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Research Article

State Consensus Analysis and Design for High-Order Discrete-Time Linear Multiagent Systems

Yanrong Ge,[1,2] Yangzhou Chen,[1] Yaxiao Zhang,[1] and Zhonghe He[1]

[1] *College of Electronic Information and Control Engineering, Beijing University of Technology, Beijing 100124, China*
[2] *College of Physics Science and Information Engineering, Hebei Normal University, Shijiazhuang 050024, China*

Correspondence should be addressed to Yangzhou Chen; yzchen@bjut.edu.cn

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The paper deals with the state consensus problem of high-order discrete-time linear multiagent systems (DLMASs) with fixed information topologies. We consider three aspects of the consensus analysis and design problem: (1) the convergence criteria of global state consensus, (2) the calculation of the state consensus function, and (3) the determination of the weighted matrix and the feedback gain matrix in the consensus protocol. We solve the consensus problem by proposing a linear transformation to translate it into a partial stability problem. Based on the approach, we obtain necessary and sufficient criteria in terms of Schur stability of matrices and present an analytical expression of the state consensus function. We also propose a design process to determine the feedback gain matrix in the consensus protocol. Finally, we extend the state consensus to the formation control. The results are explained by several numerical examples.

1. Introduction

In recent years, the consensus problem of multiagent systems (MASs) has been becoming a significant research topic because of its broad practical applications, including the work load balance in a network of parallel computers [1], the clock synchronization [2], distributed decision [3], consensus filtering and estimation in sensor networks [4–6], rendezvous, and the formation of various moving objects [7–11] such as underwater vehicles, aircrafts, satellites, mobile robots, and intelligent vehicles in automated highway systems, to name only a few. Hence, its study has captured attention of the researchers from different disciplines.

MASs are comprised of locally interacting agents equipped with dedicated sensing, computing, and communication devices. The consensus problem of MASs is to design a distributed control law for each agent, using only information from itself and its neighbors, such that all agents achieve an agreement on some quantities of interest. To design and analyse this class of systems, one needs to consider three essential elements: (1) a dynamic model describing the states of the agents, which can be either continuous time or discrete time, linear or nonlinear, homogeneous or heterogeneous, time varying or time invariant, low order or high order; (2) an information topology describing communication network between the agents, which can be either undirected or directed, fixed or switched; (3) a protocol (control input) for each of the agents describing how the agents interact on each other according to the given information topology, which can be synchronous or asynchronous, with or without time delay.

Up to now, numerous researches have been done for continuous-time MASs in different settings from the above cases [10, 12–18]. This paper focuses on the study of high-order discrete-time linear multiagent systems (DLMASs) by proposing a linear transformation to translate the consensus problem into a partial stability problem. Although this approach can be extended to any setting from the above cases, we pay our attention only to the case of fixed information topology and in the absence of time delay for giving prominence to the trait of the approach. Here we give an overview mainly to the DLMASs.

Reference [19] first proposed an interesting model for self-propelled particle systems, where all agents move in a plane with the same speed but different headings, and showed that in the model all agents might eventually move in the same direction despite the absence of centralized coordination. Reference [20] further gave a mathematically rigorous qualitative analysis. Then, a theoretical explanation was given for the consensus behavior of Vicsek model on the basis of graph theory in [21, 22]. A necessary and sufficient condition was given for the average consensus criterion in [23]. Reference [24] further considered the case of switching network topologies for the average consensus. The average consensus was investigated for the systems with uncertain communication environments and time-varying topologies in [25] and with communication constraints in [26]. Reference [27] presented convergence results for the time-varying protocol in the absence or presence of communication delays. Reference [28] proposed an asynchronous time-varying consensus protocol. Reference [29] further discussed nonlinear systems with time-dependent communication links. References [30, 31] addressed the case with both time-varying delays and switching information topologies and provided a class of effective consensus protocols by repeatedly using the same state information at two time steps.

The researches mentioned above were limited to first-order systems. The extension to second-order systems was done for the systems with time-varying delays and time-varying interaction topology in [32], and for the systems with nonuniform time-delays and dynamically changing topologies in [33].

Recent researches were turned to the high-order DLMASs in [34–39]. Reference [34] studied a class of dynamic average consensus algorithms that allow a group of agents to track the average of their reference inputs. Reference [35] proposed an observer-type protocol based on the relative outputs of neighboring agents. Reference [36] studied the convergence speed for the high-order systems with random networks and arbitrary weights. Reference [37] addressed the high-order systems with or without delays. These researches were focused on the consensus convergence criteria for the proposed protocols. Another significant topic is the design of the gain matrices of the protocols in [38, 39].

This paper deals with both analysis and design problems of the state consensus for general high-order DLMASs. Compared with the existing works, the contributions of the paper are summarized as follows. Firstly, motivated by [12], we improve the protocol by adding a self-feedback of the agent to achieve the expected consensus dynamics, whereas [13] introduced the internal model to change the given dynamic to achieve the expected consensus dynamics and [14, 15] introduced the virtual leader to guide the multiagent systems to achieve the expected consensus dynamics. Secondly, we propose a state linear transformation to translate the consensus problem into a partial stability problem. The approach is motivated by the error variable method or the state space decomposition method in [12, 16]. However, our improvement can more spontaneously and conveniently deal with various settings of the consensus problems. Based on the partial stability theory, we educe new necessary and sufficient consensus convergence criteria in terms of stability of matrices and moreover give an explicit analytical expression of the state consensus function based on the different contributions of the initial states of the agents and the protocols. Thirdly, based on stability theorem, we give a design procedure to determine the gain matrices in the protocol on the basis of algebraic Riccati inequality similarly to [38, 39]. Fourthly, we extend the state consensus results to the formation control problem.

The remainder of the paper is organized as follows. Section 2 introduces some basic concepts and notations, and formulates the problem under investigation. Section 3 firstly introduces a linear transformation which translates the consensus problem of the multiagent systems into a partial stability problem of the corresponding transformed system, and then educes a new necessary and sufficient condition for the multiagent system to achieve global state consensus and presents an analytical expression of the state consensus function. Section 4 shows a design procedure to determine the gain matrices in the state consensus protocol. Section 5 extends the approach for the analysis and design of the state consensus to the formation control problem. Section 6 gives numerical examples to explain the theoretical results. Section 7 concludes the paper. All the proofs of the results are deposited in the appendix for the sake of reading.

2. Problem Description

Before stating the consensus problem, we give some basic concepts and notations. Let $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$ be the sets of $n \times n$ real matrices and complex matrices, respectively. Matrices, if not explicitly stated, have appropriate dimensions in all settings. The superscript "T" means transpose for real matrices, and the superscript "H" means conjugate transpose for complex matrices. I_n presents the identity matrix of dimension n, and sometimes I is used for simplicity. $\mathbf{1}_N$ denotes the vector of dimension N with all entries equal to one. 0 is applied to denote zero matrices/vectors of any size, with zero components. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be Schur stable if all of its eigenvalues have magnitude less than 1. The Kronecker product is denoted by $\otimes$ and the Hadamard product by $\circ$ in [40]. The following properties of the Kronecker product will be used: (1) $(A\otimes B)(C\otimes D) = AC\otimes BD$; (2) $(A+B)\otimes C = A\otimes C + B\otimes C$; (3) $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$.

We consider DLMASs with N homogeneous agents and assume they are described by

$$x_i^+ = Ax_i + Bu_i, \quad i = 1, \ldots, N, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, (A, B) is assumed to be stabilizable, $x_i = x_i(k) \in \mathbb{R}^n$ is the state of the current time k, $x_i^+ = x_i(k+1)$ denotes the state at the next time $k+1$, and $u_i = u_i(k) \in \mathbb{R}^m$ is the control input of the current time k.

The control input u_i will be constructed based on the available information of the agent i. Let $\mathcal{N}_i$ denote the index set of the agents which can send their state information to the agent i. We call the set $\mathcal{N} = \{\mathcal{N}_i : i = 1, \ldots, N\}$ the information topology of the DLMASs (1). It is well known that one can use a digraph $G = (V, E)$ to express the information topology $\mathcal{N}$, where $V = \{1, \ldots, N\}$ is the index set of N agents, $E \subseteq V \times V$ is the set of directed edges to describe the information interaction between agents; that is, $(j, i) \in E \Leftrightarrow j \in \mathcal{N}_i$. Based on the directed edges, one can construct an adjacency matrix $A = [a_{ij}]_{N\times N}$, whose entries are defined as $a_{ij} = 0$ for $j = i$, $a_{ij} = 1$ for $j \in \mathcal{N}_i$, and $a_{ij} = 0$ for $j \notin \mathcal{N}_i$. The corresponding in-degree matrix and graph Laplacian are defined as $D = \operatorname{diag}\{\deg_1, \ldots, \deg_N\}$ and $L = D - A$, respectively, where $\deg_i = \sum_{j=1}^{N} a_{ij}$ is the in-degree of the vertex i. A directed spanning tree of the digraph G is a tree covering all the vertices of the digraph. The following results are well known.

Lemma 1 (see [22]). *The Laplacian matrix $L \in \mathbb{R}^{N\times N}$ has the following properties: (1) all of the eigenvalues of L are either in the open right half complex plan or equal to 0; (2) 0 is a simple eigenvalue of L if and only if the digraph G contains a directed spanning tree.*

Given the information topology $\mathcal{N}$, we construct the following linear consensus protocol:

$$u_i = K_1 x_i + K_2 \sum_{j\in\mathcal{N}_i} w_{ij}\left(x_j - x_i\right), \quad i = 1, \ldots, N, \tag{2}$$

where $K_1, K_2 \in \mathbb{R}^{m\times n}$ are feedback gain matrices to be determined, which are relative to the consensus state and the convergence rate, respectively. $W =: [w_{ij}]_{N\times N}$ is a weighted matrix associated with the information topology $\mathcal{N}$. For the sake of expression, we also define the weighted adjacency matrix $A_w = A \circ W$ by using Hadamard product of matrices and the weighted Laplacian $L_w = D_w - A_w$ with weights w_{ij}, where $D_w = \operatorname{diag}\{\deg_1, \ldots, \deg_N\}$ is the corresponding weighted in-degree matrix with weighted in-degrees $\deg_i = \sum_{j\in\mathcal{N}_i} w_{ij}$.

Definition 2. For the given information topology $\mathcal{N}$, the DLMASs (1) are said to achieve global state consensus via the protocol (2) if for any given initial state $x_i(0)$, $i = 1, \ldots, N$, there exists an n-dimensional vector function $\xi(k)$ depending on the initial states such that $\lim_{k\to\infty}\|x_i(k) - \xi(k)\| = 0$. The function $\xi(k)$ is called a state consensus function.

In this paper, we will address the following three aspects of the state consensus problem: (i) to give criteria of global state consensus, that is, for any given information topology $\mathcal{N}$, weighted matrix W and feedback gain matrices K_1 and K_2 to find the conditions of the DLMASs (1) achieving global state consensus via the protocol (2); (ii) to calculate the state consensus function $\xi(k)$ if the DLMASs (1) achieve global state consensus via the protocol (2); (iii) to determine the matrices K_1 and K_2 such that the DLMASs (1) achieve global state consensus via the protocol (2).

First of all, we transform the state consensus problem to the partial stability problem. Then, based on the partial stability theorem framework, we educe new necessary and sufficient consensus convergence criteria and state a procedure to determine the gain matrices in the protocol on the basis of algebraic Riccati inequality. We also give an explicit analytical expression of the state consensus function based on the respective contributions of the initial states and the protocols. Finally, we extend the results to formation control.

3. State Consensus Analysis

In this section, we first introduce a linear transformation which translates the consensus problem of the multiagent systems into a partial stability problem of the corresponding transformed system. Then, we educe a necessary and sufficient condition for the DLMASs (1) to achieve global state consensus via the protocol (2), and present an analysis expression of the state consensus function. Finally, we discuss some interesting remarks and corollaries based on the result.

Let $x = [x_1^T, \ldots, x_N^T]^T$. The dynamics of the DLMASs (1) with the protocol (2) is described by

$$x^+ = \Psi x, \tag{3}$$

where

$$\Psi = I_N \otimes \left(A + BK_1\right) - L_w \otimes BK_2. \tag{4}$$

We propose a state linear transformation for the linear system (3) as follows:

$$\bar{x} = Tx, \tag{5}$$

where the block matrix $T \in \mathbb{R}^{nN\times nN}$ is defined as

$$T =: \begin{bmatrix} \widetilde{T} \\ \mathbf{1}_N^T \otimes I_n \end{bmatrix}, \qquad \widetilde{T} = \begin{bmatrix} T_1 \\ \vdots \\ T_{N-1} \end{bmatrix}. \tag{6}$$

And the matrix $T_i = [T_{i1}, \ldots, T_{iN}]$, $i = 1, \ldots, N-1$, is chosen such that the following two conditions are satisfied:

(1) the row vectors in each of the matrices T_i are linearly independent, respectively;

(2) the identities $T_i(\mathbf{1}_N \otimes I_n) = 0$, $i = 1, \ldots, N-1$, are held.

Lemma 3. *The inverse T^{-1} of the matrix T admits the following form:*

$$\begin{aligned} T^{-1} &= \begin{bmatrix} \overline{T}_{11} & \cdots & \overline{T}_{1,N-1} & N^{-1}I_n \\ \vdots & \ddots & \vdots & \vdots \\ \overline{T}_{N-1,1} & \cdots & \overline{T}_{N-1,N-1} & N^{-1}I_n \\ \overline{T}_{N,1} & \cdots & \overline{T}_{N,N-1} & N^{-1}I_n \end{bmatrix} \\ &=: \left[\overline{T}_1 \ \cdots \ \overline{T}_{N-1} \ \ N^{-1}\mathbf{1}_N \otimes I_n\right] \\ &=: \left[\widehat{T} \ \ N^{-1}\mathbf{1}_N \otimes I_n\right], \end{aligned} \tag{7}$$

where $\overline{T}_{ij}$, $i = 1, \ldots, N$, $j = 1, \ldots, N-1$, are the $n \times n$ blocks indefinitely described.

Using the linear transformation (5), we transform the linear system (3) into the following system:

$$\bar{x}^+ = T\Psi T^{-1}\bar{x}, \tag{8}$$

or the form of two equations

$$y^{+} = \widetilde{T}\Psi\widehat{T}y + N^{-1}\widetilde{T}\Psi\left(\mathbf{1}_N \otimes I_n\right)z,$$
$$z^{+} = \left(\mathbf{1}_N^T \otimes I_n\right)\Psi\widehat{T}y + N^{-1}\left(\mathbf{1}_N^T \otimes I_n\right)\Psi\left(\mathbf{1}_N \otimes I_n\right)z, \quad (9)$$

where $\overline{x} = [y^T, z^T]^T$, $y = [\overline{x}_1^T, \ldots, \overline{x}_{N-1}^T]^T$, and $z = \overline{x}_N$.

We show that the state consensus problem of the DLMASs (1) with the protocol (2) can be transformed into a partial stability problem.

Definition 4 (see [41]). The linear system (8) is said to be asymptotically stable with respect to y (or asymptotically y-stable in short) if $\lim_{k\to\infty} y(k) = 0$ for any bounded initial state $\overline{x}(0)$ of the system (8).

Lemma 5. *Under the given information topology $\mathcal{N}$, the DLMASs* (1) *achieve global state consensus via the protocol* (2) *if and only if the equilibrium point $\overline{x} = 0$ of the linear system* (8) *is asymptotically y-stable. Moreover, the state consensus function of the agents is $\xi(k) = N^{-1}\sum_{i=1}^{N} x_i(k) = N^{-1}\overline{x}_N(k)$.*

Lemma 5 builds a bridge between the consensus problem and the partial stability problem. Now we focus on the asymptotical y-stability of the linear system (8). We can verify the following lemma.

Lemma 6. *The system* (9) *is of the following form:*

$$y^{+} = \overline{A}y, \quad y \in \mathbb{R}^{n(N-1)},$$
$$z^{+} = \overline{C}y + \overline{D}z, \quad z \in \mathbb{R}^{n}, \quad (10)$$

where $\overline{A} = \widetilde{T}(I_N \otimes (A + BK_1) - L_w \otimes BK_2)\widehat{T}$, $\overline{C} = -(\mathbf{1}_N^T L_w \otimes BK_2)\widehat{T}$, and $\overline{D} = A + BK_1$.

Combining Lemma 5 with Lemma 6, we directly get the following theorem.

Theorem 7. *Under the given information topology $\mathcal{N}$, the DLMASs* (1) *achieve global state consensus via the protocol* (2) *if and only if matrix $\overline{A}$ in* (10) *is Schur stable. Moreover, the state consensus function is*

$$\xi(k) = N^{-1}\left(\left(\sum_{j=0}^{k-1}(A + BK_1)^j\overline{C}\,\overline{A}^{k-1-j}\right)\widetilde{T} + \mathbf{1}_N^T \otimes (A + BK_1)^k\right)x(0). \quad (11)$$

Subsequently, we give some interesting remarks and corollaries based on the result.

Remark 8. Since $\widehat{T}\widetilde{T} = (I_N - N^{-1}\mathbf{1}_N\mathbf{1}_N^T) \otimes I_n$, the result of Theorem 7 is in fact independent of the choice of the matrix T although both $\overline{A}$ and formula (11) in Theorem 7 contain $\widetilde{T}$ and $\widehat{T}$. Hence, for simplicity, we take it in the following form:

$$T = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \otimes I_n := \begin{bmatrix} \widetilde{T}_0 \\ \mathbf{1}_N^T \end{bmatrix} \otimes I_n. \quad (12)$$

The corresponding inverse matrix is

$$T^{-1} = \frac{1}{N}\begin{bmatrix} N-1 & N-2 & \cdots & 1 & 1 \\ -1 & N-2 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2 & \cdots & 1 & 1 \\ -1 & -2 & \cdots & -(N-1) & 1 \end{bmatrix} \otimes I_n := \begin{bmatrix} \widehat{T}_0 & N^{-1}\mathbf{1}_N \end{bmatrix} \otimes I_n. \quad (13)$$

Thus, we can write $\overline{A}$ and $\overline{C}$ into

$$\overline{A} = I_{N-1} \otimes (A + BK_1) - \widetilde{T}_0 L_w \widehat{T}_0 \otimes BK_2,$$
$$\overline{C} = -\mathbf{1}_N^{\mathrm{T}} L_w \widehat{T}_0 \otimes BK_2. \quad (14)$$

Corollary 9. *Under the given information topology $\mathcal{N}$, the DLMASs* (1) *achieve global consensus via the protocol* (2) *if and only if all the matrices $A + BK_1 - \lambda_i BK_2$ are Schur stable, where λ_i, $i = 1, \ldots, N-1$, are the eigenvalues of the matrix $\widetilde{T}_0 L_w \widehat{T}_0$. Moreover, the state consensus function is expressed by*

$$\xi(k) = \left(\eta^T \otimes (A + BK_1)^k\right)x(0), \quad i = 1, \ldots, N, \quad (15)$$

where η satisfies $\eta^T L_w = 0$ and $\eta^T \mathbf{1}_N = 1$.

One can verify that as $k \to \infty$ the state consensus functions in formulas (11) and (15) are the same.

Remark 10. From Schur stability of $\overline{A}$ in the formula (14), we can conclude that if $A + BK_1$ is not Schur stable, it is a necessary condition of the consensus that the digraph G expressing the information topology $\mathcal{N}$ has a directed spanning tree. In fact, since the condition of directed spanning tree is equivalent to Hurwitz stability of $-\widetilde{T}_0 L_w \widehat{T}_0$, a lack of directed spanning tree means that $-\widetilde{T}_0 L_w \widehat{T}_0$ has a zero eigenvalue. In this case, we transform $\overline{A}$ into its Jordan form via the matrix $U \otimes I_n$, where U is the matrix such that $U^{-1}\widetilde{T}_0 L_w \widehat{T}_0 U = J$ is the Jordan form, and thus we have

$$\left(U^{-1} \otimes I_n\right)\overline{A}\left(U \otimes I_n\right) = I_{N-1} \otimes (A + BK_1) - J \otimes BK_2. \quad (16)$$

One can verify that the eigenvalues of $A + BK_1$ are the members of the eigenvalues of $\overline{A}$, and thus $\overline{A}$ is not Schur stable if $A + BK_1$ is not Schur stable. On the other hand, if $A + BK_1$ is Schur stable, one can take $K_2 = 0$ to make the DLMASs (1) achieve global consensus, which implies that for any initial states all the agents always converge to the equilibrium point 0.

Hence, from the formula (16) we can educe another global consensus criterion.

Corollary 11. *If $A + BK_1$ is not Schur stable, then under the given information topology $\mathcal{N}$, the DLMASs* (1) *achieve global state consensus via the protocol* (2) *if and only if $\widetilde{T}_0 L_w \widehat{T}_0$ has $N-1$ eigenvalues with positive real part λ_i, $i = 1, \ldots, N-1$, and the matrices $A + BK_1 - \lambda_i BK_2$, $i = 1, \ldots, N-1$, are Schur stable.*

Remark 12. If the state consensus function in (11) is a constant vector equal to the average of the initial states of all the agents, the consensus is called the average consensus. From the formula (11) we educe the following result on the average consensus.

Corollary 13. *The DLMASs* (1) *achieve global average consensus via the protocol* (2) *if and only the matrix $\overline{A}$ is Schur stable and $(\sum_{j=0}^{k-1}(A+BK_1)^j \overline{C}\,\overline{A}^{k-1-j})\widetilde{T} + \mathbf{1}_N^T \otimes (A+BK_1)^k = \mathbf{1}_N^T \otimes I_n$.*

If $A + BK_1 = I_n$, then the last condition in Corollary 13 becomes $\mathbf{1}_N^T L_w = 0$, or equivalently, the digraph G is either undirected connected or directed strong connected and balanced. More specially, if L_w is a symmetric matrix (equivalently, the digraph G becomes undirected connected), the condition $\mathbf{1}_N^T L_w = 0$ is satisfied and thus the average consensus is achieved.

Remark 14. When $A = I_n$, $K_1 = 0$, and $B = I_n$, the DLMASs (1) are called a single-integrator one. In this case, $\overline{A} = I_{(N-1)n} - \widetilde{T}_0 L_w \widehat{T}_0 \otimes K_2$ and $\overline{C} = -\mathbf{1}_N^T L_w \widehat{T}_0 \otimes K_2$. We educe the following result.

Corollary 15. *Under the given information topology $\mathcal{N}$, the single-integrator DLMASs* (1) *achieve global state consensus via the protocol* (2) *if and only if the following two conditions are held simultaneously: (1) the matrix $-\widetilde{T}_0 L_w \widehat{T}_0$ is Hurwitz stable; that is, the digraph G admits a directed spanning tree; (2) the products $\lambda_i \mu_j$, $i = 1, \ldots, N-1$, $j = 1, \ldots, n$, are in the open unit circle of the complex plane with the centre at $(1, 0)$, where λ_i, $i = 1, \ldots, N-1$, are the eigenvalues of the matrix $\widetilde{T}_0 L_w \widehat{T}_0$ and μ_j, $j = 1, \ldots, n$, are the eigenvalues of the matrix K_2. The corresponding consensus function* (11) *becomes a constant vector*

$$\xi = N^{-1} \left(\mathbf{1}_N^T \otimes I_n - \mathbf{1}_N^T L_w \widehat{T}_0 \left(\widetilde{T}_0 L_w \widehat{T}_0 \right)^{-1} \widetilde{T}_0 \otimes I_n \right) x(0). \tag{17}$$

Moreover, the single-integrator DLMASs (1) *achieve global average consensus via the protocol* (2) *if and only both of the above conditions are satisfied and in addition $\mathbf{1}_N^T L_w = 0$.*

Remark 16. When $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \otimes I_n$, $B = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes I_n$, and $K_1 = 0$, the DLMASs (1) are called a double-integrator one, whose state vector can be seen as consisting of the position and velocity in the n dimensional space $\mathbb{R}^n$.

Corollary 17. *Under the given information topology $\mathcal{N}$, the double-integrator DLMASs* (1) *achieve global state consensus via the protocol* (2) *if and only if $\overline{A}$ is Schur stable. Moreover, the consensus function is*

$$\begin{aligned}
\xi(k) &= N^{-1} \left\{ \left(\overline{C} \left(I_{2(N-1)n} - \overline{A} \right)^{-1} + \begin{bmatrix} 0 & kI_n \\ 0 & 0 \end{bmatrix} \overline{C} \left(I_{2(N-1)n} - \overline{A} \right)^{-1} \right.\right. \\
&\qquad \left. - \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix} \overline{C} \left(I_{2(N-1)n} - \overline{A} \right)^{-2} \right) \widetilde{T} \\
&\qquad \left. + \mathbf{1}_N^T \otimes \begin{bmatrix} I_n & kI_n \\ 0 & I_n \end{bmatrix} \right\} x(0).
\end{aligned} \tag{18}$$

The consensus function above can be decomposed into the position consensus function

$$\begin{aligned}
\xi_1(k) = N^{-1} &\left\{ \left(\begin{bmatrix} I_n & 0 \end{bmatrix} \overline{C} \left(I_{2(N-1)n} - \overline{A} \right)^{-1} \right.\right. \\
&+ \begin{bmatrix} 0 & kI_n \end{bmatrix} \overline{C} \left(I_{2(N-1)n} - \overline{A} \right)^{-1} \\
&\left. - \begin{bmatrix} 0 & I_n \end{bmatrix} \overline{C} \left(I_{2(N-1)n} - \overline{A} \right)^{-2} \right) \widetilde{T} \\
&\left. + \mathbf{1}_N^T \otimes \begin{bmatrix} I_n & kI_n \end{bmatrix} \right\} x(0),
\end{aligned} \tag{19}$$

which is a linear function of discrete time k, and the constant velocity consensus function is as follows:

$$\begin{aligned}
\xi_2(k) = N^{-1} &\left\{ \left(\begin{bmatrix} 0 & I_n \end{bmatrix} \overline{C} \left(I_{2(N-1)n} - \overline{A} \right)^{-1} \right) \widetilde{T} \right. \\
&\left. + \mathbf{1}_N^T \otimes \begin{bmatrix} 0 & I_n \end{bmatrix} \right\} x(0).
\end{aligned} \tag{20}$$

Similarly, we can define the velocity average consensus, that is, if the DLMASs (1) achieve global consensus via the protocol (2), and the velocity consensus function is a constant vector equal to the average of the initial velocities of all the agents.

Corollary 18. *Under the given information topology $\mathcal{N}$, the double-integrator DLMASs* (1) *achieve global velocity average consensus via the protocol* (2) *if and only the matrix $\overline{A}$ is Schur stable and $\begin{bmatrix} 0 & I_n \end{bmatrix} \overline{C} = 0$.*

It is obvious that if $\mathbf{1}_N^T L_w = 0$, then $\overline{C} = 0$; that is, the last condition in Corollary 18 is satisfied, and thus, the state consensus function becomes

$$\xi(k) = N^{-1} \mathbf{1}_N^T \otimes \begin{bmatrix} I_n & kI_n \\ 0 & I_n \end{bmatrix} x(0). \tag{21}$$

4. Design of Gain Matrices

In this section, we discuss the third problem, that is, how to determine the weighted matrix W and the gain matrices K_1 and K_2, such that the DLMASs (1) achieve global state consensus via the protocol (2).

Theorem 7 shows that the matrices W, K_1, and K_2 should be taken to ensure that the matrix $\overline{A}$ is Schur stable. Furthermore, from Corollary 11 we see that if the matrix W with respect to the information topology $\mathcal{N}$ has been given, we need only to design the gain matrices K_1 and K_2 to ensure that the matrices $A + BK_1 - \lambda_i BK_2$ are Schur stable, where λ_i, $i = 1, \ldots, N-1$, are the eigenvalues of the matrix $\widetilde{T}_0 L_w \widehat{T}_0$. The matrix K_1 is often taken to obtain an expected consensus dynamics. The matrix K_2 is designed to achieve state consensus and expected convergence rate. Its design needs the following lemma.

Lemma 19 (see [38]). *Supposing that the matrix $\widehat{A} = A + BK_1$ is not Schur stable but $(\widehat{A}, B)$ is stabilizable, then there is a critical value $\delta_c \in (0,1]$, such that for any number δ with $0 < \delta < \delta_c$ the modified Riccati inequality*

$$\widehat{A}^T P \widehat{A} - P - \left(1 - \delta^2\right) \widehat{A}^T PB \left(B^T PB\right)^{-1} B^T P \widehat{A} < 0 \qquad (22)$$

admits a positive definite matrix solution P, where δ_c depends on the unstable eigenvalues of the matrix $\widehat{A}$.

We define functions $\delta_i(\omega) = 1 - \omega\lambda_i$ and $\delta(\omega) = \max_{i\in\{1,\ldots,N-1\}} |\delta_i(\omega)|$. Motivated by [38], we get the following theorem.

Theorem 20. *Supposing that the matrix (A, B) is stabilizable, the gain matrix K_1 has been taken such that the expected consensus dynamic matrix $A + BK_1$ is not Schur stable, and the weighted matrix W with respect to the information topology $\mathcal{N}$ is given such that $-\widetilde{T}_0 L_w \widehat{T}_0$ is Hurwitz stable with $N-1$ eigenvalues $-\lambda_i$, $i = 1, \ldots, N-1$; then, for the DLMASs* (1) *to achieve state consensus via the protocol* (2)*, the matrix K_2 can be designed as $K_2 = \omega(B^T PB)^{-1} B^T P(A + BK_1)$, where ω is an arbitrary constant satisfying $\delta = \delta(\omega) < \delta_c$, $\delta_c \in (0, 1]$ is a critical value which depends on the unstable eigenvalues of the matrix $A + BK_1$, and $P^T = P > 0$ is a solution of the algebraic Riccati inequality* (22).

Based on Theorem 20, we give the following algorithm of determining the feedback gain matrices K_1 and K_2 in the protocol (2).

Algorithm 21. Design procedure of the gain matrices K_1 and K_2.

Step 1. Verify the stabilizability condition of (A, B) and the spanning tree condition of the information topology $\mathcal{N}$. If neither of them is satisfied, then stop. Otherwise, design the weighted Laplacian L_w such that $-\widetilde{T}_0 L_w \widehat{T}_0$ is Hurwitz stable with $N-1$ eigenvalues $-\lambda_i$, $i = 1, \ldots, N-1$.

Step 2. Design K_1 such that $\widehat{A} = A + BK_1$ is the matrix of the expected consensus dynamics of the DLMASs (1) and is not Schur stable.

Step 3. Calculate all the eigenvalues of $\widehat{A}$, which are composed of the stable eigenvalues $\lambda_i^s(\widehat{A})$, $i = 1, \ldots, n_s$ and unstable ones $\lambda_i^u(\widehat{A})$, $i = 1, \ldots, n_u$, $n_s + n_u = n$.

Step 4. Calculate the critical value $\delta_c \in (0, 1]$. If B is invertible, then $\delta_c = (\max_{i=\{1,\ldots,n_u\}} |\lambda_i^u(\widehat{A})|)^{-1}$. If B is of rank one, then $\delta_c = (\prod_{i=\{1,\ldots,n_u\}} |\lambda_i^u(\widehat{A})|)^{-1}$. Otherwise, apply Wonham decomposition to the unstable part (A_u, B_u) of $(\widehat{A}, B)$ to convert the multiple input system to m single input subsystems, where m is the number of the Jordan blocks of matrix A_u. Specifically, there is a nonsingular real matrix Q with a compatible dimension such that $\widetilde{A} = Q^{-1} A_u Q$ and $\widetilde{B} = Q^{-1} B_u$ take the form

$$\widetilde{A} = \begin{bmatrix} A_1 & * & * & * \\ 0 & A_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \qquad \widetilde{B} = \begin{bmatrix} b_1 & * & * & * \\ 0 & b_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_m \end{bmatrix}, \quad (23)$$

where the symbol $*$ denotes possibly nonzero parts and (A_j, b_j) with $A_j \in \mathbb{R}^{n_j \times n_j}$ and $b_j \in \mathbb{R}^{n_j}$ for all $j \in \{1, \ldots, m\}$ is controllable and $\sum_{j=1}^m n_j = n_u$. In this case, δ_c is lower bounded by $\delta_c \geq (\prod_i |\lambda_i^u(A_{m^*})|)^{-1}) = \delta_c'$, where the index m^* is defined by $m^* = \arg\max_{j=\{1,\ldots,m\}} (\prod_i |\lambda_i^u(A_j)|)$ and A_j is the Jordan block of the unstable part of matrix $\widehat{A}$.

Step 5. Calculate the value ω such that $\delta(\omega) = \max_{i\in\{1,\ldots,N-1\}} |\delta_i(\omega)|$.

Step 6. Solve (22) with $\delta = \delta(\omega)$ for a positive definite matrix P.

Step 7. Calculate the matrix $K_2 = \omega(B^T PB)^{-1} B^T P(A + BK_1)$.

5. Application to Formation Control

In this section, the consensus approach is modified to solve the formation control problem of the DLMASs (1). Let $h = \begin{bmatrix} h_1^T & h_2^T & \cdots & h_N^T \end{bmatrix}^T \in \mathbb{R}^{nN}$ describe a constant formation of the agent network in a reference coordinate frame, where $h_i \in \mathbb{R}^n$ is the formation variable corresponding to the agent i. The variable $h_i - h_j$ denotes the relative formation vector between the agents i and j, which is assumed to be independent of the reference coordinate.

We modify the consensus protocol (2) and propose a distributed formation protocol as follows:

$$u_i = K_1 x_i + K_2 \sum_{j\in\mathcal{N}_i} w_{ij} \left(x_j - x_i - \left(h_j - h_i\right)\right), \quad i = 1, \ldots, N. \qquad (24)$$

Definition 22. Under the given information topology $\mathcal{N}$, the DLMASs (1) achieve the given formation h via the protocol

(24) if $\|(x_i(k) - x_j(k)) - (h_i - h_j)\| \rightarrow 0$ as $k \rightarrow \infty$, for all $i, j = 1, \ldots, N$, that is, if there is a function $\xi(k)$ such that $\|x_i(k) - h_i - \xi(k)\| \rightarrow 0$ as $k \rightarrow \infty$, for all $i = 1, \ldots, N$, where $\xi(k)$ is called reference state consensus function.

Theorem 23. *Under the given information topology $\mathcal{N}$, the DLMASs* (1) *achieve the formation h via the protocol* (24) *if and only if the matrix $I_{N-1} \otimes (A + BK_1) - \widetilde{T}_0 L_w \widehat{T}_0 \otimes BK_2$ is Schur stable and $(\widetilde{T}_0 \otimes (A + BK_1 - I_n))h = 0$. Moreover, the reference state consensus function is*

$$\xi(k) = N^{-1}\left(\left(\sum_{j=0}^{k-1}(A + BK_1)^j \overline{C}\,\overline{A}^{k-1-j}\right)\widetilde{T} + \mathbf{1}_N^T \otimes (A + BK_1)^k\right) x(0). \tag{25}$$

Similarly to Corollary 11, we get the following corollary for the formation control.

Corollary 24. *Under the given information topology $\mathcal{N}$, the DLMASs* (1) *achieve the formation h via the protocol* (24) *if and only if all the matrices $A + BK_1 - \lambda_i BK_2$ are Schur stable and $(\widetilde{T}_0 \otimes (A + BK_1 - I_n))h = 0$, where λ_i, $i = 1, \ldots, N-1$, are the eigenvalues of the matrix $\widetilde{T}_0 L_w \widehat{T}_0$.*

Remark 25. Note that not all kinds of formation structure can be achieved for the DLMASs (1) by using the protocol (24). The achievable formation structures have to satisfy the constraints $(\widetilde{T}_0 \otimes (A + BK_1 - I_n))h = 0$. The formation protocol (24) for a given achievable formation h can be constructed analogously by using the Algorithm 21 in Section 4.

6. Numerical Examples

In this section, we give some illustrative examples.

Example 26 (state consensus analysis). We consider the DLMASs (1) consisting of four agents described by the following matrices:

$$A = \begin{bmatrix} 2 & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{-1}{2} & \frac{\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} \\ 1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ \frac{-1}{2} \\ 1 \end{bmatrix}. \tag{26}$$

Supposed that we are given the information topology $\mathcal{N} = \{\{4\}, \{1\}, \{2\}, \{3\}\}$, the weights $w_{ij} = 0.5$, and the gain matrices $K_1 = [-1, -\sqrt{2}/2, \sqrt{2}/2]$ and $K_2 = [0.4444, -0.4714, 0.1571]$, we are required to verify the consensus convergence.

The matrix $\overline{A}$ is Schur stable since its eigenvalues are $0.3482 \pm 0.7182i$, $0.8746 \pm 0.2786i$, $0.7885 \pm 0.4643i$, $0.7728 \pm 0.5782i$, and 0.0001. Therefore, according to Theorem 7, the DLMASs (1) with the matrices in (26) achieve global consensus via the protocol (2) under the given information topology $\mathcal{N}$ and the gain matrix K_1, K_2. For the initial states $x_1(0) = [0, 0.6, 0.6]^T$, $x_2(0) = [1.2, 1.5, 1.8]^T$, $x_3(0) = [1.8, 2.4, 2.1]^T$, and $x_4(0) = [3.0, 2.7, 3.6]^T$, the corresponding state consensus function in (11) is $\xi(k) = [1.5, 2.7094\cos(0.7854k - 1.6296), 2.7094\cos(0.7854k - 0.0588)]^T$. Figure 1 shows the state trajectories of the agents and the trajectory of the state consensus function marked by circles.

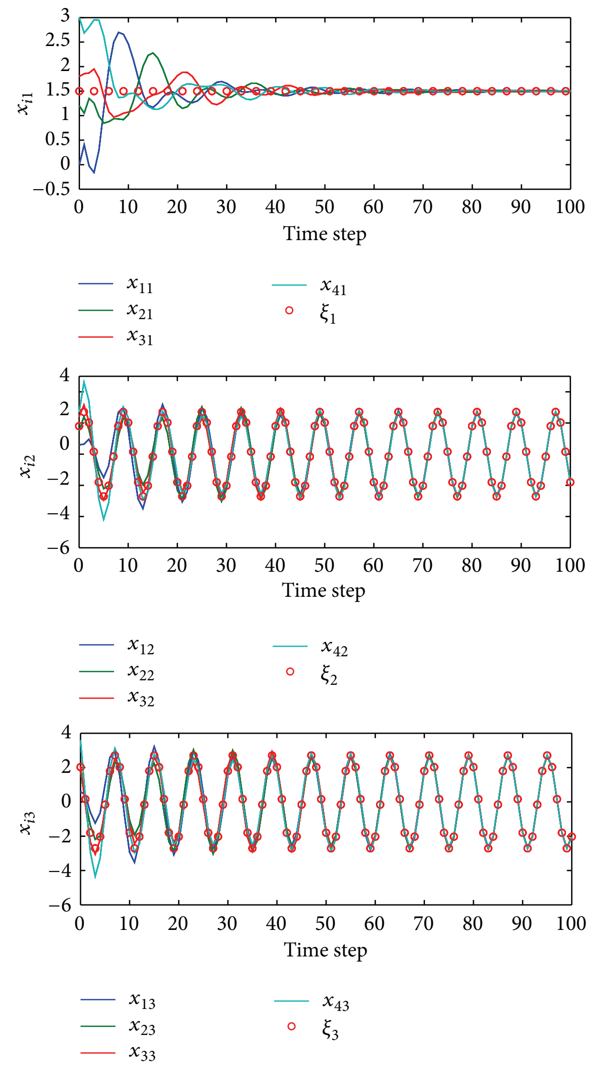

Figure 1: State trajectories of DLMASs.

Example 27 (gain matrices design for a formation control). The consensus problem of multiagent systems has many practical applications, such as formation control of mobile robots and cooperative control of unmanned airborne vehicles. In this example, we consider that DLMASs consisting of four mobile robots are described as

$$\begin{aligned} x_i^+ &= x_i + v_i, \\ v_i^+ &= v_i + u_i, \quad i = 1, \ldots, 4, \end{aligned} \tag{27}$$

where $x_i \in \mathbb{R}^2$, $v_i \in \mathbb{R}^2$, and $u_i \in \mathbb{R}^2$ are the position, the velocity, and the acceleration input of the robot i, respectively.

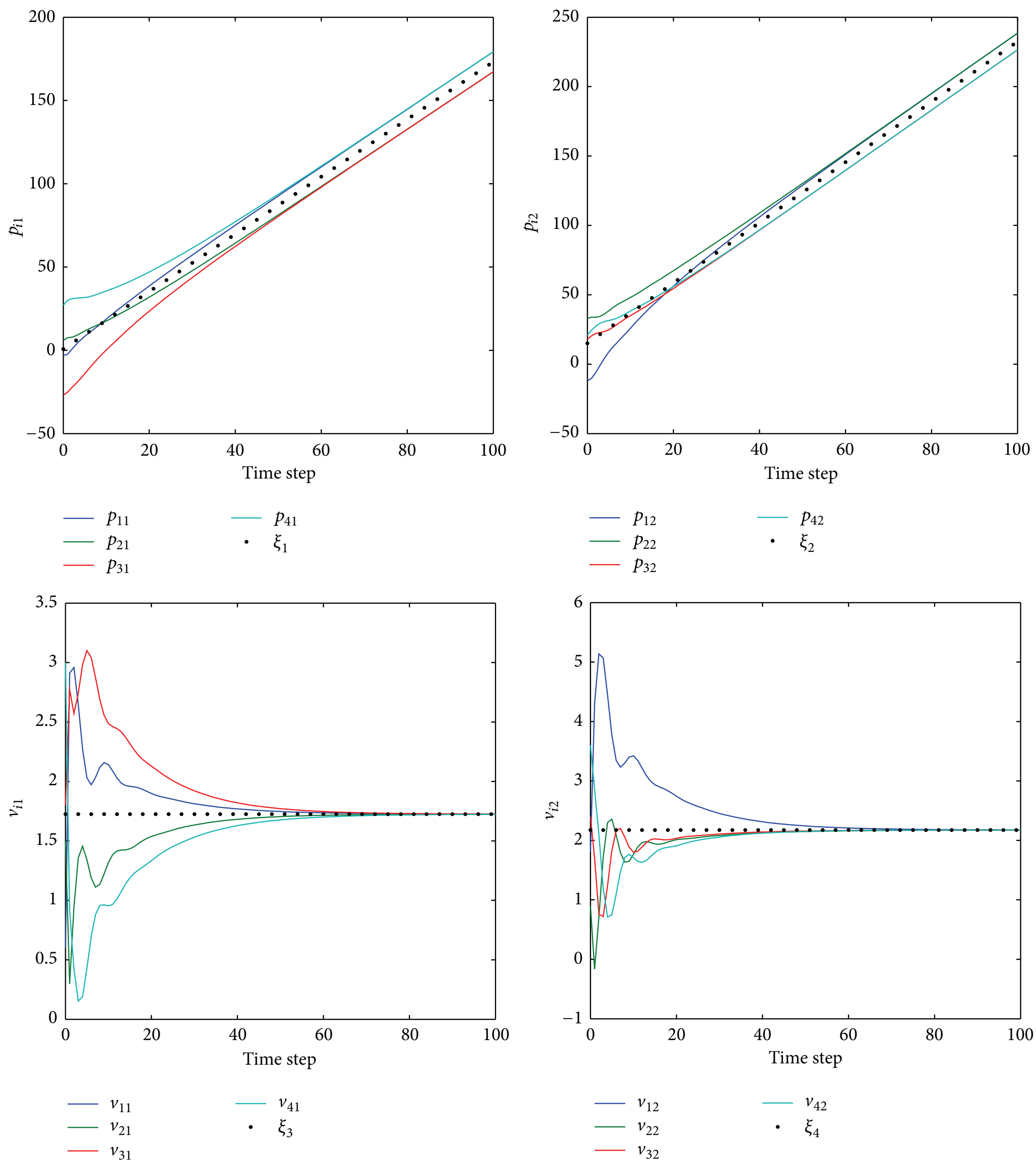

Figure 2: Position and velocity trajectories of mobile robots.

The information topology and the weights are the same as in Example 26. The eigenvalues of the matrix $\widetilde{T}_0 L_w \widehat{T}_0$ are $\lambda_1 = 0.5 + 0.5i$, $\lambda_2 = 0.5 - 0.5i$, and $\lambda_3 = 1$. We choose $h_1 = [6, 6, 0, 0]^T$, $h_2 = [-6, 6, 0, 0]^T$, $h_3 = [-6, -6, 0, 0]^T$, and $h_4 = [6, -6, 0, 0]^T$ for the formation h of the mobile robots. Note that (A, B) in (27) is stabilizable. The eigenvalues of matrix A are 1 (4 multiples). We assume that the expected consensus dynamics of the agents is not changed, and thus, we have $K_1 = 0$. It is easy to verify that the formation h satisfies the constraints $(\widetilde{T}_0 \otimes (A + BK_1 - I_n))h = 0$. One can get $\delta_c = 1$ through Step 4 of the Algorithm 21. By Step 5, one can obtain $\omega = 1.0$, and the corresponding $\delta(\omega) = \sqrt{2}/2 < \delta_c$. The modified Riccati inequality (22) admits a solution

$$P = \begin{bmatrix} 88 & 17 & 0 & 0 \\ 17 & 248 & 0 & 0 \\ 0 & 0 & 88 & 17 \\ 0 & 0 & 17 & 248 \end{bmatrix} \times 10^{-4}. \tag{28}$$

The corresponding control gain is calculated as follows:

$$K_2 = \begin{bmatrix} 0 & 0 & 0.0686 & 1.0686 \\ 0.0686 & 1.0686 & 0 & 0 \end{bmatrix}. \tag{29}$$

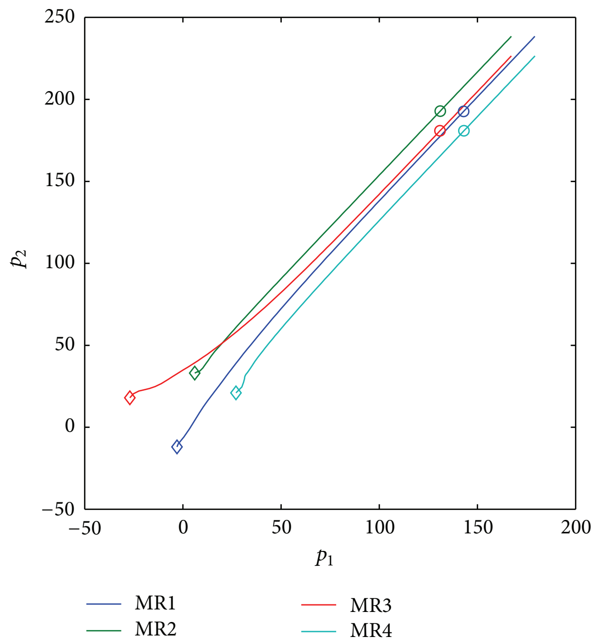

Figure 3: Formation of four mobile robots.

The eigenvalues of matrix $\overline{A}$ are $0.5336 \pm 0.5392i$ (2 multiples), $0.9321 \pm 0.0049i$ (2 multiples), 0.9314 (2 multiples), and 0 (2 multiples), that is, the matrix $\overline{A}$ is Schur stable. By Theorem 23, the system (27) via the protocol (24) achieves the formation h.

For the initial states $x_1(0) = [-9, 0.6, -18, 1.8]^T$, $x_2(0) = [12, 1.5, 27, 0.9]^T$, $x_3(0) = [-21, 1.8, 24, 2.4]^T$, and $x_4(0) = [21, 3, 27, 3.6]^T$, we get the reference state consensus function $\xi(k) = [0.75 + 1.725k, 1.725, 15 + 2.175k, 2.175]^T$ according to (25). Figure 2 shows the position and velocity trajectories of the mobile robots and the trajectory of the reference state consensus function marked by dots, and Figure 3 shows the formation evolution trajectories. From Figure 3 it is clear to see that the mobile robots achieve the expected formation, where the symbol diamond denotes the initial position, and the symbol circle denotes the position at the moment $k = 80$ of the mobile robots, respectively.

7. Conclusions

We considered the state consensus problem of high-order discrete-time linear multiagent systems with fixed directed information topology. A linear transformation approach was proposed to translate the consensus problem of multiagent systems into a partial stability problem of the corresponding transformed systems. We have shown that the approach is powerful in dealing with the three aspects of the consensus problem: (1) the criteria of global state consensus, (2) the calculation of the state consensus function, and (3) the determination the weighted matrix and the feedback gain matrix. Precisely, we have educed new necessary and sufficient consensus criteria in terms of Schur stability of a matrix related to the weighted Laplacian matrix and presented an analytical expression of the state consensus function. In addition, we have stated a design process of determining the feedback gain matrix under the condition of each agent being stabilizable. The consensus algorithm has been further applied to solve the formation control problem of multiagent systems.

Though the work in this paper focuses on the high-order discrete-time linear multiagent systems with fixed information topology and without time delay, it is undoubted that the approach can be easily extended to more complex cases, which will be dealt with in the future works.

Appendix

Proof of Lemma 3. Assuming the inverse to be $T^{-1} = [\overline{T}_1 \ \cdots \ \overline{T}_{N-1} \ \overline{T}_N]^T$ with columns $\overline{T}_i$, $i = 1, \ldots, N$, we prove that the equality $\overline{T}_N = N^{-1}\mathbf{1}_N \otimes I_n$ is correct. Since the matrix T^T is invertible and thus each column of $\overline{T}_N$ can be linearly represented by the columns of the matrix T^T, so the matrix $\overline{T}_N$ can be represented by $q_1 + q_2$, where each column of the matrix q_1 can be linearly represented by matrices T_i^T, $i = 1, \ldots, N-1$, and q_2 by those of the matrix $\mathbf{1}_N \otimes I_n$. Left multiplying $\overline{T}_N = q_1 + q_2$ by q_1^T gets $q_1^T\overline{T}_N = q_1^T q_1 + q_1^T q_2$. Because of $TT^{-1} = I_{Nn}$, one has $T_i\overline{T}_N = 0$, $i = 1, \ldots, N-1$, which implies $q_1^T\overline{T}_N = 0$. On the other hand, from the equalities $T_i(\mathbf{1}_N \otimes I_n) = 0$, $i = 1, \ldots, N-1$, one gets $T_i q_2 = 0$, $i = 1, \ldots, N-1$, which means $q_1^T q_2 = 0$. Hence, from the equality $q_1^T\overline{T}_N = q_1^T q_1 + q_1^T q_2$, one deduces $q_1^T q_1 = 0$, that is, $q_1 = 0$ and thus $\overline{T}_N = q_2$. In other words, one can write $\overline{T}_N$ into $\overline{T}_N = (\mathbf{1}_N \otimes I_n)\alpha$, where α is a matrix of the order $n \times n$. From the identity $T\overline{T} = I_{Nn}$, one has $(\mathbf{1}_N \otimes I_n)^T = I_n$, and thus, from $\overline{T}_N = (\mathbf{1}_N \otimes I_n)\alpha$ one can get $I_n = N\alpha$, that is, $\alpha = N^{-1}I_n$. Finally, one has the expression $\overline{T}_N = N^{-1}\mathbf{1}_N \otimes I_n$. □

Proof of Lemma 5. In fact, if there is $\xi(k; x(0))$ such that $\lim_{k\to\infty}\|x_i(k; x_i(0)) - \xi(k; x(0))\| = 0$, $i = 1, \ldots, N$, it follows that $\lim_{k\to\infty}\|\overline{x}_i(k)\| = 0$, $i = 1, \ldots, N-1$, in virtue of $\overline{x}_i = T_i(x - \mathbf{1}_N \otimes \xi)$, $i = 1, \ldots, N$, and therefore the necessary has been proved. Conversely, by virtue of Lemma 3, one can verify $x_i = \sum_{j=1}^{N-1}\overline{T}_{ij}\overline{x}_j + N^{-1}\overline{x}_N$, $i = 1, \ldots, N-1$. So from $\lim_{k\to\infty}\|\overline{x}_i(k)\| = 0$, $i = 1, \ldots, N-1$, it follows that $\lim_{k\to\infty}\|x_i(k) - \xi(k)\| = 0$, $i = 1, \ldots, N$, where $\xi(k) = N^{-1}\sum_{i=1}^{N} x_i(k) = N^{-1}\overline{x}_N(k)$, and thus, the sufficiency has been verified. □

Proof of Lemma 6. By observation, we only need to show that $\widetilde{T}\Psi(\mathbf{1}_N \otimes I_n) = 0$, $(\mathbf{1}_N^T \otimes I_n)\Psi\widehat{T} = -(\mathbf{1}_N^T L_w \otimes BK_2)\widehat{T}$, and $(\mathbf{1}_N^T \otimes I_n)\Psi(\mathbf{1}_N \otimes I_n) = N(A + BK_1)$. In fact, since $L_w\mathbf{1}_N = 0$, $(\mathbf{1}_N^T \otimes I_n)\widehat{T} = 0$, and $\widetilde{T}(\mathbf{1}_N \otimes I_n) = 0$, we have that

$$
\begin{aligned}
&\widetilde{T}\Psi(\mathbf{1}_N \otimes I_n) \\
&\quad = \widetilde{T}\left(I_N \otimes (A + BK_1) - L_w \otimes BK_2\right)(\mathbf{1}_N \otimes I_n) \\
&\quad = \widetilde{T}\left(I_N \otimes (A + BK_1)\right)(\mathbf{1}_N \otimes I_n) \\
&\quad = \widetilde{T}\left(\mathbf{1}_N \otimes (A + BK_1)\right) = 0,
\end{aligned}
$$

$$\begin{aligned}
&\left(\mathbf{1}_N^T \otimes I_n\right)\Psi\widehat{T}\\
&\quad=\left(\mathbf{1}_N^T \otimes I_n\right)\left(I_N \otimes (A+BK_1) - L_w \otimes BK_2\right)\widehat{T}\\
&\quad=\left(\mathbf{1}_N^T \otimes (A+BK_1) - \mathbf{1}_N^T L_w \otimes BK_2\right)\widehat{T}\\
&\quad=-\left(\mathbf{1}_N^T L_w \otimes BK_2\right)\widehat{T},\\
&\left(\mathbf{1}_N^T \otimes I_n\right)\Psi\left(\mathbf{1}_N \otimes I_n\right)\\
&\quad=\left(\mathbf{1}_N^T \otimes I_n\right)\left(I_N \otimes (A+BK_1) - L_w \otimes BK_2\right)\left(\mathbf{1}_N \otimes I_n\right)\\
&\quad=\left(\mathbf{1}_N^T \otimes I_n\right)\left(I_N \otimes (A+BK_1)\right)\left(\mathbf{1}_N \otimes I_n\right)\\
&\quad=\left(\mathbf{1}_N^T \mathbf{1}_N\right)\otimes (A+BK_1) = N(A+BK_1).
\end{aligned} \tag{A.1}$$

Thus, system (9) becomes

$$\begin{aligned}
y^+ &= \widetilde{T}\Psi\widehat{T}y,\\
z^+ &= -\left(\mathbf{1}_N^T L_w \otimes BK_2\right)\widehat{T}y + (A+BK_1)z.
\end{aligned} \tag{A.2}$$

Denoting $\overline{A} = \widetilde{T}\Psi\widehat{T}$, $\overline{C} = -(\mathbf{1}_N^T L_w \otimes BK_2)\widehat{T}$, and $\overline{D} = A+BK_1$, where $\Psi = I_N \otimes (A+BK_1) - L_w \otimes BK_2$, we get (10). □

Proof of Theorem 7. The necessary and sufficient condition is verified directly by using the Lemmas 5 and 6. Now we focus on the calculation of the state consensus function. The first equation in (10) gives $y(k) = \overline{A}^k y(0)$. From the second equation in (10) we have

$$\begin{aligned}
z(k) &= \overline{C}y(k-1) + \overline{D}z(k-1)\\
&= \overline{C}y(k-1) + \overline{D}\,\overline{C}y(k-2) + \cdots\\
&\quad + \overline{D}^{k-1}\overline{C}y(0) + \overline{D}^k z(0).
\end{aligned} \tag{A.3}$$

Substituting $y(k) = \overline{A}^k y(0)$, $y(0) = \widetilde{T}x(0)$, $\overline{D} = A + BK_1$ and $z(0) = (\mathbf{1}_N^T \otimes I_n)x(0)$ into the above formula, one gets $z(k) = ((\sum_{j=0}^{k-1}(A+BK_1)^j \overline{C}\,\overline{A}^{k-1-j})\widetilde{T} + \mathbf{1}_N^T \otimes (A+BK_1)^k)x(0)$. Thus, by Lemma 5 one has the state consensus function in (11). □

Proof of Corollary 9. First of all, one easily verifies that the Schur stability of $\overline{A}$ is equivalent to Schur stability of all the matrices $A + BK_1 - \lambda_i BK_2$ by transforming $\overline{A}$ into its Jordan form. We focus on the calculation of the state consensus function. Rewrite the system (3) as $x^+ = (I_N \otimes (A+BK_1) - L_w \otimes BK_2)x$. So for the left eigenvector η with the property $\eta^T \mathbf{1}_N = 1$ of the Laplacian L_w with respect to the zero eigenvalue, we obtain $(\eta^T \otimes I_n)x^+ = (A+BK_1)(\eta^T \otimes I_n)x$. When the consensus is achieved, we have that $\xi(k) = (\eta^T \otimes I_n)x(k)$ and thus the state consensus function is $\xi(k) = (A + BK_1)^k(\eta^T \otimes I_n)x(0)$, which can be written into the form (15). □

Proof of Corollary 15. In this case, the two conditions in Corollary 11 become: (1) the matrix $-\widetilde{T}_0 L_w \widehat{T}_0$ is Hurwitz stable, that is, the digraph G admits a directed spanning tree; (2) for all the eigenvalues λ_i, $i = 1, \ldots, N-1$, of the matrix $\widetilde{T}_0 L_w \widehat{T}_0$, all the matrix $I_n - \lambda_i K_2$ are Schur stable. Let μ_j, $j = 1, \ldots, n$, be the eigenvalues of the matrix K_2. Hence, the Schur stability of the matrix $I_n - \lambda_i K_2$ is equivalent to that the products $\lambda_i \mu_j$, $i = 1, \ldots, N-1$, $j = 1, \ldots, n$, are in the open unit circle of the complex plane with the centre at $(1, 0)$.

Now we calculate the state consensus function. In this case, the state consensus function becomes $\xi(k) = N^{-1}\{\overline{C}(I_{(N-1)n} - \overline{A}^k)(I_{(N-1)n} - \overline{A})^{-1}\widetilde{T} + \mathbf{1}_N^T \otimes I_n\}x(0)$. Since $\overline{A}$ is Schur stable, instead of the previous state consensus function $\xi(k)$, we take the following state consensus value $\xi = N^{-1}\{\overline{C}(I_{(N-1)n} - \overline{A})^{-1}\widetilde{T} + \mathbf{1}_N^T \otimes I_n\}x(0)$. Since $\widetilde{T} = \widetilde{T}_0 \otimes I_n$, $\overline{A} = I_{(N-1)n} - \widetilde{T}_0 L_w \widehat{T}_0 \otimes K_2$, and $\overline{C} = -\mathbf{1}_N^T L_w \widehat{T}_0 \otimes K_2$, and noticing that $(\widetilde{T}_0 L_w \widehat{T}_0 \otimes K_2)^{-1} = (\widetilde{T}_0 L_w \widehat{T}_0)^{-1} \otimes K_2^{-1}$, the previous state consensus value can be written into (17). □

Proof of Corollary 17. Since

$$A^j = \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix} \otimes I_n, \tag{A.4}$$

Equation (11) becomes

$$\begin{aligned}
\xi(k) &= N^{-1}\Bigg(\Bigg(\sum_{j=0}^{k-1}\begin{bmatrix} I_n & jI_n \\ 0 & I_n \end{bmatrix}\overline{C}\,\overline{A}^{k-1-j}\Bigg)\widetilde{T}\\
&\quad + \mathbf{1}_N^T \otimes \begin{bmatrix} I_n & kI_n \\ 0 & I_n \end{bmatrix}\Bigg)x(0)\\
&= N^{-1}\Bigg(\Bigg(\overline{C}\sum_{j=0}^{k-1}\overline{A}^{k-1-j} + \sum_{j=0}^{k-1}\begin{bmatrix} 0 & jI_n \\ 0 & 0 \end{bmatrix}\overline{C}\,\overline{A}^{k-1-j}\Bigg)\widetilde{T}\\
&\quad + \mathbf{1}_N^T \otimes \begin{bmatrix} I_n & kI_n \\ 0 & I_n \end{bmatrix}\Bigg)x(0).
\end{aligned} \tag{A.5}$$

Letting

$$X = \sum_{j=0}^{k-1}\begin{bmatrix} 0 & jI_n \\ 0 & 0 \end{bmatrix}\overline{C}\,\overline{A}^{k-1-j} \tag{A.6}$$

and $j' = j + 1$, we get

$$\begin{aligned}
X &= \sum_{j'=1}^{k}\begin{bmatrix} 0 & (j'-1)I_n \\ 0 & 0 \end{bmatrix}\overline{C}\,\overline{A}^{k-1-j'+1}\\
&= \sum_{j'=1}^{k}\begin{bmatrix} 0 & j'I_n \\ 0 & 0 \end{bmatrix}\overline{C}\,\overline{A}^{k-1-j'+1} - \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}\overline{C}\left(I_{2(N-1)n} - \overline{A}^k\right)\\
&= \sum_{j'=0}^{k-1}\begin{bmatrix} 0 & j'I_n \\ 0 & 0 \end{bmatrix}\overline{C}\,\overline{A}^{k-1-j'+1} + \begin{bmatrix} 0 & kI_n \\ 0 & 0 \end{bmatrix}\overline{C}\\
&\quad - \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}\overline{C}\left(I_{2(N-1)n} - \overline{A}^k\right)\left(I_{2(N-1)n} - \overline{A}\right)^{-1}
\end{aligned}$$

$$= X\overline{A} + \begin{bmatrix} 0 & kI_n \\ 0 & 0 \end{bmatrix}\overline{C} - \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}\overline{C}\left(I_{2(N-1)n} - \overline{A}^k\right) \times \left(I_{2(N-1)n} - \overline{A}\right)^{-1}. \tag{A.7}$$

Thus, we have the following:

$$X = \begin{bmatrix} 0 & kI_n \\ 0 & 0 \end{bmatrix}\overline{C}\left(I_{2(N-1)n} - \overline{A}\right)^{-1} - \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}\overline{C}\left(I_{2(N-1)n} - \overline{A}^k\right)\left(I_{2(N-1)n} - \overline{A}\right)^{-2}. \tag{A.8}$$

Substituting it into (A.5), we get

$$\begin{aligned} \xi(k) &= N^{-1}\left\{\left(\overline{C}\left(I_{2(N-1)n} - \overline{A}^k\right)\left(I_{2(N-1)n} - \overline{A}\right)^{-1}\right.\right. \\ &\quad + \begin{bmatrix} 0 & kI_n \\ 0 & 0 \end{bmatrix}\overline{C}\left(I_{2(N-1)n} - \overline{A}\right)^{-1} \\ &\quad \left. - \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}\overline{C}\left(I_{2(N-1)n} - \overline{A}^k\right)\left(I_{2(N-1)n} - \overline{A}\right)^{-2}\right)\widetilde{T} \\ &\quad \left. + \mathbf{1}_N^T \otimes \begin{bmatrix} I_n & kI_n \\ 0 & I_n \end{bmatrix}\right\} x(0). \end{aligned} \tag{A.9}$$

If $\overline{A}$ is Schur stable, the consensus function above can be replaced by

$$\begin{aligned} \xi(k) &= N^{-1}\left\{\left(\overline{C}\left(I_{2(N-1)n} - \overline{A}\right)^{-1} + \begin{bmatrix} 0 & kI_n \\ 0 & 0 \end{bmatrix}\overline{C}\left(I_{2(N-1)n} - \overline{A}\right)^{-1}\right.\right. \\ &\quad \left. - \begin{bmatrix} 0 & I_n \\ 0 & 0 \end{bmatrix}\overline{C}\left(I_{2(N-1)n} - \overline{A}\right)^{-2}\right)\widetilde{T} \\ &\quad \left. + \mathbf{1}_N^T \otimes \begin{bmatrix} I_n & kI_n \\ 0 & I_n \end{bmatrix}\right\} x(0). \end{aligned} \tag{A.10}$$

□

Proof of Theorem 20. Supposed that the matrices L_w and K_1 have been given, by Corollary 11, we need only to verify that the matrix K_2 ensures that all the matrices $A + BK_1 - \lambda_i BK_2$ are Schur stable, where $\lambda_i, i = 1, \ldots, N-1$, are the eigenvalues of the matrix $\widetilde{T}_0 L_w \widehat{T}_0$.

It is clear that there exists $\omega > 0$ such that $|\delta_i(\omega)| \le \delta(\omega) < \delta_c$, $i = 1, \ldots, N-1$. We can verify that the following systems $\xi_i(k+1) = (A + BK_1 - \lambda_i BK_2)\xi_i(k)$, $i = 1, \ldots, N-1$, admit a common Lyapunov function $V(\xi_i) = \xi_i^H P \xi_i$. In fact, let $K_2 = \omega(B^T PB)^{-1}B^T P(A + BK_1)$, then we have that

$$\begin{aligned} \Delta V(\xi_i) &= V(\xi_i(k+1)) - V(\xi_i(k)) \\ &= \xi_i^H(k)\left(\left((A + BK_1) - \lambda_i\omega B\left(B^T PB\right)^{-1}B^T P(A + BK_1)\right)^H\right. \\ &\quad \times P\left((A + BK_1) - \lambda_i\omega B(B^T PB)^{-1}B^T P\right. \\ &\quad \left.\left. \times (A + BK_1)\right) - P\right)\xi_i(k) \\ &= \xi_i^H(k)\left((A + BK_1)^T P(A + BK_1) - P\right. \\ &\quad - \left(\left(\lambda_i^H + \lambda_i\right)\omega + \lambda_i^H\lambda_i\omega^2\right) \\ &\quad \left.\times(A + BK_1)^T PB\left(B^T PB\right)^{-1}B^T P(A + BK_1)\right)\xi_i(k) \\ &= \xi_i^H(k)\left((A + BK_1)^T P(A + BK_1) - P\right. \\ &\quad - \left(1 - |\delta_i(\omega)|^2\right)(A + BK_1)^T PB\left(B^T PB\right)^{-1} \\ &\quad \left.\times B^T P(A + BK_1)\right)\xi_i(k) \\ &\le \xi_i^H(k)\left((A + BK_1)^T P(A + BK_1) - P\right. \\ &\quad - (1 - \delta_i(\omega))(A + BK_1)^T PB\left(B^T PB\right)^{-1} \\ &\quad \left.\times B^T P(A + BK_1)\right)\xi_i(k) \\ &< 0. \end{aligned} \tag{A.11}$$

That is, all the matrices $A + BK_1 - \lambda_i BK_2$, $i = 1, \ldots, N-1$, are Schur stable. □

Proof of Theorem 23. Let $\widetilde{x}_i = x_i - h_i$, $i = 1, \ldots, N$. Then the DLMASs (1) reach the formation h if and only if $\|(\widetilde{x}_i(k) - \widetilde{x}_j(k)\| \to \infty$ as $k \to \infty$, for all $i, j = 1, \ldots, N$. So the formation problem on the variables x_i is transformed to the consensus problem on the variables $\widetilde{x}_i$.

From (1), one gets $\widetilde{x}_i^+ = x_i^+ - h_i = A(\widetilde{x}_i + h_i) + Bu_i - h_i$, $i = 1, \ldots, N$. Substituting the protocol (24) and writing it into the vector form, we have that

$$\widetilde{x}^+ = \left(I_N \otimes (A + BK_1) - L_w \otimes BK_2\right)\widetilde{x} + \left(I_N \otimes (A + BK_1 - I_n)\right)h. \tag{A.12}$$

Introducing the state linear transformation $\overline{x} = T\widetilde{x}$ for the linear system (A.12), one obtains

$$\overline{x}^+ = T\left(I_N \otimes (A + BK_1) - L_w \otimes BK_2\right)T^{-1}\overline{x} + T\left(I_N \otimes (A + BK_1 - I_n)\right)h. \tag{A.13}$$

Let $y = [\overline{x}_1^T, \ldots, \overline{x}_{N-1}^T]^T$, then

$$y^+ = \left(I_{N-1} \otimes (A + BK_1) - \widetilde{T}_0 L_w \widehat{T}_0 \otimes BK_2\right)y + \left(\widetilde{T}_0 \otimes I_N\right)\left(I_N \otimes (A + BK_1 - I_n)\right)h \tag{A.14}$$

is held. Since $y = 0$ must be the equilibrium point if the DLMASs (1) reach the formation h, one has $(\widetilde{T}_0 \otimes (A + BK_1 - I_n))h = 0$. The residuary proof is similar to that of Theorem 7. □

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