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Research Article

On Multiple Convolutions and Time Scales

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The properties of the multiple Laplace transform and convolutions on a time scale are studied. Further, some related results are also obtained by utilizing the double Laplace transform. We also provide an example in order to illustrate the main result.

1. Introduction

A time scale $\top$ is a nonempty closed subset of the real numbers $\mathbb{R}$ and it has the topology that it inherits from the real numbers with the standard topology. It is also well known that if the time scale is the set of real numbers, then the dynamic equation is a differential equation, whilst if the time scale is the set of integers, then the dynamic equation is reduced to a difference equation. Thus integral transform of differential and integral calculus on time scales allows us to develop a theory of differential equations. For the single Laplace transform on time scales, see [1–3]. Similarly, in [4–8], the authors defined the time scale to be any nonempty closed subset of the real numbers and provided the motivation and formulation of delta derivatives on a time scale as well as the properties of delta derivatives and integrals. In [2], Bohner and Peterson defined the Laplace transform of a time scale for function of single variable as follows.

Definition 1. For $f : \top \to \mathbb{C}$, the time scale or generalized Laplace transform of f, denoted by $L[f(t)]$ or $F(z)$, is given by

$$L\left[f(t)\right](z) = \int_0^\infty e^\sigma_{\ominus z}(t,0)\, f(t)\,\Delta t, \quad \text{for } z \in D\{f\}, \quad (1)$$

where $D\{f\}$ consists of all $z \in \mathbb{C}$ for which the improper integral exists and $1 + \mu_1 z \neq 0$ for all $t \in \top$.

Definition 2. The function $f : \top \to \mathbb{R}$ is said to be of exponential type I if there exist constants $M, c > 0$ such that $|f(t)| \leq Me^{ct}$. Furthermore, f is said to be of exponential type II if there exist constants M, $c > 0$ such that $|f(t)| \leq Me_c(t,0)$.

The following two theorems were proved in [3].

Theorem 3. *The integral $\int_0^\infty e^\sigma_{\ominus z}(t,0) f(t)\Delta t$ converges absolutely for $z \in D$ if $f(t)$ is of exponential type II with exponential constant c. For more details, see [3].*

Theorem 4. *Let $\alpha \in \mathbb{R}$ be regressive. Then*

(I)

$$L\left\{e_\alpha(t,0)\right\}(z) = \frac{1}{z-\alpha}, \quad (2)$$

provided that $\lim_{t\to\infty} e_{\alpha\ominus z}(t,0) = 0$,

(II)

$$L\left\{\cos_\alpha(t,0)\right\}(z) = \frac{z}{z^2+\alpha^2}, \quad (3)$$

provided that $\lim_{t\to\infty} e_{i\alpha\ominus z}(t,0) = \lim_{t\to\infty} e_{-i\alpha\ominus z}(t,0) = 0$,

(III)

$$L\left\{\sin_\alpha(t,0)\right\}(z) = \frac{\alpha}{z^2+\alpha^2}, \quad (4)$$

provided that $\lim_{t\to\infty} e_{i\alpha\ominus z}(t,0) = \lim_{t\to\infty} e_{-i\alpha\ominus z}(t,0) = 0$.

The following theorem was also studied by Bohner and Peterson in [2].

Theorem 5. *If $f:\mathbb{T}\to\mathbb{R}$ is such that f^{Δ} is regulated, then*

$$L\left[f^{\Delta}\right](z)=zL\left[f(t)\right](z)-f(0), \tag{5}$$

for all $z\in D\{f\}$ such that $\lim_{t\to\infty}\{e_{\ominus z}(t)f(t)\}=0$.

Throughout the study, the following notation will be useful, where $\mathbb{T}_1,\mathbb{T}_2$ are time scales and

$$\begin{aligned}e_{a\oplus b}(t_1,t_2)&=e_b(t_1)e_b(t_2),\\ e_{\ominus z_1\ominus z_2}(t_1,t_2)&=e_{\ominus z_1}(t_1)e_{\ominus z_2}(t_2).\end{aligned} \tag{6}$$

In [2], further it was proved that

$$e^{\sigma}_{\ominus z}=\frac{e_{\ominus z_1}}{(1+\mu z)} \tag{7}$$

was satisfied, where $1+\mu_1 z\neq 0$, by using the property

$$e^{\sigma}_{z}=(1+\mu z)e_z,\qquad \ominus z=-\frac{z}{1+z},\qquad p\ominus q=\frac{p-q}{(1+\mu q)}. \tag{8}$$

Later in [9], Bohner and Guseinov defined the single convolution operation on time scale of two functions $f,g:\mathbb{T}\to\mathbb{R}$ by the following formula:

$$(f*g)(t)=\int_{t_0}^{t}\widehat{f}(t,\sigma(s))g(s)\Delta s. \tag{9}$$

In this study, we introduce double Laplace transform on time scales and study some of the properties in solving partial differential equations. Further, we also extend the Δ-Laplace transform, which is given by Jackson in [10], to the multiple Δ-Laplace transform as follows.

Definition 6. Let $0\in\mathbb{T}_1,\mathbb{T}_2$, $\sup\{\mathbb{T}_1,\mathbb{T}_2\}=\infty$, and $t_1\in\mathbb{T}_1,t_2\in\mathbb{T}_2$. The Δ-double Laplace transform of the function $f(t_1,t_2)$ (for $f\in C(\mathbb{T}_1\times\mathbb{T}_2)$) with respect to t_1,t_2 is given by

$$\begin{aligned}&L_{t_1}L_{t_2}\left[f(t_1,t_2)\right](z_1,z_2)\\ &\quad=\iint_0^{\infty}e^{\sigma_1\sigma_2}_{\ominus z_1\ominus z_2}(t_1,t_2)f(t_1,t_2)\Delta_1t_1\Delta_2t_2\end{aligned} \tag{10}$$

provided that the integral exists, further $(1+\mu_1z_1)\neq 0$, and $(1+\mu_2z_2)\neq 0$, for all $(t_1,t_2)\in(\mathbb{T}_1\times\mathbb{T}_2)_+$.

Lemma 7. *If $z_1,z_2\in\mathbb{C}$ are regressive, then*

$$e^{\sigma_1\sigma_2}_{\ominus z_1\ominus z_2}=\frac{e_{\ominus z_1\ominus z_2}}{(1+\mu_1z_1)(1+\mu_2z_2)}. \tag{11}$$

Proof. By using the above analysis, we have

$$\begin{aligned}e^{\sigma_1\sigma_2}_{\ominus z_1\ominus z_2}&=((1+\mu_1(\ominus z_1)))e_{\ominus z_1}(1+\mu_2(\ominus z_2))e_{\ominus z_2}\\ &=\left[1-\mu_1\frac{z_1}{1+\mu_1z_1}\right]\left[1-\mu_2\frac{z_2}{1+\mu_2z_2}\right]e_{\ominus z_1e_{\ominus z_2}}\\ &=\frac{e_{\ominus z_1\ominus z_2}}{(1+\mu_1z_1)(1+\mu_2z_2)}.\end{aligned} \tag{12}$$

□

As an example, we can easily compute $L_{t_2}L_{t_1}[e_{a\oplus b}(t_1,t_2,0,0)](z_1,z_2)$, where a,b are constants such that $\lim_{t_1\to\infty,\,t_2\to\infty}e_{a\oplus b\ominus z_1\ominus z_2}(t_1,t_2,0,0)=0$.

We have

$$\begin{aligned}&L_{t_2}L_{t_1}\left[e_{a\oplus b}(t_1,t_2,0,0)\right](z_1,z_2)\\ &=\iint_0^{\infty}e_{a\oplus b}(t_1,t_2,0,0)e^{\sigma_1\sigma_2}_{\ominus z_1\ominus z_2}(t_1,t_2)\Delta_1t_1\Delta_2t_2\\ &=\iint_0^{\infty}\left(1+\mu_{\sigma_1}(t_1)\ominus z_1(t_1)\right)\left(1+\mu_{\sigma_2}(t_2)\ominus z_2(t_2)\right)e_{a\oplus b}\\ &\qquad\times(t_1,t_2,0,0)e_{\ominus z_1\ominus z_2}(t_1,t_2)\Delta_1t_1\Delta_2t_2\\ &=\int_0^{\infty}\left(1-\frac{\mu_{\sigma_1}(t_1)z_1}{1+\mu_{\sigma_1}\ominus z_1}\right)\\ &\quad\times\left[\int_0^{\infty}\left(1-\frac{\mu_{\sigma_1}(t_2)z_2}{1+\mu_{\sigma_2}\ominus z_2}\right)e_{b\ominus z_2}(t_1,t_2,0,0)\Delta_2t_2\right]e_{a\ominus z_1}\\ &\quad\times(t_1,t_2)\Delta_1t_1\\ &=\int_0^{\infty}\left(\frac{1}{1+\mu_{\sigma_1}\ominus z_1}\right)\\ &\quad\times\left[\int_0^{\infty}\left(\frac{1}{1+\mu_{\sigma_2}\ominus z_2}\right)e_{b\ominus z_2}(t_1,t_2,0,0)\Delta_2t_2\right]\\ &\quad\times e_{a\ominus z_1}(t_1,t_2,0,0)\Delta_1t_1\\ &=\frac{1}{a-z_1}\int_0^{\infty}\left[\frac{1}{b-z_2}\int_0^{\infty}b\ominus z_2e_{b\ominus z_2}(t_1,t_2,0,0)\Delta_2t_2\right]\\ &\qquad\times a\ominus z_1e_{a\ominus z_1}(t_1,t_2,0,0)\Delta_1t_1\\ &=\frac{1}{a-z_1}\int_0^{\infty}\left[\frac{1}{b-z_2}\int_0^{\infty}\left(e_{b\ominus z_2}(t_1,t_2,0,0)\right)_{\Delta_2t_2}\Delta_2t_2\right]\\ &\qquad\times\left(e_{a\ominus z_1}(t_1,t_2,0,0)\right)_{\Delta_1t_1}\Delta_1t_1\\ &=\frac{1}{(z_1-a)(z_2-b)}.\end{aligned} \tag{13}$$

For $t_1\in\mathbb{T}_1$ and $t_2\in\mathbb{T}_2$ with $t_1\geq\alpha$ and $t_2\geq\beta$, we can easily see that

$$\begin{aligned}\cosh_{(\alpha\oplus\beta)}(t_1,t_2)&=\frac{e_{(\alpha t_1\oplus\beta t_2)}+e_{-(\alpha t_1\oplus\beta t_2)}}{2},\\ \sinh_{(\alpha\oplus\beta)}(t_1,t_2)&=\frac{e_{(\alpha t_1\oplus\beta t_2)}-e_{-(\alpha t_1\oplus\beta t_2)}}{2},\\ \cos_{(\alpha\oplus\beta)}(t_1,t_2)&=\frac{e_{i(\alpha t_1\oplus\beta t_2)}+e_{-i(\alpha t_1\oplus\beta t_2)}}{2},\\ \sin_{(\alpha\oplus\beta)}(t_1,t_2)&=\frac{e_{i(\alpha t_1\oplus\beta t_2)}-e_{-i(\alpha t_1\oplus\beta t_2)}}{2i},\end{aligned} \tag{14}$$

and then the double Laplace transforms are given by

$$L_{t_2}L_{t_1}\left[\cosh_{(\alpha\oplus\beta)}(t_1,t_2)\right] = \frac{1/((z_1-\alpha)(z_2-\beta))+1/((z_1+\alpha)(z_2+\beta))}{2} = \frac{z_1z_2+\alpha\beta}{(z_2^2-\beta^2)(z_1^2-\alpha^2)}, \tag{15}$$

$$L_{t_2}L_{t_1}\left[\sinh_{(\alpha\oplus\beta)}(t_1,t_2)\right] = \frac{1/((z_1-\alpha)(z_2-\beta))-1/((z_1+\alpha)(z_2+\beta))}{2} = \frac{\beta z_1+\alpha z_2}{(z_2^2-\beta^2)(z_1^2-\alpha^2)}. \tag{16}$$

Similarly, we have

$$L_{t_2}L_{t_1}\left[\cos_{(\alpha\oplus\beta)}(t_1,t_2)\right] = \frac{z_1z_2-\alpha\beta}{(z_2^2+\beta^2)(z_1^2+\alpha^2)}, \quad L_{t_2}L_{t_1}\left[\sin_{(\alpha\oplus\beta)}(t_1,t_2)\right] = \frac{z_1z_2+\alpha\beta}{(z_2^2+\beta^2)(z_1^2+\alpha^2)}. \tag{17}$$

Proposition 8. *Suppose* $\mathbb{T}_1$, $\mathbb{T}_2$ *are two time scales such that* $\mu_1 < N_1$ *and* $\mu_1 < N_2$ *for some* $N_1, N_2 \in \mathbb{R}$ *and all* $(t_1,t_2) \in \mathbb{T}_1\times\mathbb{T}_2$. *Let* $f:(\mathbb{T}_1\times\mathbb{T}_2)_+ \to \mathbb{R}$ *and*

$$\lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{e_{\ominus z_1\ominus z_2}(t_1,t_2)\, f(\sigma(t_1),\sigma(t_2),0,0)\right\} = 0. \tag{18}$$

Then

$$\lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{e_{\ominus z_1\ominus z_2}(t_1,t_2)\, f(t_1,t_2,0,0)\right\} = 0. \tag{19}$$

Proof. Since

$$\lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{e_{\ominus z_1\ominus z_2}(t_1,t_2)\, f(\sigma(t_1),\sigma(t_2),0,0)\right\} = 0 \tag{20}$$

and on using the following relation:

$$e_{\ominus p_1\ominus p_2}(\sigma(t_1),\sigma(t_2),0,0) = \frac{1}{(1+\mu_1p_1)(1+\mu_2p_2)}e_{\ominus p_1\ominus p_2}(t_1,t_2,0,0) = \frac{\ominus p_1\ominus p_2}{p_1p_2}e_{\ominus p_1\ominus p_2}(t_1,t_2,0,0), \tag{21}$$

we have

$$\lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{\frac{\ominus z_1(t_1)\ominus z_2(t_2)}{z_1z_2}e_{\ominus z_1\ominus z_2}(t_1,t_2)\, f(t_1,t_2,0,0)\right\} = 0,$$

$$\lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{\frac{\ominus z_1(t_1)\ominus z_2(t_2)}{z_1z_2(1+\mu_1z_1)(1+\mu_2z_2)}e_{\ominus z_1\ominus z_2} \times (t_1,t_2)\, f(t_1,t_2,0,0)\right\} = 0,$$

$$\lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{\frac{1}{(1+\mu_1z_1)(1+\mu_2z_2)}e_{\ominus z_1\ominus z_2} \times (t_1,t_2)\, f(t_1,t_2,0,0)\right\} = 0. \tag{22}$$

Since $\mu_1(t_2) < N_1$ and $\mu_1(t_2) < N_2$, we get

$$\lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left|\left\{\frac{e_{\ominus z_1\ominus z_2}(t_1,t_2)\, f(t_1,t_2,0,0)}{(1+\mu_1z_1)(1+\mu_2z_2)}\right\}\right| > \left|\frac{\lim_{t_1\to\infty,\, t_2\to\infty}\left\{e_{\ominus z_1\ominus z_2}(t_1,t_2)\, f(t_1,t_2,0,0)\right\}}{(1+N_1z_1)(1+N_2z_2)}\right| > 0. \tag{23}$$

Thus we obtain that

$$\frac{1}{(1+N_1z_1)(1+N_2z_2)} \times \lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{e_{\ominus z_1\ominus z_2}(t_1,t_2)\, f(t_1,t_2,0,0)\right\} = 0 \tag{24}$$

and it follows that

$$\lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{e_{\ominus z_1\ominus z_2}(t_1,t_2)\, f(t_1,t_2,0,0)\right\} = 0. \tag{25}$$

□

Theorem 9. *Let* $f:(\mathbb{T}_1\times\mathbb{T}_2)_+ \to \mathbb{R}$ *be regulated and let*

$$F(t_1,t_2) = \iint_0^{t_1} f(\zeta,\eta)\,\Delta\zeta\,\Delta\eta \tag{26}$$

for $(t_1,t_2)\in(\mathbb{T}_1\times\mathbb{T}_2)_+$. *Then*

$$L_{t_1}L_{t_2}\left[f(t_1,t_2)\right](z_1,z_2) = \frac{1}{z_1z_2}L_{t_1}L_{t_2}\left[f(t_1,t_2)\right](z_1,z_2) - \lim_{\substack{t_1\to\infty\\ t_2\to\infty}} e_{\ominus z_1\ominus z_2}(t_1,t_2)\, f(t_1,t_2,0,0), \tag{27}$$

for $z_1, z_2 \neq 0$.

Proof. By using the definition of the double Laplace transform on a time scale, we have

$$L_{t_1}L_{t_2}\left[f\left(t_1,t_2\right)\right]\left(z_1,z_2\right) = \iint_0^\infty e^{\sigma_1\sigma_2}_{\ominus z_1\ominus z_2}\left(t_1,t_2\right)F\left(t_1,t_2,0,0\right)\Delta_1 t_1\Delta_2 t_2, \tag{28}$$

and on using (21), we have

$$L_{t_1}L_{t_2}\left[f\left(t_1,t_2\right)\right]\left(z_1,z_2\right) = \frac{1}{z_1z_2}\iint_0^\infty e_{\ominus z_1\ominus z_2}\left(t_1,t_2\right)\left(\ominus z_1\ominus z_2\right) \times F\left(t_1,t_2,0,0\right)\Delta_1 t_1\Delta_2 t_2. \tag{29}$$

Now, applying integration by parts, and on using the fact that $F(0,0) = 0$ together with the fundamental theorem of calculus, we obtain that

$$L_{t_1}L_{t_2}\left[f\left(t_1,t_2\right)\right]\left(z_1,z_2\right) = \frac{1}{z_1z_2}L_{t_1}L_{t_2}\left[f\left(t_1,t_2\right)\right]\left(z_1,z_2\right) - \lim_{\substack{t_1\to\infty\\ t_2\to\infty}} e_{\ominus z_1\ominus z_2}\left(t_1,t_2\right)f\left(t_1,t_2,0,0\right). \tag{30}$$

□

Example 10. Let $a\in\mathbb{T}_1$, $b\in\mathbb{T}_2$, $a,b>0$, and $H_{a,b}(t_1,t_2)=0$ for $t_1\in\mathbb{T}_1$ and $t_2\in\mathbb{T}_2$, $t_1<a$, and $t_2<b$ whilst $H_{a,b}(t_1,t_2)=1$ for $t_1\in\mathbb{T}_1$, $t_2\in\mathbb{T}_2$ and $t_1\ge a$, $t_2\ge b$, where $H_{a,b}(t_1,t_2)=H_{a,}(t_1)\otimes H_b(t_2)$ and the symbol $\otimes$ denotes the tensor product. Then we have

$$L_{t_1}L_{t_2}\left[H_{a,b}\left(t_1,t_2\right)\right]\left(z_1,z_2\right) = \iint_0^\infty e^{\sigma_1\sigma_2}_{\ominus z_1\ominus z_2}\left(t_1,t_2\right)H_{a,b}\left(t_1,t_2\right)\Delta_1 t_1\Delta_2 t_2 = \frac{e_{\ominus z_1}(a)\,e_{\ominus z_2}(b)}{z_1z_2}. \tag{31}$$

Now assume that $f(t_1,t_2):\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{C}$ such that f^{Δ_1} is continuous. Then the Δ-double Laplace transform

$$L_{t_1}L_{t_2}\left[f^{\Delta_1}\left(t_1,t_2\right)\right]\left(z_1,z_2\right) = z_1\,\overline{\overline{F}}\left(z_1,z_2\right) - \overline{F}\left(0,z_2\right) \tag{32}$$

holds for those regressive $z_1,z_2\in\mathbb{C}$ with respect to t_1 and t_2 which further satisfies

$$\lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{e_{\ominus z_1\ominus z_2}\left(t_1,t_2\right)f\left(t_1,t_2\right)\right\} = 0. \tag{33}$$

We next extend the result that was proved by Davis et al. in [3].

Proposition 11. *Let $(a,b)\in(\mathbb{T}_1\times\mathbb{T}_2)_+$, $a,b>0$. Assume that $f(t_1,t_2)$ is one of the following functions:*

$$e_{a\oplus b}\left(t_1,t_2,0,0\right),\quad \sinh_{(\alpha\oplus\beta)}\left(t_1+t_2\right),\quad \cosh_{(\alpha\oplus\beta)}\left(t_1+t_2\right),\quad \cos_{(\alpha\oplus\beta)}\left(t_1+t_2\right),\quad \text{or } \sin_{(\alpha\oplus\beta)}\left(t_1+t_2\right). \tag{34}$$

If z_1,z_2,α, and β are regressive and satisfy

$$\lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{e_{\alpha\ominus z_1\beta\ominus z_2}\left(t_1,t_2\right)f\left(t_1,t_2,a,b\right)\right\} = \lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{e_{i\alpha\ominus z_1 i\beta\ominus z_2}\left(t_1,t_2\right)f\left(t_1,t_2,a,b\right)\right\} = \lim_{\substack{t_1\to\infty\\ t_2\to\infty}}\left\{e_{-i\alpha\ominus z_1-i\beta\ominus z_2}\left(t_1,t_2\right)f\left(t_1,t_2,a,b\right)\right\}, \tag{35}$$

then

$$L_{t_1}L_{t_2}\left[H_{a,b}\left(t_1,t_2\right)f\left(t_1,t_2,a,b\right)\right]\left(z_1,z_2\right) = e_{\ominus z_1\ominus z_2}(a,b,0,0)\,L_{t_1}L_{t_2}\left[f\left(t_1,t_2,0,0\right)\right]. \tag{36}$$

Proof. Let us study the case $f(t_1,t_2,a,b) = e_{\alpha\oplus\beta}(t_1,t_2,a,b)$. By using (21), we get

$$e_{\alpha\oplus\beta}\left(t_1,t_2,a,b\right)e_{\ominus z_1\ominus z_2}\left(\sigma\left(t_1\right),\sigma\left(t_2\right),0,0\right) = \frac{e_{\alpha\oplus\beta}\left(t_1,t_2,a,b\right)e_{\ominus z_1\ominus z_2}\left(t_1,t_2,0,0\right)}{\left(1+\mu_1z_1\right)\left(1+\mu_2z_2\right)} \tag{37}$$

and by using Theorem 4 and (21) the right hand side of equation (37) is

$$= \frac{e_{\ominus z_1\ominus z_2}(a,b,0,0)}{\left(\alpha-z_1\right)\left(\beta-z_2\right)}\left(\frac{\left(\alpha-z_1\right)\left(\beta-z_2\right)}{\left(1+\mu_1z_1\right)\left(1+\mu_2z_2\right)}\right) \times e_{\alpha\ominus z_1\beta\ominus z_2}\left(t_1,t_2,a,b\right) = \frac{e_{\ominus z_1\ominus z_2}(a,b,0,0)}{\left(\alpha-z_1\right)\left(\beta-z_2\right)}\left(\alpha\ominus z_1\right)\left(\beta\ominus z_2\right)e_{\alpha\ominus z_1\beta\ominus z_2}\left(t_1,t_2,a,b\right). \tag{38}$$

Then

$$L_{t_1}L_{t_2}\left[H_{a,b}\left(t_1,t_2\right)f\left(t_1,t_2,a,b\right)\right]\left(z_1,z_2\right) = \iint_0^\infty e_{\ominus z_1\ominus z_2}\left(\sigma\left(t_1\right),\sigma\left(t_2\right),0,0\right)H_{a,b}\left(t_1,t_2\right)e_{\alpha\oplus\beta} \times \left(t_1,t_2,a,b\right)\Delta_1 t_1\Delta_2 t_2$$
$$= \frac{e_{\ominus z_1\ominus z_2}(a,b,0,0)}{\left(\alpha-z_1\right)\left(\beta-z_2\right)}\iint_0^\infty H_{a,b}\left(t_1,t_2\right) \times \left(\left(\alpha\ominus z_1\right)\left(\beta\ominus z_2\right)\right)\left(t_1,t_2\right) \times e_{\alpha\ominus z_1\beta\ominus z_2}\left(t_1,t_2,a,b\right) \times \Delta_1 t_1\Delta_2 t_2$$

$$
\begin{aligned}
&= \frac{e_{\ominus z_1 \ominus z_2}(a,b,0,0)}{(\alpha - z_1)(\beta - z_2)} \int_b^\infty \int_a^\infty ((\alpha \ominus z_1)(\beta \ominus z_2)) \\
&\qquad \times e_{\alpha \ominus z_1 \beta \ominus z_2}(t_1,t_2,a,b) \\
&\qquad \times \Delta_1 t_1 \Delta_2 t_2 \\
&= \frac{e_{\ominus z_1 \ominus z_2}(a,b,0,0)}{(\alpha - z_1)(\beta - z_2)} \\
&= e_{\ominus z_1 \ominus z_2}(a,b,0,0)\, L_{t_1} L_{t_2}\left[e_{\alpha \oplus \beta}(t_1,t_2,0,0)\right].
\end{aligned} \tag{39}
$$

□

Theorem 12. *Let $f(t_1,t_2) : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{C}$ and suppose that f^{Δ_i} and f^{Δ_j} are continuous for all $i = 1,2,\dots n$ and $j = 1,2,\dots,m$, respectively; then the Δ-double Laplace transforms for f^{Δ_i} and f^{Δ_j} are given by*

$$
\begin{aligned}
&L_{t_1} L_{t_2}\left[f^{\Delta_n^1}(t_1,t_2)\right](z_1,z_2) \\
&\quad = z_1^n \overline{\overline{F}}(z_1,z_2) - z_1^{n-1}\overline{F}(0,z_2) - \cdots - \overset{-\Delta_1^{n-1}}{F}(0,z_2),
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&L_{t_1} L_{t_2}\left[f^{\Delta_m^2}(t_1,t_2)\right](z_1,z_2) \\
&\quad = z_2^m \overline{\overline{F}}(z_1,z_2) - z_2^{m-1}\overline{F}(z_1,0) - \cdots - \overset{-\Delta_2^{m-1}}{F}(z_1,0),
\end{aligned} \tag{41}
$$

for all $z_1, z_2 \in \mathbb{C}$ with

$$
\begin{aligned}
&\lim_{t_1 \to \infty} \left\{e_{\ominus z_1 \ominus z_2}(t_1,t_2) f(t_1,t_2)\right\} \\
&\quad = \lim_{t_1 \to \infty} \left\{e_{\ominus z_1 \ominus z_2}(t_1,t_2)\ f^{\Delta_n^1}(t_1,t_2)\right\} \cdots \\
&\quad = \lim_{t_1 \to \infty} \left\{e_{\ominus z_1 \ominus z_2}(t_1,t_2)\ f^{\Delta_{n-1}^1}(t_1,t_2)\right\} = 0, \\
&\lim_{t_2 \to \infty} \left\{e_{\ominus z_1 \ominus z_2}(t_1,t_2) f(t_1,t_2)\right\} \\
&\quad = \lim_{t_2 \to \infty} \left\{e_{\ominus z_1 \ominus z_2}(t_1,t_2)\ f^{\Delta_m^2}(t_1,t_2)\right\} \cdots \\
&\quad = \lim_{t_2 \to \infty} \left\{e_{\ominus z_1 \ominus z_2}(t_1,t_2)\ f^{\Delta_{m-1}^2}(t_1,t_2)\right\} = 0,
\end{aligned} \tag{42}
$$

respectively.

Proof. Let $n = 1$. On using the definition of the Δ-single Laplace transform and integrating parts of the function $f^{\Delta_1^1}(t_1,t_2)$, we have

$$
\begin{aligned}
&L_{t_1}\left[f^{\Delta_1^1}(t_1,t_2)\right](z_1) \\
&\quad = \int_0^\infty e_{\ominus z_1}^{\sigma_1}\ f^{\Delta_1^1}(t_1,t_2)\,\Delta t_1 \\
&\quad = \left[f(t_1,t_2)\right]_0^\infty + z_1 \int_0^\infty e_{\ominus z_1}^{\sigma_1}\ f(t_1,t_2)\,\Delta t_1 \\
&\quad = -f(0,t_2) + z_1 \overline{F}(z_1,t_2).
\end{aligned} \tag{43}
$$

Then the single Laplace transform with respect to t_1 is given by

$$
L_{t_1}\left[f^{\Delta_1^1}(t_1,t_2)\right](z_1,z_2) = z_1 \overline{F}(z_1,t_2) - f(0,t_2). \tag{44}
$$

Similarly, Laplace transform for (44) with respect to t_2 is given by

$$
L_{t_2} L_{t_1}\left[f^{\Delta_1^1}(t_1,t_2)\right](z_1) = z_1 \overline{\overline{F}}(z_1,z_2) - \overline{F}(0,z_2). \tag{45}
$$

Then (45) is called the Δ-double Laplace transform for the function $f^{\Delta_1^1}(t_1,t_2)$. Similarly, if $m = 1$ in (41), the Δ-double Laplace transform for the function $f^{\Delta_1^2}(t_1,t_2)$ is given by

$$
L_{t_2} L_{t_1}\left[f^{\Delta_1^2}(t_1,t_2)\right](z_1,z_2) = z_2 \overline{\overline{F}}(z_1,z_2) - \overline{F}(z_1,0). \tag{46}
$$

Now, in order to obtain the Δ-double Laplace transform for the function

$$
f^{\Delta_2^1}(t_1,t_2) = \frac{\partial^2 f(t_1,t_2)}{\Delta_1 t_1^2} \tag{47}
$$

with respect to t_1, t_2, we proceed as follows: first of all by taking the Δ-single Laplace transform with respect to the variable t_1 and on using the fact that

$$
\begin{aligned}
&\lim_{t_1 \to \infty} \left\{e_{\ominus z_1}(t_1)\, f^{\Delta_1^1}(t_1,t_2)\right\} = 0, \\
&\lim_{t_1 \to \infty} \left\{e_{\ominus z_1}(t_1)\, f(t_1,t_2)\right\} = 0,
\end{aligned} \tag{48}
$$

we have

$$
\begin{aligned}
&L_{t_1}\left[f^{\Delta_2^1}(t_1,t_2)\right](z_1) \\
&\quad = \left[f^{\Delta_1^1}(t_1,t_2)\right]_0^\infty + z_1 \int_0^\infty e_{\ominus z_1}^{\sigma_1}\ f^{\Delta_1^1}(t_1,t_2)\,\Delta t_1 \\
&\quad = -f^{\Delta_1^1}(0,t_2) + z_1 \int_0^\infty e_{\ominus z_1}^{\sigma_1}\ f^{\Delta_1^1}(t_1,t_2)\,\Delta t_1.
\end{aligned} \tag{49}
$$

Thus on using (44), we obtain that

$$
\begin{aligned}
&L_{t_1}\left[f^{\Delta_2^1}(t_1,t_2)\right](z_1) \\
&\quad = z_1^2 \overline{F}(z_1,t_2) - z_1 \overline{F}(0,t_2) - f^{\Delta^1}(0,t_2).
\end{aligned} \tag{50}
$$

In a similar way, the single Laplace transform with respect to t_2 is given by

$$
\begin{aligned}
&L_{t_2} L_{t_1}\left[f^{\Delta_2^1}(t_1,t_2)\right](z_1,z_2) \\
&\quad = z_1^2 \overline{\overline{F}}(z_1,z_2) - z_1 \overline{F}(0,z_2) - \overline{F}^{\Delta^1}(0,z_2).
\end{aligned} \tag{51}
$$

Further, the Δ-double Laplace transform for the function $f^{\Delta_2^2}(t_1,t_2) = \partial^2 f(t_1,t_2)/\Delta_2 t_2^2$ with respect to t_1, t_2 is given by

$$
\begin{aligned}
&L_{t_2} L_{t_1}\left[\ f^{\Delta_2^2}(t_1,t_2)\right](z_1,z_2) \\
&\quad = z_2^2 \overline{\overline{F}}(z_1,z_2) - z_2 \overline{F}(z_1,0) - \overline{F}^{\Delta^1}(z_1,0).
\end{aligned} \tag{52}
$$

Finally, note that we can generalize the proof for n, m. □

Next, we define the double convolution of two time scale functions as follows.

Definition 13. The double convolution of the function $f, g : \mathbf{T}_1 \times \mathbf{T}_2 \to \mathbb{R}$ is given by

$$(f**g)(t_1,t_2) = \iint_0^{t_1} f(\zeta,\eta)\, g(t_1,t_2,\sigma_1(\zeta),\sigma_2(\eta))\, \Delta\zeta\, \Delta\eta. \tag{53}$$

In the next theorem we prove the properties of the double convolution on a time scale as follows.

Theorem 14. *Let $f, g, h : \mathbf{T}_1 \times \mathbf{T}_2 \to \mathbb{R}$ be integrable functions on a time scale. Then the following properties are satisfied:*

$$\begin{gathered}(f**g)(t_1,t_2) = (g**f)(t_1,t_2),\\ (f**g)**h = f**(g**h).\end{gathered} \tag{54}$$

Proof. We can easily show that convolution is commutative by using the definition of the double convolution

$$\begin{aligned}(f&**g)(t_1,t_2)\\ &= \iint_0^{t_1} f(\zeta,\eta)\, g(t_1,t_2,\sigma_1(\zeta),\sigma_2(\eta))\, \Delta\zeta\, \Delta\eta\\ &= \iint_0^{t_1} f(t_1,t_2,\sigma_1(\zeta),\sigma_2(\eta))\, g(\zeta,\eta)\, \Delta\zeta\, \Delta\eta.\end{aligned} \tag{55}$$

For the proof of the associative law, we also use the definition as follows:

$$\begin{aligned}(f&**g)**h\\ &= \iint_0^{t_1} (f**g)(t_1,t_2,\sigma_1(\zeta),\sigma_2(\eta))\, h(\zeta,\eta)\, \Delta\zeta\, \Delta\eta\\ &= \iint_0^{t_1} \int_{\sigma_2(\eta)}^{t_1} \int_{\sigma_1(\zeta)}^{t_1} f(t_1,t_2,\sigma_1(\alpha),\sigma_2(\beta))\\ &\qquad \times g(\alpha,\beta,\sigma_1(\zeta),\sigma_2(\eta))\\ &\qquad \times h(\zeta,\eta)\, \Delta\zeta\, \Delta\eta\, \Delta\alpha\, \Delta\beta\\ &= \iint_0^{t_1} f(t_1,t_2,\sigma_1(\alpha),\sigma_2(\beta))\, (f**g)(\alpha,\beta)\, \Delta\alpha\, \Delta\beta\\ &\qquad \times f**(g**h)(t_1,t_2).\end{aligned} \tag{56}$$

□

Theorem 15 (convolution theorem). *Let $f, g : \mathbf{T}_1 \times \mathbf{T}_2 \to \mathbb{C}$ be regulated functions. Then the double Laplace transform of the double convolution is given by*

$$L_{t_2} L_{t_1} [(f**g)(t_1,t_2); z_1, z_2] = F(z_1,z_2)\, G(z_1,z_2). \tag{57}$$

Proof. By using the definition of the double Laplace transform on a time scale, we obtain that

$$\begin{aligned}L_{t_2}&L_{t_1} [(f**g)(t_1,t_2); z_1, z_2]\\ &= \iint_0^{\infty} e^{\sigma_1\sigma_2}_{\ominus z_1 \ominus z_2}(t_1,t_2,0,0)\, [(f**g)(t_1,t_2)]\, \Delta_1 t_1 \Delta_2 t_2\\ &= \iint_0^{\infty} \Bigg[\iint_0^{t_1} f(\zeta,\eta)\, g(t_1,t_2,\sigma_1(\zeta),\sigma_2(\eta))\\ &\qquad \times \Delta\zeta\, \Delta\eta\, e^{\sigma_1\sigma_2}_{\ominus z_1 \ominus z_2}(t_1,t_2,0,0)\Bigg]\, \Delta_1 t_1 \Delta_2 t_2\\ &= \iint_0^{\infty} f(\zeta,\eta) \Bigg[\int_{\sigma_1(\zeta)}^{\infty} \int_{\sigma_2(\eta)}^{\infty} g(t_1,t_2,\sigma_1(\zeta),\sigma_2(\eta))^{\sigma_1\sigma_2}_{\ominus z_1 \ominus z_2}\\ &\qquad \times (t_1,t_2,0,0)\, \Delta_1 t_1 \Delta_2 t_2\Bigg]\, \Delta\zeta\, \Delta\eta\\ &= \iint_0^{\infty} f(\zeta,\eta) \big[L_{t_2} L_{t_1} \big[H_{\sigma_1(\zeta),\sigma_2(\eta)}(t_1,t_2)\\ &\qquad \times g(t_1,t_2,\sigma_1(\zeta),\sigma_2(\eta))\big]\big]\, \Delta\zeta\, \Delta\eta,\end{aligned} \tag{58}$$

where $H_{\sigma_1(\zeta),\sigma_2(\eta)}(t_1,t_2) = H_{\sigma_1(\zeta),}(t_1) \otimes H_{\sigma_2(\zeta),}(t_2)$ and the symbol $\otimes$ denotes the tensor product. Thus we have

$$\begin{aligned}L_{t_2}&L_{t_1} [(f**g)(t_1,t_2); z_1, z_2]\\ &= \iint_0^{\infty} f(\zeta,\eta) \big[G(z_1,z_2)^{\sigma_1\sigma_2}_{\ominus z_1 \ominus z_2}(\zeta,\eta,0,0)\big]\, \Delta\zeta\, \Delta\eta\\ &= G(z_1,z_2) \iint_0^{\infty} f(\zeta,\eta)^{\sigma_1\sigma_2}_{\ominus z_1 \ominus z_2}(\zeta,\eta,0,0)\, \Delta\zeta\, \Delta\eta\\ &= F(z_1,z_2)\, G(z_1,z_2).\end{aligned} \tag{59}$$

□

Finally, we give the solution to the wave equation in one dimension as follows. Consider the wave equation in the form of

$$u^{\Delta_1\Delta_1} - u^{\Delta_2\Delta_2} = 0, \tag{60}$$

under the conditions

$$\begin{gathered}u(x,0) = e_1(x),\, u(0,t) = e_1(t),\\ u^{\Delta_2}(0,t) = e_1(t),\, u^{\Delta_1}(x,0) = e_1(x).\end{gathered} \tag{61}$$

Then by formally taking the double Laplace transform, we get

$$\begin{aligned}L_{t_2}&L_{t_1} [u(t,x); z_1, z_2]\\ &= \frac{z_1+1}{(z_2-1)(z_1^2-z_2^2)} - \frac{z_2+1}{(z_1-1)(z_1^2-z_2^2)}.\end{aligned} \tag{62}$$

By using partial fractions, we have

$$\begin{aligned}
& L_{t_2}L_{t_1}\left[u(t,x);z_1,z_2\right] \\
&\quad = \frac{1}{2(z_2-1)(z_1-z_2)} + \frac{1}{2(z_2-1)(z_1+z_2)} \\
&\qquad + \frac{1}{2z_2(z_2-1)(z_1-z_2)} - \frac{1}{2z_2(z_2-1)(z_1+z_2)} \\
&\qquad - \frac{1}{2(z_1-1)(z_1-z_2)} + \frac{1}{2(z_1-1)(z_1+z_2)} \\
&\qquad - \frac{1}{2z_1(z_1-1)(z_1-z_2)} - \frac{1}{2z_1(z_1-1)(z_1+z_2)}.
\end{aligned} \tag{63}$$

Thus the solution to the above equation is given by

$$u(t,x) = e_{1\oplus 1}(t,x,0,0). \tag{64}$$

In particular, if we consider the case $\mathbf{T}_1 \times \mathbf{T}_2 = \mathbb{R}\times\mathbb{R}$, $\sigma(t) = t$, $\sigma(x) = x$, for all $(t,x) \in \mathbf{T}_1 \times \mathbf{T}_2$, then $e_{a\oplus b}(t,x,0,0) = e^{ax+bt}$ for any constant $a, b \in \mathbb{R}$ and so the solution is given by

$$u(t,x) = e^{t+x}. \tag{65}$$

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of paper.

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