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Research Article

Lamb Modes for an Isotropic Incompressible Plate

Takasar Hussain, Muhammad Afzal Awan, Moniba Shams, and Faiz Ahmad

School of Natural Sciences, National University of Sciences and Technology, Sector H-12, Islamabad 44000, Pakistan

Correspondence should be addressed to Takasar Hussain; htakasar@yahoo.com

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Lamb modes for an incompressible isotropic plate behave in a manner different from those for a compressible plate. The plateau region disappears and anomalous behavior of modes does not exist.

1. Introduction

For an infinite isotropic plate of thickness $2h$, the dispersion relation for symmetric Lamb modes is given by [1]

$$\frac{\tan(qh)}{\tan(ph)} = -\frac{4k^2pq}{(q^2-k^2)^2}, \tag{1}$$

where

$$p = \sqrt{\frac{\omega^2}{c_L^2} - k^2}, \qquad q = \sqrt{\frac{\omega^2}{c_T^2} - k^2}. \tag{2}$$

In (1), c_T and c_L, respectively, denote the phase speeds of the transverse and longitudinal bulk waves in the material. Also ω and k, respectively, denote the frequency and the wave number of the mode. The phase velocity, c, of a mode is given by

$$c = \frac{\omega}{k}. \tag{3}$$

Differentiation with respect to ω gives

$$\frac{dc}{d\omega} = \frac{1}{k} - \frac{\omega}{k^2}\frac{dk}{d\omega}, \qquad \frac{dc}{d\omega} = \frac{1}{k}\left(1 - \frac{c}{c_g}\right), \tag{4}$$

where $c_g = d\omega/dk$ denotes the group velocity.

When the dispersion curves corresponding to (1) are plotted, depicting the normalized phase velocity as a function of dimensionless wave number hk, the phase speeds of all modes except the lowest S_0 mode, asymptotically approaches c_T as the frequency ω tends to infinity, whereas that of the S_0 approaches c_R, the speed of the Rayleigh wave in the material. Near their crossing points of the line $c = c_L$, the curves flatten out before embarking once again on the downward journey. This flat region is called the plateau region [2] as shown in Figure 1.

If the normalized phase velocity is plotted as a function of frequency, by using (1), the S_1 mode in most materials, exhibits a region in which phase velocity is directed opposite to the group velocity [2]. A mode possessing such a region is called an anomalous mode.

Tolstoy and Üsdin [3] were the first to observe this phenomenon for S_1 mode for an isotropic material with $\nu = 1/4$. Negishi [4] observed this phenomenon in aluminum. Prada et al. [5] observed the anomalous behavior of Lamb modes in $\omega - k$ plane for Duralumin plate. For an isotropic material with Poisson's ratio $\nu = 1/3$, each of the mode S_{3n+1}, $n = 0, 1, 2, \ldots$ is found to be anomalous. Theoretical explanation of this peculiar behavior of the anomalous modes has been presented in Werby and Überall [2]. Shuvalov and Poncelet [6] have investigated the dispersion relation of a plate of unrestricted anisotropy. They identify anomalous modes by looking at the sign of the coefficient of $\omega_n(k) - \omega_n(0)$ for small values of the wave number k. Recently Hussain and Ahmad [7] considered zero-group velocity (ZGV) points in the spectrum of Lamb modes in compressible orthotropic plate. It was found that, in addition to modes with a single

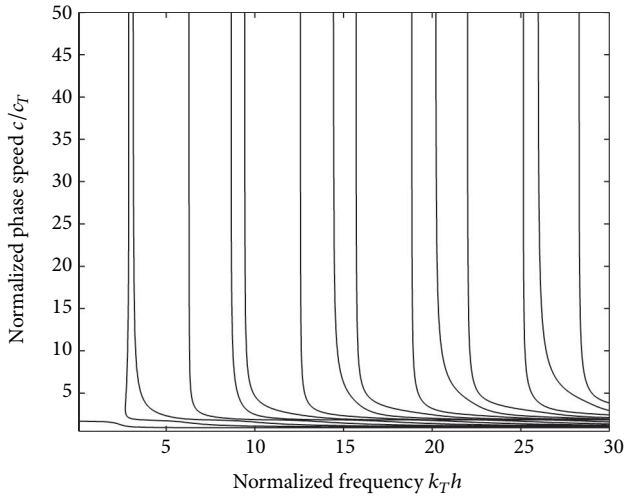

Figure 1: Spectrum of Lamb modes for a steel plate having plateau region.

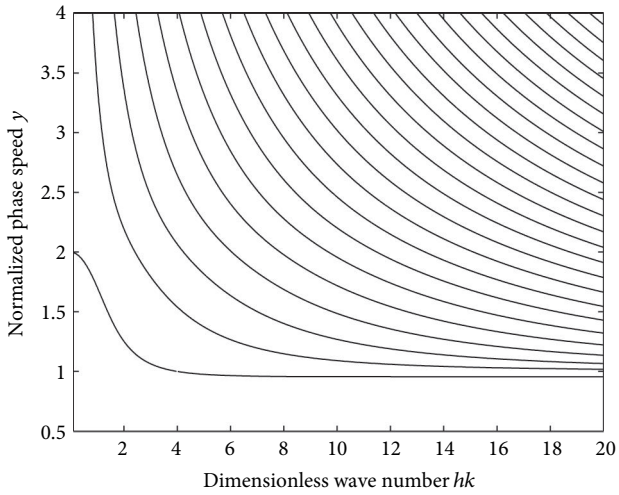

Figure 2: Symmetric Lamb modes for an incompressible isotropic plate in $k - c$ plane.

ZGV point, a large number of modes exist with multiple points.

In this paper we have studied the Lamb modes for an incompressible isotropic material for anomalous behavior of the modes. In [8, 9], details of governing equations for incompressible solids have been given. A cartesian coordinate system, $x_1x_2x_3$, is chosen in such a way that x_2-axis is normal to the plate surface. With the (x_1, x_2) as plane of motion, the displacement components (u_1, u_2, u_3) are such that

$$u_i = u_i(x_1, x_2, t), \quad i = 1, 2, \quad u_3 \equiv 0, \tag{5}$$

and the incompressibility condition, $\nabla \cdot \mathbf{u} = 0$, reduces to

$$u_{1,1} + u_{2,2} = 0. \tag{6}$$

The constitutive relation for an incompressible elastic material is given by

$$\sigma_{ij} = -p\delta_{ij} + c_{ijkl}\epsilon_{kl}, \tag{7}$$

where σ_{ij} is the stress tensor, p is an arbitrary hydrostatic pressure associated with the incompressibility constraint, c_{ijkl} are the elastic constants, and ϵ_{kl} is the strain tensor. Traction free boundary conditions lead to the following dispersion relation for symmetric modes:

$$\frac{\tan(s_1kh)}{\tan(s_2kh)} = \frac{s_1\left(1 - s_2^2\right)^2}{s_2\left(1 - s_1^2\right)^2}, \tag{8}$$

[8], where s_1^2 and s_2^2 are roots of the following quadratic equation:

$$s^4 + \left(2 - \frac{\rho c^2}{\mu}\right)s^2 + \left(1 - \frac{\rho c^2}{\mu}\right) = 0, \tag{9}$$

hence, $s_1^2 = -1$, $s_2^2 = \rho c^2/\mu - 1$, so (8) becomes

$$\frac{\tan\left(\sqrt{y^2 - 1}x\right)}{\tanh(x)} = \frac{4\sqrt{y^2 - 1}}{\left(2 - y^2\right)^2}, \tag{10}$$

where we have defined the dimensionless wave number hk by x and normalized velocity (c/c_T) $(c_T = \sqrt{\mu/\rho})$ by y.

The dispersion curves are plotted using a recent technique of Honarvar et al. [10]. It is found that there exists no mode with anomalous behavior. This result is derived analytically.

Rogerson [8] and Ogden and Roxburgh [9] discussed the dispersion relation asymptotically. The main result of the present work is to keep focus on the fact that incompressibility constraint in an isotropic material suppresses the ZGV phenomenon which exists in S_1 and several other modes of a compressible material [2]. It appears that absence of ZGV Lamb modes in an incompressible isotropic plate, at least for materials with $\nu = 0$ or 1, was known in Russia in the 1980s [11]. However, to the best of our knowledge, there does not exist any analytical proof of this fact in the literature.

2. Dispersion Curves

Shape of the dispersion curves, in $k - c$ plane, for a compressible plate, was discussed by Ahmad [12]. The plateau region occurs because, as can be easily shown, the slope of the nth mode, when $y = \kappa$, is given by

$$\left.\frac{dy}{dx}\right|_{y=\kappa} = -\frac{\left(\kappa^2 - 1\right)^{3/2}\left(\kappa^2 - 2\right)^2}{n\kappa^4\pi}. \tag{11}$$

It is clear that for large n, the slope, while remaining negative approaches zero.

The dispersion curves for an incompressible isotropic plate, in the $k - c$ plane, are shown in Figure 2.

All curves, except the lowest S_0 mode, approach the line $y = 1$ corresponding to $c = c_T$. Also the phase speed of S_0 mode approaches the speed of the Rayleigh wave which is the unique root of the equation as follow:

$$4\sqrt{1 - \frac{c^2}{c_T^2}} = \left(2 - \frac{c^2}{c_T^2}\right)^2. \tag{12}$$

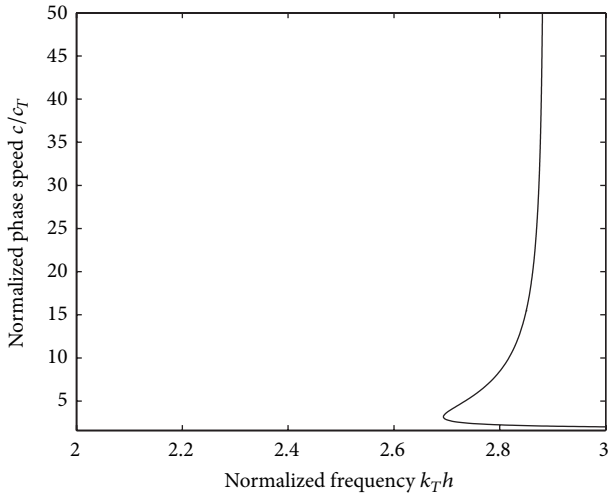

FIGURE 3: The S_1 Lamb mode for a steel plate in $\omega - c$ plane.

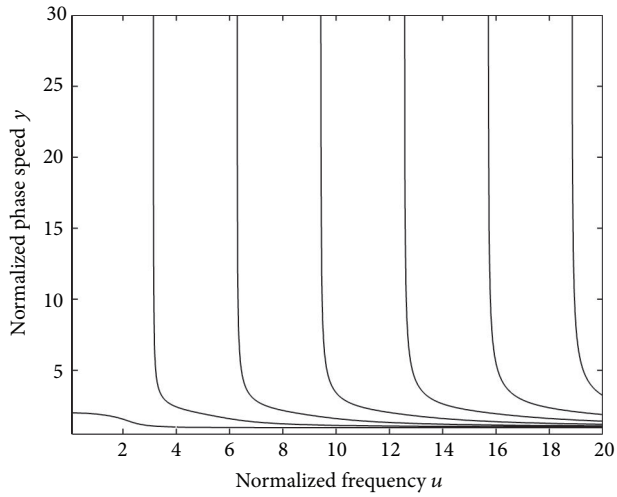

FIGURE 4: Symmetric Lamb modes for an incompressible isotropic plate in $u - y$ plane.

Explanation of the above features follows on the same lines as for a compressible plate [12].

Since (10) does not depend on c_L, so $y = \kappa$ has no significance, and there is no plateau region for curves in Figure 2. The equation does not involve any material parameter; hence, Figure 2 represents the dispersion curves for all incompressible isotropic materials.

It is well known [2–6] that the Lamb modes of a compressible isotropic plate in $\omega - c$ plane exhibit a phenomenon known as "anomalous dispersion." The S_1 mode in steel plate, as in most other materials, undergoes a turning point when $k_T = 2.686$, and group velocity is negative in the interval $[2.686, 2.873]$. This is highlighted in Figure 3.

A question naturally arises whether the anomaly persists even when the incompressibility constraint is enforced. To verify this define

$$u = \frac{\omega h}{c_T} = \frac{ckh}{c_T} = yx. \tag{13}$$

Equation (10) in terms of u and y becomes

$$\frac{\tan\left(\sqrt{y^2-1}u/y\right)}{\tanh\left(u/y\right)} = \frac{4\sqrt{y^2-1}}{\left(2-y^2\right)^2}. \tag{14}$$

Dispersion curves in the $u - y$ plane, which is the same as $\omega - c$ plane are shown in Figure 4.

The anomalous behavior of the S_1 mode disappears. Analytically this can be seen by examining slope of the modes given by (14) first when $y \gg 1$, secondly when $y \to 1^+$.

We rewrite (14) in the form

$$\begin{aligned} f(u,y) &= \tan\left(\sqrt{y^2-1}u/y\right)\left(2-y^2\right)^2 \\ &\quad - \tanh\left(\frac{u}{y}\right) 4\sqrt{y^2-1} = 0. \end{aligned} \tag{15}$$

For $y \gg 1$, (14) becomes

$$4\tanh\left(\frac{u}{y}\right) \simeq \tan(u)\, y^3. \tag{16}$$

Since for large y

$$\tanh\left(\frac{u}{y}\right) \simeq 0, \tag{17}$$

hence, (16) leads to

$$\tan(u) \simeq 0, \tag{18}$$

or

$$u_n = n\pi + \epsilon, \quad n = 0, 1, 2, 3, \ldots, \tag{19}$$

where ϵ is an infinitesimally small positive number.

Now we will calculate dy/du by the following formula:

$$\frac{dy}{du} = -\frac{\partial f/\partial u}{\partial f/\partial y}. \tag{20}$$

We will show that $dy/du < 0$ for large value of y for all modes.

For large value of y, the partial derivatives can be approximated as follows:

$$\begin{aligned} \frac{\partial f}{\partial u} &\simeq \left(4 - y^4\right)\cos u, \\ \frac{\partial f}{\partial y} &\simeq -\left(uy\cos u + 4y^3 \sin u\right). \end{aligned} \tag{21}$$

Then

$$\left.\frac{dy}{du}\right|_{u=u_n} \simeq -\frac{y^3}{u_n + 4y^2 \tan u_n}. \tag{22}$$

Since $u_n + 4y^2 \tan u_n > 0$, therefore

$$\frac{dy}{du} < 0 \tag{23}$$

for all modes.

Now let

$$y^2 = 1 + \epsilon^2. \tag{24}$$

Dispersion relation (14) can be approximated in the following way:

$$\frac{\tan(\epsilon u)}{\tanh u} \simeq \frac{4\epsilon}{\left(1-\epsilon^2\right)^2} \simeq 4\epsilon\left(1+2\epsilon^2\right) \simeq 4\epsilon, \qquad \tan(\epsilon u) \simeq 4\epsilon \tanh u. \tag{25}$$

The derivative dy/du can be easily shown to be, for large u_n, as

$$\frac{dy}{du} \simeq \frac{\epsilon^2}{-u_n + 4}. \tag{26}$$

Thus, the modes S_n $(n \neq 0)$ start off with negative slopes and the slope retains its sign till the end. Therefore, no anomalous behavior is expected.

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References

[1] J. D. Achenbach, *Wave Propagation in Elastic Solids*, chapter 6, North-Holland, Amsterdam, The Netherlands, 1990.

[2] M. F. Werby and H. Überall, "The analysis and interpretation of some special properties of higher order symmetric Lamb waves: the case for plates," *Journal of the Acoustical Society of America*, vol. 111, no. 6, pp. 2686–2691, 2002.

[3] I. Tolstoy and E. Usdin, "Wave propagation in elastic plates: low and high mode dispersion," *Journal of the Acoustical Society of America*, vol. 29, pp. 37–42, 1957.

[4] K. Negishi, "Negative group velocities of Lamb waves," *Journal of the Acoustical Society of America*, vol. 64, supplement 1, p. S63, 1978.

[5] C. Prada, D. Clorennec, and D. Royer, "Local vibration of an elastic plate and zero-group velocity Lamb modes," *Journal of the Acoustical Society of America*, vol. 124, no. 1, pp. 203–212, 2008.

[6] A. L. Shuvalov and O. Poncelet, "On the backward Lamb waves near thickness resonances in anisotropic plates," *International Journal of Solids and Structures*, vol. 45, no. 11-12, pp. 3430–3448, 2008.

[7] T. Hussain and F. Ahmad, "Lamb modes with multiple zero-group velocity points in an orthotropic plate," *Journal of the Acoustical Society of America*, vol. 32, pp. 641–645, 2012.

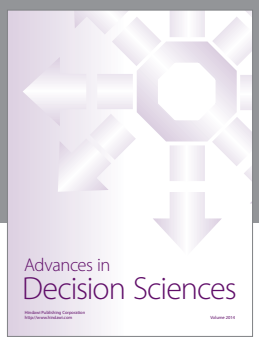
[8] G. A. Rogerson, "Some asymptotic expansions of the dispersion relation for an incompressible elastic plate," *International Journal of Solids and Structures*, vol. 34, no. 22, pp. 2785–2802, 1997.

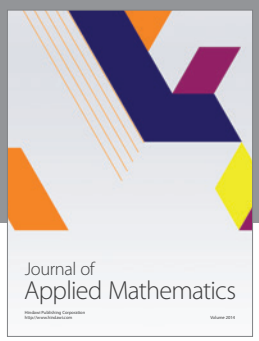
[9] R. W. Ogden and D. G. Roxburgh, "The effect of pre-stress on the vibration and stability of elastic plates," *International Journal of Engineering Science*, vol. 31, no. 12, pp. 1611–1639, 1993.

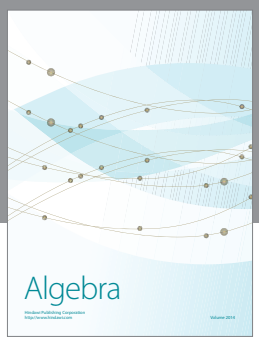
[10] F. Honarvar, E. Enjilela, and A. N. Sinclair, "An alternative method for plotting dispersion curves," *Ultrasonics*, vol. 49, no. 1, pp. 15–18, 2009.

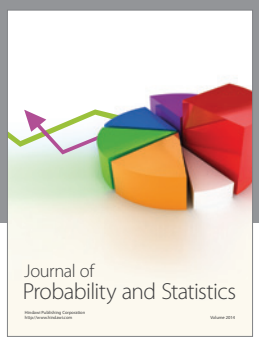
[11] E. Glushkov, private communication.

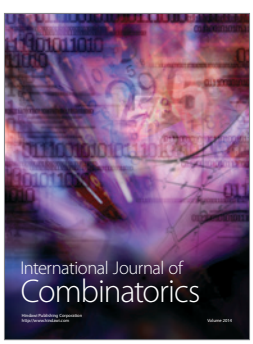
[12] F. Ahmad, "Shape of dispersion curves in the Rayleigh-Lamb spectrum," *Archives of Mechanics*, vol. 56, no. 2, pp. 157–165, 2004.

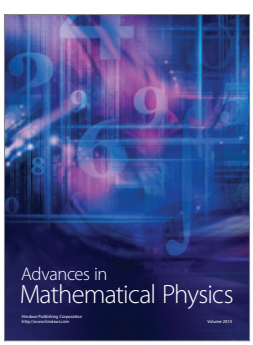


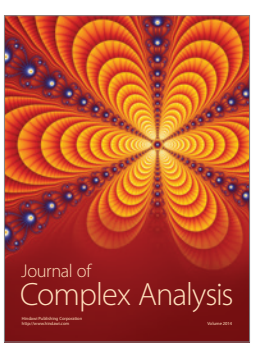


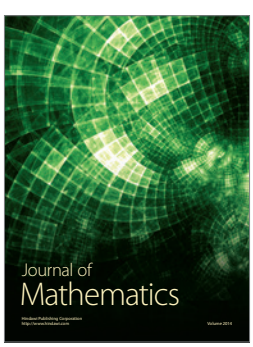


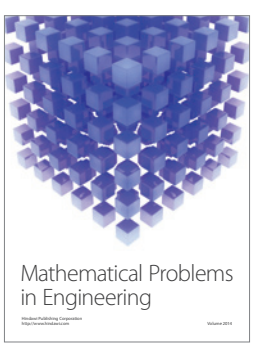


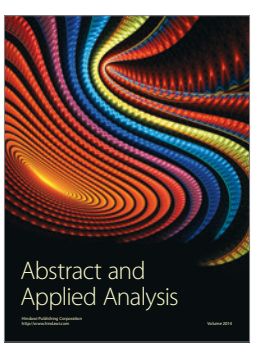


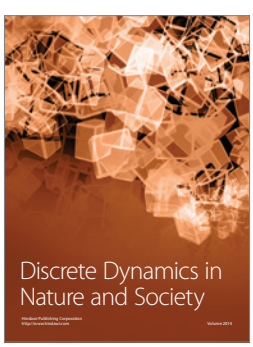


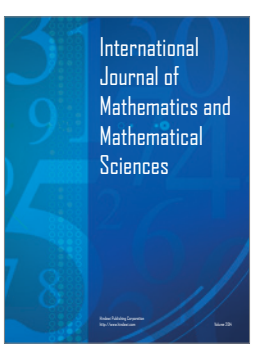


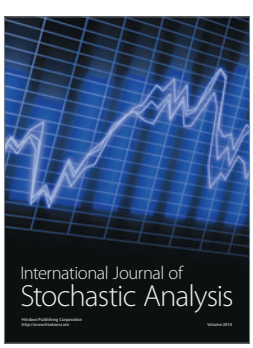












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