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Research Article

Conservation Laws for a Generalized Coupled Korteweg-de Vries System

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We construct conservation laws for a generalized coupled KdV system, which is a third-order system of nonlinear partial differential equations. We employ Noether's approach to derive the conservation laws. Since the system does not have a Lagrangian, we make use of the transformation $u = U_x$, $v = V_x$ and convert the system to a fourth-order system in U, V. This new system has a Lagrangian, and so the Noether approach can now be used to obtain conservation laws. Finally, the conservation laws are expressed in the u, v variables, and they constitute the conservation laws for the third-order generalized coupled KdV system. Some local and infinitely many nonlocal conserved quantities are found.

1. Introduction

The generalized coupled KdV system given by [1]

$$\begin{aligned} u_t + au_{xxx} - buu_x + cvv_x &= 0, \\ v_t + dv_{xxx} - euv_x + fu_x v &= 0, \end{aligned} \tag{1}$$

where a, b, c, d, e, and f are real constants, describes the interaction of two long waves, whose dispersion relations are different. For the case when $f = 0$, soliton solutions have been obtained in [2, 3]. Many other special cases of (1) have been considered in the literature, and various methods have been used to find its exact solutions. See, for example, [4–11].

In this study, we consider a special case of the generalized coupled KdV system given by

$$\begin{aligned} u_t + au_{xxx} + buu_x + cvv_x &= 0, \\ v_t + dv_{xxx} + cuv_x + cu_x v &= 0 \end{aligned} \tag{2}$$

and construct conservation laws for (2). Recently, the conservation laws of system (2) for special values of the constants $a = d = -1$ and $b = c = -6$ were derived in [12] using the multiplier approach.

Many nonlinear partial differential equations (PDEs) of mathematical physics and engineering are continuity equations, which express conservation of mass, momentum, energy, or electric charge. It is well known that conservation laws play a crucial role in the solution and reduction of PDEs. For variational problems the conservation laws can be constructed by means of the Noether theorem [13]. The application of the Noether theorem depends upon the existence of a Lagrangian. However, there are nonlinear differential equations that do not have a Lagrangian. In such instances, researchers have developed several methods to derive conserved quantities for such equations. See, for example, [14–20].

The organization of this paper is as follows. In Section 2 we briefly recall some notations and fundamental relations concerning the Noether symmetries approach, which we utilize in the same section to obtain the Noether symmetries and the corresponding conserved vectors. The concluding remarks are summarized in Section 3.

2. Conservation Laws of Coupled KdV Equations

In this section we derive the conservation laws for the generalized coupled KdV system (2). This system does not have a Lagrangian. In order to apply the Noether theorem we transform our system (2) to a fourth-order system, using the transformations $u = U_x$ and $v = V_x$. Then system (2) becomes

$$\begin{gathered} U_{tx} + aU_{xxxx} + bU_x U_{xx} + cV_x V_{xx} = 0, \\ V_{tx} + dV_{xxxx} + cU_x V_{xx} + cV_x U_{xx} = 0. \end{gathered} \tag{3}$$

It can readily be verified that the second-order Lagrangian for system (3) is given by

$$L = \frac{1}{2}\left(aU_{xx}^2 + dV_{xx}^2 - \frac{1}{3}bU_x^3 - cU_x V_x^2 - U_x U_t - V_t V_x\right) \tag{4}$$

because

$$\frac{\delta L}{\delta U} = 0, \qquad \frac{\delta L}{\delta V} = 0, \tag{5}$$

where $\delta/\delta U$ and $\delta/\delta V$ are the standard Euler operators defined by

$$\begin{aligned} \frac{\delta}{\delta U} &= \frac{\partial}{\partial U} - D_t \frac{\partial}{\partial U_t} - D_x \frac{\partial}{\partial U_x} + D_t^2 \frac{\partial}{\partial U_{tt}} \\ &\quad + D_x^2 \frac{\partial}{\partial U_{xx}} + D_x D_t \frac{\partial}{\partial U_{tx}} - \cdots, \\ \frac{\delta}{\delta V} &= \frac{\partial}{\partial V} - D_t \frac{\partial}{\partial V_t} - D_x \frac{\partial}{\partial V_x} + D_t^2 \frac{\partial}{\partial V_{tt}} \\ &\quad + D_x^2 \frac{\partial}{\partial V_{xx}} + D_x D_t \frac{\partial}{\partial V_{tx}} - \cdots. \end{aligned} \tag{6}$$

Consider the vector field

$$\begin{aligned} X &= \xi^1 (t, x, U, V) \frac{\partial}{\partial t} + \xi^2 (t, x, U, V) \frac{\partial}{\partial x} \\ &\quad + \eta^1 (t, x, U, V) \frac{\partial}{\partial U} + \eta^2 (t, x, U, V) \frac{\partial}{\partial V}, \end{aligned} \tag{7}$$

which has the second-order prolongation defined by

$$\begin{aligned} X^{[2]} &= \xi^1 (t, x, U, V) \frac{\partial}{\partial t} + \xi^2 (t, x, U, V) \frac{\partial}{\partial x} \\ &\quad + \eta^1 (t, x, U, V) \frac{\partial}{\partial U} + \eta^2 (t, x, U, V) \frac{\partial}{\partial V} \\ &\quad + \zeta_t^1 \frac{\partial}{\partial U_t} + \zeta_t^2 \frac{\partial}{\partial V_t} + \zeta_x^1 \frac{\partial}{\partial U_x} + \zeta_x^2 \frac{\partial}{\partial V_x} + \cdots. \end{aligned} \tag{8}$$

Here

$$\begin{aligned} \zeta_t^1 &= D_t(\eta^1) - U_t D_t(\xi^1) - U_x D_t(\xi^2), \\ \zeta_x^1 &= D_x(\eta^1) - U_t D_x(\xi^1) - U_x D_x(\xi^2), \\ \zeta_t^2 &= D_t(\eta^2) - V_t D_t(\xi^1) - V_x D_t(\xi^2), \\ \zeta_x^2 &= D_x(\eta^2) - V_t D_x(\xi^1) - V_x D_x(\xi^2), \end{aligned}$$

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + U_t \frac{\partial}{\partial U} + V_t \frac{\partial}{\partial V} + U_{tt} \frac{\partial}{\partial U_t} \\ &\quad + V_{tt} \frac{\partial}{\partial V_t} + U_{tx} \frac{\partial}{\partial U_x} + V_{tx} \frac{\partial}{\partial V_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + U_x \frac{\partial}{\partial U} + V_x \frac{\partial}{\partial V} + U_{xx} \frac{\partial}{\partial U_x} \\ &\quad + V_{xx} \frac{\partial}{\partial V_x} + U_{tx} \frac{\partial}{\partial U_t} + V_{tx} \frac{\partial}{\partial V_t} + \cdots. \end{aligned} \tag{9}$$

The Lie-Bäcklund operator X defined in (7) is a Noether operator corresponding to the Lagrangian (4) if it satisfies

$$X^{[2]}(L) + L\left[D_t(\xi^1) + D_x(\xi^2)\right] = D_t(B^1) + D_x(B^2), \tag{10}$$

where $B^1(t, x, U, V)$, $B^2(t, x, U, V)$ are the gauge terms. Expansion of (10) yields

$$\begin{aligned} &-\frac{1}{2}U_x \left[\eta_t^1 + U_t \eta_U^1 + V_t \eta_V^1 - U_t \xi_t^1 - U_t^2 \xi_U^1 \right. \\ &\qquad \left. - U_t V_t \xi_V^1 - U_x \xi_t^2 - U_t U_t \xi_U^2 - U_x V_t \xi_V^2\right] \\ &\quad - \frac{1}{2}V_x \left[\eta_t^2 + U_t \eta_U^2 + V_t \eta_V^2 - V_t \xi_t^1 - U_t V_t \xi_U^1 \right. \\ &\qquad \left. - V_t^2 \xi_V^1 - V_x \xi_t^2 - U_t V_x \xi_U^2 - V_t V_x \xi_V^2\right] \\ &\quad - \frac{1}{2}\left(bU_x^2 + cV_x^2 + U_t\right) \\ &\quad \times \left[\eta_x^1 + U_x \eta_U^1 + V_x \eta_V^1 - U_t \xi_x^1 - U_t U_x \xi_U^1 \right. \\ &\qquad \left. - U_t V_x \xi_V^1 - U_x \xi_x^2 - U_x^2 \xi_U^2 - U_x V_x \xi_V^2\right] \\ &\quad - \frac{1}{2}\left(cU_x V_x + V_t\right) \\ &\quad \times \left[\eta_x^2 + U_x \eta_U^2 + V_x \eta_V^2 - V_t \xi_x^1 - U_x V_t \xi_U^1 \right. \\ &\qquad \left. - V_t V_x \xi_V^1 - V_x \xi_x^2 - U_x V_x \xi_U^2 - V_x^2 \xi_V^2\right] \\ &\quad + dV_{xx}\left[D_x^2(\eta^1) - U_t D_x^2(\xi^1) - U_x D_x^2(\xi^2) \right. \\ &\qquad - 2U_{tx}\left(\xi_x^1 + U_x \xi_U^1 + V_x \xi_V^1\right) \\ &\qquad \left. - 2U_{xx}\left(\xi_x^2 + U_x \xi_U^2 + V_x \xi_V^2\right)\right] \\ &\quad + aU_{xx}\left[D_x^2(\eta^2) - V_t D_x^2(\xi^1) - V_x D_x^2(\xi^2) \right. \\ &\qquad - 2V_{tx}\left(\xi_x^1 + U_x \xi_U^1 + V_x \xi_V^1\right) \\ &\qquad \left. - 2V_{xx}\left(\xi_x^2 + U_x \xi_U^2 + V_x \xi_V^2\right)\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left(aU_{xx}^2+dV_{xx}^2-\frac{1}{3}bU_x^3-cU_xV_x^2-U_xU_t-V_tV_x\right)\\
&\times\left[\xi_t^1+U_t\xi_U^1+V_t\xi_V^1+\xi_x^2+U_x\xi_U^2+V_x\xi_V^2\right]\\
&=B_t^1+U_tB_U^1+V_tB_V^1+B_x^2+U_xB_U^2+V_xB_V^2.
\end{aligned}
\tag{11}
$$

The splitting of (11) with respect to different combinations of derivatives of U and V results in an overdetermined system of PDEs for $\xi^1, \xi^2, \eta^1, \eta^2, B^1$, and B^2. Solving this system of PDEs we arrive at the following two cases for which Noether symmetries exist.

Case 1. $b \neq c$.

In this case we obtain the following Noether symmetries and gauge terms:

$$
\begin{aligned}
\xi^1 &= A_1,\\
\xi^2 &= A_2,\\
\eta^1 &= E(t),\\
\eta^2 &= F(t),\\
B^1 &= P(t,x),\\
B^2 &= -\frac{1}{2}UE'(t)-\frac{1}{2}VF'(t)+S(t,x),\\
P_t &+ S_x = 0.
\end{aligned}
\tag{12}
$$

The above results will now be used to find the components of the conserved vectors for the second-order Lagrangian. Here we can choose $P=0$, $S=0$ as they contribute to the trivial part of the conserved vector. We recall that the conserved vectors for the second-order Lagrangian are given by [13, 21]

$$
\begin{aligned}
T^1 = &-B^1+\xi^1L+W^1\left[\frac{\partial L}{\partial U_t}-D_t\frac{\partial L}{\partial U_{tt}}-D_x\frac{\partial L}{\partial U_{tx}}\cdots\right]\\
&+W^2\left[\frac{\partial L}{\partial V_t}-D_t\frac{\partial L}{\partial V_{xt}}-D_x\frac{\partial L}{\partial V_{tt}}\cdots\right]\\
&+D_t\left(W^1\right)\frac{\partial L}{\partial U_{tt}}+D_t\left(W^2\right)\frac{\partial L}{\partial V_{tt}},\\
T^2 = &-B^2+\xi^2L+W^1\left[\frac{\partial L}{\partial U_x}-D_t\frac{\partial L}{\partial U_{xt}}-D_x\frac{\partial L}{\partial U_{xx}}\cdots\right]\\
&+W^2\left[\frac{\partial L}{\partial V_x}-D_t\frac{\partial L}{\partial V_{xt}}-D_x\frac{\partial L}{\partial V_{xx}}\cdots\right]\\
&+D_x\left(W^1\right)\frac{\partial L}{\partial U_{xx}}+D_x\left(W^2\right)\frac{\partial L}{\partial V_{xx}}.
\end{aligned}
\tag{13}
$$

Here W^1 and W^2 are the Lie characteristic functions, given by $W^1=\eta^1-U_t\xi^1-U_x\xi^2$ and $W^2=\eta^2-V_t\xi^1-V_x\xi^2$. Using (13) together with (12) and $u=U_x$, $v=V_x$ we obtain the following independent conserved vectors for system (2):

$$
\begin{aligned}
T_1^1 &= \frac{1}{2}\left(au_x^2+dv_x^2-\frac{1}{3}bu^3-cuv^2\right),\\
T_1^2 &= \frac{1}{2}\int u_t dx\int u_t dx+\frac{1}{2}\left(bu^2+cv^2\right)\\
&\quad\times\int u_t dx+au_{xx}\int u_t dx\\
&\quad+\frac{1}{2}\int v_t dx\int v_t dx+dv_{xx}\int v_t dx\\
&\quad+cuv\int v_t dx-au_tu_x-dv_tu_x,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
T_2^1 &= \frac{1}{2}\left(u^2+v^2\right),\\
T_2^2 &= auu_{xx}+dvv_{xx}-\frac{1}{2}au_x^2-\frac{1}{2}dv_x^2+\frac{1}{3}bu^3+cuv^2,
\end{aligned}
\tag{15}
$$

and for the arbitrary functions $E(t)$ and $F(t)$,

$$
\begin{aligned}
T_{(E,F)}^1 &= -\frac{1}{2}uE(t)-\frac{1}{2}vF(t),\\
T_{(E,F)}^2 &= \frac{1}{2}E'(t)\int u dx+\frac{1}{2}F'(t)\int v dx\\
&\quad-\frac{1}{2}E(t)\int u_t dx-\frac{1}{2}F(t)\int v_t dx\\
&\quad-\frac{1}{2}\left(bu^2+cv^2\right)E(t)-au_{xx}E(t)\\
&\quad-dv_{xx}F(t)-cuvF(t).
\end{aligned}
\tag{16}
$$

Conserved vector (14) is a nonlocal conserved vector, and (15) is a local conserved vector for system (2). We now derive two particular cases from conserved vector (16) by letting $E(t)=1$ and $F(t)=0$, which gives a nonlocal conserved vector

$$
\begin{aligned}
T_3^1 &= -\frac{1}{2}u,\\
T_{(3)}^2 &= -\frac{1}{2}\left(bu^2+cv^2\right)-au_{xx}-\frac{1}{2}\int u_t dx,
\end{aligned}
\tag{17}
$$

and by choosing $E(t)=0$ and $F(t)=1$, we get the nonlocal conserved vector

$$
\begin{aligned}
T_4^1 &= -\frac{1}{2}v,\\
T_4^2 &= -cuv-dv_{xx}-\frac{1}{2}\int v_t dx.
\end{aligned}
\tag{18}
$$

Case 2. $b = c$.

The second case gives the following Noether symmetries and gauge terms:

$$\begin{aligned}
\xi^1 &= A_1,\\
\xi^2 &= cA_2 t + A_3,\\
\eta^1 &= A_2 x + F(t),\\
\eta^2 &= G(t),\\
B^1 &= -\frac{1}{2}A_2 U + P(t,x),\\
B^2 &= -\frac{1}{2}UF'(t) - \frac{1}{2}VG'(t) + R(t,x),\\
P_t + R_x &= 0.
\end{aligned} \tag{19}$$

Again we can set $P = 0$ and $R = 0$ as they contribute to the trivial part of the conserved vector. The independent conserved vectors for system (2), in this case, are

$$\begin{aligned}
T_1^1 &= \frac{1}{2}\left(au_x^2 + dv_x^2 - \frac{1}{3}bu^3 - cuv^2\right),\\
T_1^2 &= \frac{1}{2}\int u_t dx \int u_t dx + \frac{1}{2}\left(bu^2 + cv^2\right)\int u_t dx\\
&\quad + au_{xx}\int u_t dx + \frac{1}{2}\int v_t dx \int v_t dx\\
&\quad + dv_{xx}\int v_t dx + cuv\int v_t dx - au_t u_x - dv_t u_x,\\
T_2^1 &= \frac{1}{2}\left(-xu + ctu^2 + ctv^2 + \int u dx\right),\\
T_2^2 &= au_x + actuu_{xx} + cdtvv_{xx} + c^2tuv^2 - axu_{xx} + \frac{1}{3}cbtu^3\\
&\quad - \frac{1}{2}\left(actu_x^2 + cdtv_x^2 + bxu^2 + cxv^2 + x\int u_t dx\right),
\end{aligned} \tag{20}$$

$$\begin{aligned}
T_3^1 &= \frac{1}{2}\left(u^2 + v^2\right),\\
T_3^2 &= auu_{xx} + dvv_{xx} - \frac{1}{2}au_x^2 - \frac{1}{2}dv_x^2 + \frac{1}{3}bu^3 + cuv^2,
\end{aligned} \tag{21}$$

and for the arbitrary functions $E(t)$ and $F(t)$, we obtain

$$\begin{aligned}
T_{(E,F)}^1 &= -\frac{1}{2}uE(t) - \frac{1}{2}vF(t),\\
T_{(E,F)}^2 &= \frac{1}{2}E'(t)\int u dx + \frac{1}{2}F'(t)\int v dx\\
&\quad - \frac{1}{2}E(t)\int u_t dx - \frac{1}{2}F(t)\int v_t dx\\
&\quad - \frac{1}{2}\left(bu^2 + cv^2\right)E(t) - au_{xx}E(t)\\
&\quad - dv_{xx}F(t) - cuvF(t).
\end{aligned} \tag{22}$$

Conserved vectors (20) are nonlocal, whereas (21) is a local conserved vector for system (2). Conserved vector (22) for $E(t) = 1$ and $F(t) = 0$ gives a nonlocal conserved vector

$$\begin{aligned}
T_3^1 &= -\frac{1}{2}u,\\
T_3^2 &= -\frac{1}{2}\left(bu^2 + cv^2\right) - au_{xx} - \frac{1}{2}\int u_t dx,
\end{aligned} \tag{23}$$

and for $E(t) = 0$ and $F(t) = 1$ it gives a nonlocal conserved vector

$$\begin{aligned}
T_4^1 &= -\frac{1}{2}v,\\
T_4^2 &= -cuv - dv_{xx} - \frac{1}{2}\int v_t dx.
\end{aligned} \tag{24}$$

We note that for arbitrary values of $E(t)$ and $F(t)$ infinitely many nonlocal conservation laws exist for system (2).

3. Conclusion

In this paper we studied the third-order generalized coupled Korteweg-de Vries system (2). This system did not have a Lagrangian. In order to apply Noether theorem the transformations $u = U_x$ and $v = V_x$ were utilized, and the system was transformed to fourth-order system (3) in U and V variables. This system admitted the Lagrangian (4). Noether theorem was then used to derive the conservation laws in U and V variables. Finally, by reverting back to our original variables u and v we obtained the conservation laws for the third-order generalized coupled KdV system (2). The conservation laws obtained consisted of some local and infinite number of nonlocal conserved vectors.

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