{1}, r_{2}, \quad j=p h, p l, \tag{22}
\end{equation*}
$$

(4) transmission ratio without loop-power conditions

$$
\begin{gather*}
-32 \leq i_{1}+i_{2}-i_{1} i_{2} \leq-30, \\
0<\frac{i_{2}-i_{1} i_{2}}{i_{1}+i_{2}-i_{1} i_{2}}<1, \tag{23}
\end{gather*}
$$

Table 3: Parameters of structure after optimization.

|  | High-speed stage |  | Low-speed stage <br> Planet gear | Ring gear | Sun gear |
| :--- | :---: | :---: | :---: | :---: | :---: |



FIGURE 3: Response of low-speed stage carrier without optimization.


Figure 4: Response of low-speed stage carrier after optimization.
where

$$
\begin{equation*}
i_{n}=-\frac{Z_{r n}}{Z_{s n}}, \quad n=1,2 \tag{24}
\end{equation*}
$$

(5) minimum tooth thickness constraints

$$
\begin{equation*}
S_{a i} \geq 0.4 m_{j}, \quad i=s_{1}, s_{2}, r_{1}, r_{2}, p h, p l ; j=1,2 . \tag{25}
\end{equation*}
$$

## 5. Results and Discussions

Some more logical parameters of the closed-form planetary gear set are obtained according to optimization constraints conditions previously mentioned: $m_{1}=2, m_{2}=3, Z_{s 1}=17$, $Z_{s 2}=18, \alpha_{1}=\alpha_{2}=20^{\circ}$. Further, other parameters of transmission can be computed as shown in Table 3. Total
mass of gear set is 63.5 kg . Comparing the results between before and after the optimization, it is clear that transmission system reduces the total mass of $8.6 \%$ and the dynamic characteristic significantly improves as well. It is also evident from the comparison of Figure 3 with Figure 4 that the displacement of each structure significantly reduces at the same time. It is obvious that the scope of the amplitude of displacement of low-stage carrier is -3 to 3 in Figure 4(a), while that in Figure 3(a) is -5 to 3.

## 6. Conclusions

Considering the gear backlash, time-varying mesh stiffness and excitation fluctuation, and so forth, a discrete nonlinear dynamic model of a two stages closed-form planetary set was proposed in this study. In order to facilitate the analysis and comparison of system response between before and after optimization, nonlinear differential equations of motion of the dynamic model were solved using a Runge-Kutta numerical integration method. For optimization of transmission system, the analytical solutions were obtained. The total optimization objective function is obtained using the normalized weights method.

A case shows that the total mass of transmission system has significantly reduced and dynamic characteristic has distinctly improved. Effectiveness of the dynamic model and the dynamic optimization mathematic model is demonstrated.

## Appendix

Consider the following:

$$
\begin{align*}
& G(\mu)= \begin{cases}\left(\frac{2}{\pi}\right)\left(\mu \sin ^{-1} \mu+\sqrt{1-\mu^{2}}\right), & |\mu| \leq 1 \\
|\mu|, & |\mu|>1\end{cases} \\
& H(\mu)= \begin{cases}-1, & \mu<-1 \\
\left(\frac{2 \mu}{\pi}\right)\left(\sin ^{-1} \mu+\sqrt{1-\mu^{2}}\right), & |\mu| \leq 1 \\
1, & \mu>1\end{cases} \tag{A.1}
\end{align*}
$$

The elements of the stiffness matrix $\mathbf{k}_{m}$ are given as follows

$$
\begin{gathered}
k(1,1)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{s 1}}+\frac{1}{M_{c 1}}+\frac{1}{M_{p h}}\right) \\
k(1, i)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{s 1}}+\frac{1}{M_{c 1}}\right), \quad i=2,3 \\
k(1,4)=\frac{k_{r 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 1}}-\frac{1}{M_{p h}}\right) \\
k(1, i)=\frac{k_{r 1 n}}{M_{c 1} \omega_{n}^{2}}, \quad i=5,6 \\
k(1,7)=-\frac{k_{c 1 s 2}}{M_{c 1} \omega_{n}^{2}}
\end{gathered}
$$

$$
\begin{aligned}
& k(2, i)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{s 1}}+\frac{1}{M_{c 1}}\right), \quad i=1,3, \\
& k(2,7)=-\frac{k_{c 1 s 2}}{M_{c 1} \omega_{n}^{2}}, \\
& k(2,2)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{s 1}}+\frac{1}{M_{c 1}}+\frac{1}{M_{p h}}\right), \\
& k(2, i)=\frac{k_{r 1 n}}{M_{c 1} \omega_{n}^{2}}, \quad i=4,6, \\
& k(2,5)=\frac{k_{r 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 1}}-\frac{1}{M_{p h}}\right), \\
& k(2, i)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{s 1}}+\frac{1}{M_{c 1}}\right), \quad i=1,3, \\
& k(2,2)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{s 1}}+\frac{1}{M_{c 1}}+\frac{1}{M_{p h}}\right), \\
& k(2, i)=\frac{k_{r 1 n}}{M_{c 1} \omega_{n}^{2}}, \quad i=4,6, \\
& k(2,5)=\frac{k_{r 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 1}}-\frac{1}{M_{p h}}\right), \\
& k(2,7)=-\frac{k_{c 1 s 2}}{M_{c 1} \omega_{n}^{2}}, \\
& k(3, i)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{s 1}}+\frac{1}{M_{c 1}}\right), \quad i=1,2, \\
& k(3,3)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{s 1}}+\frac{1}{M_{c 1}}+\frac{1}{M_{p h}}\right), \\
& k(3, i)=\frac{k_{r 1 n}}{M_{c 1} \omega_{n}^{2}}, \quad i=4,5, \\
& k(3,6)=\frac{k_{r 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 1}}-\frac{1}{M_{p h}}\right), \\
& k(3,7)=-\frac{k_{c 1 s 2}}{M_{c 1} \omega_{n}^{2}}, \\
& k(4,1)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 1}}-\frac{1}{M_{p h}}\right), \\
& k(4, i)=\frac{k_{s 1 n}}{M_{c 1} \omega_{n}^{2}}, \quad i=2,3, \\
& k(4,7)=-\frac{k_{c 1 s 2}}{M_{c 1} \omega_{n}^{2}}, \\
& k(4,4)=\frac{k_{r 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{p h}}+\frac{1}{M_{c 1}}+\frac{1}{M_{r 1}}\right),
\end{aligned}
$$

$$
\begin{aligned}
& k(4, i)=\frac{k_{r 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 1}}+\frac{1}{M_{r 1}}\right), \quad i=5,6, \\
& k(4,15)=-\frac{k_{r 1 r 2}}{M_{r 1} \omega_{n}^{2}}, \\
& k(5, i)=\frac{k_{s 1 n}}{M_{c 1} \omega_{n}^{2}}, \quad i=1,3, \\
& k(5,2)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 1}}-\frac{1}{M_{p h}}\right), \\
& k(5, i)=\frac{k_{r 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 1}}+\frac{1}{M_{r 1}}\right), \quad i=5,6, \\
& k(5,5)=\frac{k_{r 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{p h}}+\frac{1}{M_{c 1}}+\frac{1}{M_{r 1}}\right), \\
& k(5,7)=-\frac{k_{c 1 s 2}}{M_{c 1} \omega_{n}^{2}}, \\
& k(5,15)=-\frac{k_{r 1 r 2}}{M_{r 1} \omega_{n}^{2}}, \\
& k(6, i)=\frac{k_{s 1 n}}{M_{c 1} \omega_{n}^{2}}, \quad i=1,2, \\
& k(6,3)=\frac{k_{s 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 1}}-\frac{1}{M_{p h}}\right), \\
& k(6, i)=\frac{k_{r 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 1}}+\frac{1}{M_{r 1}}\right), \quad i=4,5, \\
& k(6,6)=\frac{k_{r 1 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{p h}}+\frac{1}{M_{c 1}}+\frac{1}{M_{r 1}}\right), \\
& k(6,7)=-\frac{k_{c 1 s 2}}{M_{c 1} \omega_{n}^{2}}, \\
& k(6,15)=-\frac{k_{r 1 r 2}}{M_{r 1} \omega_{n}^{2}}, \\
& k(7, i)=-\frac{k_{s 1 n}}{M_{c 1} \omega_{n}^{2}}, \quad i=1,2,3, \\
& k(7, i)=-\frac{k_{r 1 n}}{M_{c 1} \omega_{n}^{2}}, \quad i=4,5,6, \\
& k(7,7)=\frac{k_{c 1 s 2}}{\omega_{n}^{2}}\left[\frac{1}{M_{c 1}}+\frac{1}{M_{s 2}}\left(\frac{r_{c 1} \cos \alpha}{r_{s 2}}\right)^{2}\right], \\
& k(7, i)=-\frac{k_{s 2 n}}{M_{s 2} \omega_{n}^{2}} \frac{r_{c 1} \cos \alpha}{r_{s 2}}, \quad i=9,10,11, \\
& k(8,8)=\frac{k_{c 2}}{M_{c 2} \omega_{n}^{2}}, \\
& k(8, i)=-\frac{k_{s 2 n}}{M_{c 2} \omega_{n}^{2}}, \quad i=9,10,11,
\end{aligned}
$$

$$
\begin{align*}
& k(12,15)=\frac{k_{r 1 r 2} r_{r 1}}{M_{r 2} \omega_{n}^{2} r_{r 2}}, \\
& k(12,9)=\frac{k_{s 2 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 2}}-\frac{1}{M_{p l}}\right) \text {, } \\
& k(12, i)=\frac{k_{s 2 n}}{M_{c 2} \omega_{n}^{2}}, \quad i=10,11, \\
& k(12,12)=\frac{k_{r 2 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{p l}}+\frac{1}{M_{c 2}}\right), \\
& k(12, i)=\frac{k_{r 2 n}}{M_{c 2} \omega_{n}^{2}}, \quad i=13,14, \\
& k(13,8)=-\frac{k_{c 2}}{M_{c 2} \omega_{n}^{2}}, \\
& k(13,10)=\frac{k_{s 2 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 2}}-\frac{1}{M_{p l}}\right), \\
& k(13, i)=\frac{k_{s 2 n}}{M_{c 2} \omega_{n}^{2}}, \quad i=9,11, \\
& k(13,13)=\frac{k_{r 2 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{p l}}+\frac{1}{M_{c 2}}\right), \\
& k(13, i)=\frac{k_{r 2 n}}{M_{c 2} \omega_{n}^{2}}, \quad i=12,14, \\
& k(13,15)=\frac{k_{r 1{ }_{2}} r_{r 1}}{M_{r 2} \omega_{n}^{2} r_{r 2}}, \\
& k(14,8)=-\frac{k_{c 2}}{M_{c 2} \omega_{n}^{2}}, \\
& k(14,11)=\frac{k_{s 2 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{c 2}}-\frac{1}{M_{p l}}\right), \\
& k(14, i)=\frac{k_{s 2 n}}{M_{c 2} \omega_{n}^{2}}, \quad i=9,10, \\
& k(14,14)=\frac{k_{r 2 n}}{\omega_{n}^{2}}\left(\frac{1}{M_{p l}}+\frac{1}{M_{c 2}}\right), \\
& k(14, i)=\frac{k_{r 2 n}}{M_{c 2} \omega_{n}^{2}}, \quad i=12,13, \\
& k(14,15)=\frac{k_{r 1 r 2} r_{r 1}}{M_{r 2} \omega_{n}^{2} r_{r 2}}, \\
& k(15, i)=-\frac{k_{r 1 n}}{M_{r 1} \omega_{n}^{2}}, \quad i=4,5,6, \\
& k(15, i)=\frac{k_{r 2 n} r_{r 1}}{M_{r 2} r_{r 2} \omega_{n}^{2}}, \quad i=12,13,14, \\
& k(15,15)=\frac{k_{r 1 r 2}}{\omega_{n}^{2}}\left(\frac{1}{M_{r 1}}+\frac{r_{r 1}}{M_{r 2} r_{r 2}}\right) . \tag{A.2}
\end{align*}
$$

The elements of $\mathbf{k}_{m}$ not listed previously are taken as zeros. The matrix $\mathbf{C}$ is in the similar form of $\mathbf{k}_{m}$ and no longer is listed in detail here.

## Notations

HBM: Harmonic balance method
b: Half of clearance (backlash)
$k$ : Gear mesh stiffness
$r$ : Ring gear
$g: \quad$ Discontinuous displacement function
I: Polar mass moment of inertia
$i$ : Transmission ratio
c: Carrier
$m$ : Actual mass
$\bar{t}$ : $\quad$ Dimensional time
$u$ : Actual equivalent transverse displacement in the direction of pressure line
$U: \quad$ Relative equivalent transverse displacement in the direction of pressure line
$s$ : $\quad$ Sun gear
$n_{p}$ : Number of planet
T: Torque
$\theta$ : Angular displacement
C: Damping coefficient
$Z$ : Number of gear
$r$ : Base cycle radius
M: Equivalent mass
$t$ : Nondimensional time
$\alpha$ : Pressure angle.

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## References

[1] F. Cunliffe, J. D. Smith, and D. B. Welbourn, "Dynamic tooth loads in epicyclic gears," Transactions of the ASME Journal of Engineering for Industry, vol. 96, no. 2, pp. 578-584, 1974.
[2] M. Botman, "Epicyclic gear vibrations," Transactions of the ASME Journal of Engineering for Industry, vol. 98, no. 3, pp. 811815, 1976.
[3] G. Antony, "Gear vibration-investigation of the dynamic behavior of one stage epicyclic gears," AGMA Technical Paper 88-FTM-12, 1988.
[4] A. Kahraman, "Free torsional vibration characteristics of compound planetary gear sets," Mechanism and Machine Theory, vol. 36, no. 8, pp. 953-971, 2001.
[5] T. Sun and H. Hu, "Nonlinear dynamics of a planetary gear system with multiple clearances," Mechanism and Machine Theory, vol. 38, no. 12, pp. 1371-1390, 2003.
[6] A. Al-shyyab and A. Kahraman, "A non-linear dynamic model for planetary gear sets," Proceedings of the Institution of Mechanical Engineers K: Journal of Multi-body Dynamics, vol. 221, no. 4, pp. 567-576, 2007.
[7] A. Al-Shyyab, K. Alwidyan, A. Jawarneh, and H. Tlilan, "Nonlinear dynamic behaviour of compound planetary gear trains: model formulation and semi-analytical solution," Proceedings of the Institution of Mechanical Engineers K: Journal of Multi-body Dynamics, vol. 223, no. 3, pp. 199-210, 2009.
[8] T. Zeng Bao, Z. Yi Fang, and L. Wei Zhong, "Many stage helical gear train system optimal design which takes the dynamic properties as an objective function," Chinese Journal of Mechanical Engineering, vol. 30, no. 5, pp. 66-75, 1994 (Chinese).
[9] X. Guan Wei, Y. Jia Jun, X. LaiYuan et al., "Dynamic optimal design on planetary driver," Mechanical Science and Technology, vol. 17, no. 2, pp. 209-217, 1998 (Chinese).
[10] G. W. Blankenship and A. Kahraman, "Steady state forced response of a mechanical oscillator with combined parametric excitation and clearance type non-linearity," Journal of Sound and Vibration, vol. 185, no. 5, pp. 743-765, 1995.
[11] H. Guang and W. Gui Cheng, "Dynamics optimization design of closed epicyclic gear trains," Mechanical Science and Technology, vol. 23, no. 9, pp. 1131-1134, 2004 (Chinese).
[12] D.-T. Qin, X.-G. Gu, J.-H. Wang, and J.-G. Liu, "Dynamic analysis and optimization of gear trains in a megawatt level wind turbine," Journal of Chongqing University, vol. 32, no. 4, pp. 408414, 2009 (Chinese).


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