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Research Article

Controllability of Fractional Neutral Stochastic Integro-Differential Systems with Infinite Delay

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This paper is concerned with the controllability of a class of fractional neutral stochastic integro-differential systems with infinite delay in an abstract space. By employing fractional calculus and Sadovskii's fixed point principle without assuming severe compactness condition on the semigroup, a set of sufficient conditions are derived for achieving the controllability result.

1. Introduction

It is well known that the fractional calculus is a classical mathematical notion and is a generalization of ordinary differentiation and integration to arbitrary (noninteger) order. Nowadays, studying fractional-order calculus has become an active research field [1–7]. Much effort has been devoted to apply the fractional calculus to networks control. For example, Chen et al. [8], Delshad et al. [9], and Wang and Zhang [10] studied the synchronization for fractional-order complex dynamical networks; Zhang et al. [11] investigated a fractional order three-dimensional Hopfield neural network and pointed out that chaotic behaviors can emerge in a fractional network; Kaslik and Sivasundaram [12] discussed the local stability for fractional-order neural networks of Hopfield type by applying the linear stability theory of fractional-order system.

One of the emerging branches of this study is the theory of fractional evolution equations, say, evolution equations, where the integer derivative with respect to time is replaced by a derivative of fractional order. The increasing interest in this class of equations is motivated both by their application to problems from fluid dynamic traffic model, viscoelasticity, heat conduction in materials with memory, electrodynamics with memory, and also because they can be employed to approach nonlinear conservation laws (see [13] and references therein). In addition, neutral stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both science and engineering, for instance, in the theory development in Gurtin and Pipkin [14] and Nunziato [15] for the description of heat conduction in materials with fading memory. It should be pointed out that the deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. We mention here the recent papers [16, 17] concerning the existence of mild solutions of fractional stochastic systems.

As one of the fundamental concepts in mathematical control theory, controllability plays an important role both in deterministic and stochastic control problems such as stabilization of unstable systems by feedback control. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability problems for different nonlinear stochastic systems in infinite dimensional spaces have been extensively studied in many papers; see [18–22] and references therein. We would also like to mention that the controllability for stochastic systems with infinite delay has been investigated by Balasubramaniam et al. [23, 24] and Ren et al. [25] using some abstract spaces. Nevertheless, to

the best of our knowledge, it seems that little is known about the controllability of fractional neutral stochastic differential equations with infinite delay, and the aim of this paper is to close this gap.

In this paper, we are interested in the controllability of a class of fractional neutral stochastic integro-differential systems with infinite delay of the followin form:

$$\begin{aligned} \frac{{}^cD^\alpha}{dt^\alpha}\left[x(t)+G(t,x_t)\right] &= -Ax(t)+Bu(t) \\ &\quad + \int_0^t \sigma(t,s,x_s)\,dW(s), \quad t\in J, \\ x(t) &= \phi(t), \quad t\in(-\infty,0]. \end{aligned} \tag{1}$$

Here, $J := [0,b]$, $b > 0$. $x(\cdot)$ takes value in a real separable Hilbert space $\mathbb{H}$ with inner product $(\cdot,\cdot)$ and norm $|\cdot|_{\mathbb{H}}$. The fractional derivative ${}^cD^\alpha$, $\alpha \in (1/2,1]$ is understood in the Caputo sense. $-A : \mathscr{D}(-A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a analytic semigroup of a bounded linear operator $S(t)$, $t \geq 0$, on $\mathbb{H}$. Let $\mathbb{K}$ be another separable Hilbert space with inner product $(\cdot,\cdot)_{\mathbb{K}}$ and norm $\|\cdot\|_{\mathbb{K}}$. W is a given $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},P)$. The control function $u(\cdot)$ takes value in $L^2(J,\mathbb{U})$ of admissible control functions for a separable Hilbert space $\mathbb{U}$, and B is a bounded linear operator from $\mathbb{U}$ into $\mathbb{H}$. The histories $x_t : \Omega \to C_h$ defined by $x_t = \{x(t+\theta), \theta \in (-\infty,0]\}$ belong to the phase space C_h, which will be defined in Section 2. The initial data $\phi = \{\phi(t), t \in (-\infty,0]\}$ is an $\mathscr{F}_0$-measurable, C_h-valued random variable independent of W with finite second moments, and $G : J\times C_h \to \mathbb{H}$, $\sigma : J\times J\times\mathbb{H} \to \mathscr{L}_2^0(\mathbb{K},\mathbb{H})$ are appropriate mappings specified later (here, $\mathscr{L}_2^0(\mathbb{K},\mathbb{H})$ denotes the space of all Q-Hilbert-Schmidt operators from $\mathbb{K}$ into $\mathbb{H}$, which is going to be defined later).

The structure of this paper is as follows. In Section 2, we briefly present some basic notations and preliminaries. The controllability result of system (1) is investigated by means of Sadovskii's fixed point theorem and operator theory in Section 3. Conclusion is given in Section 4.

2. Preliminaries

For more details in this section, we refer the reader to Pazy [26], Da Prato and Zabczyk [27], and Samko et al. [28]. Throughout this paper, $(\mathbb{H},|\cdot|_{\mathbb{H}})$ and $(\mathbb{K},\|\cdot\|_{\mathbb{K}})$ denote two real separable Hilbert spaces. We denote by $\mathscr{L}(\mathbb{K},\mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|\cdot\|$. In this paper, we use the symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises.

Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},P)$ be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and $\mathscr{F}_0$ contains all P-null sets. $W = (W_t)_{t\geq 0}$ is a Q-Wiener process defined on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},P)$ with covariance operator Q such that $\operatorname{Tr} Q < \infty$. We assume that there exist a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1,2,\ldots$, and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(W(t),e)_{\mathbb{K}} = \sum_{k=1}^{\infty}\sqrt{\lambda_k}(e_k,e)_{\mathbb{K}}\beta_k(t), \quad e\in\mathbb{K},\ t\geq 0. \tag{2}$$

Let $\mathscr{L}_2^0 = \mathscr{L}_2(Q^{1/2}\mathbb{K},\mathbb{H})$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}\mathbb{K}$ to $\mathbb{H}$ with the inner product $\langle\varphi,\phi\rangle_{\mathscr{L}_2^0} = \operatorname{Tr}[\varphi Q\phi*]$.

Suppose that $0\in\rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$, then the semigroup $S(\cdot)$ is uniformly bounded. That is to say, $\|S(t)\| \leq M$, $t \geq 0$, for some constant $M > 0$. Then, for $\alpha \in (0,1]$, it is possible to define the fractional power operator A^α as a closed linear operator on its domain $\mathscr{D}(A^\alpha)$. Furthermore, the subspace $\mathscr{D}(A^\alpha)$ is dense in $\mathbb{H}$, and the expression

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x\in\mathscr{D}(A^\alpha), \tag{3}$$

defines a norm on $\mathbb{H}_\alpha := \mathscr{D}(A^\alpha)$. The following properties are well known.

Lemma 1 (Pazy [26]). *Suppose that the preceding conditions are satisfied.*

(a) *If $0 < \beta < \alpha \leq 1$, then $\mathbb{H}_\alpha \subset \mathbb{H}_\beta$ and the embedding is compact whenever the resolvent operator of A is compact.*

(b) *For every $\alpha \in (0,1]$, there exists a positive constant C_α such that*

$$\|A^\alpha S(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad t > 0. \tag{4}$$

Let now us recall some basic definitions and results of fractional calculus.

Definition 2. The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^\alpha f = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \quad t>0,\ \alpha>0, \tag{5}$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 3. The Caputo derivative of order α with the lower limit 0 for a function f can be written as

$$\begin{aligned} {}^cD^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}ds \\ &= I^{n-\alpha}f^n(t), \quad t>0,\ 0\leq n-1<\alpha<n. \end{aligned} \tag{6}$$

If f is an abstract function with values in $\mathbb{H}$, then the integrals that appear in the previous definitions are taken in Bochner's sense.

Assume that $h:(-\infty,0]\to(0,+\infty)$ with $l=\int_{-\infty}^{0}h(t)dt<+\infty$ is a continuous function. Recall that the abstract phase space $\mathscr{C}_h$ is defined by

$$\mathscr{C}_h=\left\{\varphi:(-\infty,0]\longrightarrow\mathbb{H},\ \text{for any } a>0,\left(E\left|\varphi(\theta)\right|^2\right)^{1/2}\right.$$
$$\text{is abounded and measurable function on } [-a,0],$$
$$\left.\int_{-\infty}^{0}h(s)\sup_{s\le\theta\le0}\left(E\left|\varphi(\theta)\right|^2\right)^{1/2}ds<+\infty\right\}. \tag{7}$$

If $\mathscr{C}_h$ is endowed with the norm

$$\|\varphi\|_{\mathscr{C}_h}=\int_{-\infty}^{0}h(s)\sup_{s\le\theta\le0}\left(E\left|\varphi(\theta)\right|^2\right)^{1/2}ds,\quad \varphi\in\mathscr{C}_h, \tag{8}$$

then $(\mathscr{C}_h,\|\cdot\|_{\mathscr{C}_h})$ is a Banach space (see Liu et al. [29]).

At the end of this section, we recall the fixed point theorem of Sadovskii [30].

Lemma 4. *Let Φ be a condensing operator on a Banach space $\mathbb{H}$; that is, Φ is continuous, and take bounded sets into bounded sets, and $\mu(\Phi(B))\le\mu(B)$ for every bounded set B of $\mathbb{H}$ with $\mu(B)>0$. If $\Phi(N)\subset N$ for a convex, closed, and bounded set N of $\mathbb{H}$, then Φ has a fixed point in $\mathbb{H}$ (where $\mu(\cdot)$ denotes Kuratowski's measure of noncompactness.)*

3. Main Results

In this section, we obtain controllability of system (1). We first present the definition of mild solutions.

Definition 5. An $\mathbb{H}$-valued stochastic process $\{x(t),t\in(-\infty,b]\}$ is said to be a mild solution of system (1) if

(i) $x(t)$ is $\mathscr{F}_t$-adapted and measurable for each $t\ge0$;

(ii) $x(t)$ is continuous on $[0,b]$ almost surely and for each $s\in[0,t)$, the function $(t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)$ is integrable such that the following stochastic integral equation is verified:

$$\begin{aligned}x(t)&=S_\alpha(t)\left[\phi(0)+G(0,\phi)\right]-G(t,x_t)\\&\quad-\int_0^t(t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)\,ds\\&\quad+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)Bu(s)\,ds\\&\quad+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)\\&\qquad\times\left[\int_0^s\sigma(s,\tau,x_\tau)\,dW(\tau)\right]ds;\end{aligned} \tag{9}$$

(iii) $x(t)=\phi(t)$ on $(-\infty,0]$ with $\|\phi\|^2_{\mathscr{C}_h}<\infty$,

where

$$\begin{aligned}S_\alpha(t)x&=\int_0^\infty\eta_\alpha(\theta)S(t^\alpha\theta)x\,d\theta,\\T_\alpha(t)x&=\alpha\int_0^\infty\theta\eta_\alpha(\theta)S(t^\alpha\theta)x\,d\theta\end{aligned} \tag{10}$$

with η_α a probability density function defined on $(0,\infty)$.

Definition 6. System (1) is said to be controllable on the interval J, if for every initial stochastic process $\phi\in\mathscr{C}_h$ defined on $(-\infty,0]$, there exists a stochastic control $u\in L^2(J,\mathbb{U})$, which is adapted to the filtration $\{\mathscr{F}_t\}_{t\ge0}$ such that the mild solution $x(t)$ of (1) satisfies $x(b)=x^*$, where x^* and b are preassigned terminal state and time, respectively.

The following properties of $S_\alpha(t)$ and $T_\alpha(t)$ that appeared in Zhou and Jiao [7] are useful.

Lemma 7. *Under previous assumptions on $S(t)$, $t\ge0$, and A, then*

(i) *for any fixed $t\ge0$, $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators such that for any $x\in\mathbb{H}$, $|S_\alpha(t)x|_{\mathbb{H}}\le M|x|_{\mathbb{H}}$, $|T_\alpha(t)x|_{\mathbb{H}}\le(M\alpha/\Gamma(1+\alpha))|x|_{\mathbb{H}}$;*

(ii) *$S_\alpha(t)$ and $T_\alpha(t)$ are strongly continuous;*

(iii) *for any $x\in\mathbb{H}$, $\beta\in(0,1)$, and $\theta\in(0,1]$, one has*

$$\begin{aligned}AT_\alpha(t)x&=A^{1-\beta}T_\alpha(t)A^\beta x,\\\left\|A^\theta T_\alpha(t)\right\|&\le\frac{\alpha C_\theta}{t^{\alpha\theta}}\frac{\Gamma(2-\theta)}{\Gamma(1+\alpha(1-\theta))},\quad t\in[0,b].\end{aligned} \tag{11}$$

In this paper, we will work under the following assumptions.

(A_0) $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators $S(t)$ in $\mathbb{H}$; $S(t)$, $t>0$, is continuous in the uniform operator topology and $\|S(t)\|\le M$ for some $M>0$.

(A_1) The function $G:J\times\mathscr{C}_h\to\mathbb{H}$ is continuous, and there exist some constants $M_G>0$, $\beta\in(0,1)$ with $\alpha\beta>1/2$, such that G is $\mathbb{H}_\beta$-valued and

$$\begin{aligned}&E\left|A^\beta G(t,x)-A^\beta G(t,y)\right|^2_{\mathbb{H}}\\&\qquad\le M_G\|x-y\|^2_{\mathscr{C}_h},\quad x,y\in\mathscr{C}_h,\ t\in J,\\&E\left|A^\beta G(t,x)\right|^2_{\mathbb{H}}\le M_G\left(\|x\|^2_{\mathscr{C}_h}+1\right).\end{aligned} \tag{12}$$

(A_2) $\sigma:J\times J\times\mathscr{C}_h\to L(\mathbb{K},\mathbb{H})$ satisfies the following:

(2a) for each $(t,s)\in D:=J\times J$, $\sigma(t,s,\cdot):\mathscr{C}_h\to L(\mathbb{K},\mathbb{H})$ is continuous and for each $x\in\mathscr{C}_h$, $\sigma(\cdot,\cdot,x):D\to L(\mathbb{K},\mathbb{H})$ is strongly measurable;

(2b) there is a positive integrable function $m \in L^1([0,b])$ and a continuous nondecreasing function $\Xi : [0,\infty) \to (0,\infty)$ such that for every $(t,s,x) \in J \times J \times \mathscr{C}_h$, we have

$$\int_0^t E|\sigma(t,s,x)|^2_{\mathscr{L}_2^0} ds \le m(t)\,\Xi\left(\|x\|^2_{\mathscr{C}_h}\right), \qquad \liminf_{r\to\infty} \frac{\Xi(r)}{r} ds = \Lambda < \infty; \tag{13}$$

(2c) for any $x, y \in \mathscr{C}_h$, $t \ge 0$, there exists a positive constant M_σ such that

$$\int_0^t E\left|\sigma(t,s,x) - \sigma(t,s,y)\right|^2_{\mathscr{L}_2^0} ds \le M_\sigma \|x-y\|^2_{\mathscr{C}_h}. \tag{14}$$

(A_3) The linear operator W from $L^2(J,\mathbb{U})$ into $\mathbb{H}$ defined by

$$Wu = \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s)\, Bu(s)\, ds \tag{15}$$

has an invertible operator W^{-1} defined on $L^2(J,\mathbb{U})/\ker W$ (see [31]), and there exist a pair of constants $N_1, N_2 > 0$ such that

$$\|B\|^2 \le N_1, \qquad \left\|W^{-1}\right\| \le N_2. \tag{16}$$

(A_4) There exists a compact set $\mho \subseteq \mathbb{H}$ such that $S(t)\sigma(s,\tau,\xi) \in \mho$ for all $t \in J$, $0 \le \tau \le s \le b$ and $\xi \in \mathscr{C}_h$ with finite second moment.

(A_5) Assume that the following relationship holds:

$$20\left(1 + 5N_1N_2N\right) l^2 \times \left(M_0 + M_G P + Nb^{-1}\operatorname{Tr} Q\Lambda \int_0^b m(s)\, ds\right) < 1, \tag{17}$$

where

$$N = \frac{b^{2\alpha}}{2\alpha - 1}\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2, \qquad M_0 = \left\|A^{-\beta}\right\|^2 M_G, \tag{18}$$

$$P = \frac{\alpha^2 C^2_{1-\beta}\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{b^{2\alpha\beta}}{2\alpha\beta - 1}. \tag{19}$$

Denote by $C((-\infty,b],\mathbb{H})$ the space of all continuous $\mathbb{H}$-valued stochastic processes $\{\xi(t), t \in (-\infty,b]\}$. Let

$$\mathscr{C}_b = \left\{x : x \in C((-\infty,b],\mathbb{H}), x_0 = \phi \in \mathscr{C}_h\right\}. \tag{20}$$

Set $\|\cdot\|_b$ to be a seminorm defined by

$$\|x\|_b = \|x_0\|_{\mathscr{C}_h} + \sup_{s\in[0,b]}\left(E|x(s)|^2\right)^{1/2}, \quad x \in \mathscr{C}_b. \tag{21}$$

We have the following useful lemma that appeared in Liu et al. [29].

Lemma 8. *Assume that $x \in \mathscr{C}_b$; then, for all $t \in J$, $x_t \in \mathscr{C}_h$. Moreover,*

$$l\left(E|x(t)|^2\right)^{1/2} \le \|x_t\|_{\mathscr{C}_h} \le l \sup_{s\in[0,t]}\left(E|x(s)|^2\right)^{1/2} + \|x_0\|_{\mathscr{C}_h}, \tag{22}$$

where $l = \int_{-\infty}^0 h(s)ds < \infty$.

The main object of this paper is to explain and prove the following theorem.

Theorem 9. *Assume that assumptions (A_0)–(A_5) hold. Then, system (1) is controllable on J provided that*

$$L := 3l^2\left(1 + 3N_1N_2N\right)\left(M_0 + M_G P\right) + 9N_1N_2N^2 \operatorname{Tr} Q M_\sigma l^2 < 1. \tag{23}$$

Proof. Using assumption (A_3), for an arbitrary process $x(\cdot)$, define the control process

$$\begin{aligned} u_x(t) = W^{-1}\Big\{ & x^* - S_\alpha(b)\left[\phi(0) + G(0,\phi)\right] + G(b,x_b) \\ & + \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s)\, G(s,x_s)\, ds \\ & - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) \\ & \times \left[\int_0^s \sigma(s,\tau,x_\tau)\, dW(\tau)\right] ds\Big\}(t). \end{aligned} \tag{24}$$

We transform (1) into a fixed point problem. Consider the operator $Q : \mathscr{C}_b \to \mathscr{C}_b$ defined by

$$(Qx)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ S_\alpha(t)\left[\phi(0) + G(0,\phi)\right] - G(t,x_t) \\ \quad - \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s)\, G(s,x_s)\, ds \\ \quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\, Bu_x(s)\, ds \\ \quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \\ \qquad \times \left[\int_0^s \sigma(s,\tau,x_\tau)\, dW(\tau)\right] ds, & t \in J. \end{cases} \tag{25}$$

It follows from Hölder inequality, Lemma 7, and assumption (A_1) that

$$\begin{aligned} E\left|\int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s)\, G(s,x_s)\, ds\right|^2_{\mathbb{H}} & \le E\left[\int_0^t \left|(t-s)^{\alpha-1} A^{1-\beta} T_\alpha \right.\right. \\ & \left.\left. \times (t-s) A^\beta G(s,x_s)\right|_{\mathbb{H}} ds\right]^2 \end{aligned}$$

$$\begin{aligned}
&\le PE\left[\int_0^t \left|(t-s)^{\alpha(\beta-1)}(t-s)^{\alpha-1}\right.\right. \\
&\qquad \left.\left.\times A^{\beta}G(s,x_s)\right|_{\mathbb{H}} ds\right]^2 \\
&\le P\int_0^t (t-s)^{2(\alpha\beta-1)}ds \\
&\quad \times \int_0^t M_G\left(1+|x_s|^2_{\mathscr{C}_h}\right)ds,
\end{aligned} \tag{26}$$

which deduces that $(t-s)^{\alpha-1}AT_\alpha(t-s)G(s,x_s)$ is integrable on J by Bochner's theorem (see [32]), where P is defined in (19).

In what follows, we will show that using the control $u_x(\cdot)$, the operator Q has a fixed point, which is then a mild solution for system (1).

For $\phi \in \mathscr{C}_h$, define

$$\widetilde{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0]; \\ S_\alpha(t)\phi(0), & t \in J. \end{cases} \tag{27}$$

Then, $\widetilde{\phi} \in \mathscr{C}_b$. Set $x(t) = \widetilde{\phi}(t) + z(t)$, $t \in (-\infty, b]$. It is easy to check that x satisfies (1) if and only if $z_0 = 0$ and

$$\begin{aligned}
z(t) &= S_\alpha(t)G(0,\phi) - G\left(t,\widetilde{\phi}_t + z_t\right) \\
&\quad - \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G\left(s,\widetilde{\phi}_s + z_s\right)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)Bu_{\widetilde{\phi}+z}(s)\,ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s) \\
&\qquad \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right)dW(\tau)\right]ds,
\end{aligned} \tag{28}$$

where $u_{\widetilde{\phi}+z}$ is obtained by replacing x_t by $\widetilde{\phi}_t + z_t$ in (24). Let

$$\mathscr{C}_b^0 = \left\{z \in \mathscr{C}_b, z_0 = 0 \in \mathscr{C}_h\right\}. \tag{29}$$

For each $z \in \mathscr{C}_b^0$, we have

$$\|z\|_b = \|z_0\|_{\mathscr{C}_h} + \sup_{s\in[0,b]}\left(E|z(s)|^2\right)^{1/2} = \sup_{s\in[0,b]}\left(E|z(s)|^2\right)^{1/2}. \tag{30}$$

Thus, $(\mathscr{C}_b^0, \|\cdot\|_b)$ is a Banach space. For $q > 0$, set

$$B_q = \left\{y \in \mathscr{C}_b^0, \|y\|_b^2 \le q\right\}; \tag{31}$$

then, for each q, B_q is clearly a bounded closed convex set in $\mathscr{C}_b^0$. For $z \in B_q$, from Lemma 8, we see that

$$\begin{aligned}
\left\|z_t + \widetilde{\phi}_t\right\|^2_{\mathscr{C}_h} &\le 2\left(\|z_t\|^2_{\mathscr{C}_h} + \left\|\widetilde{\phi}_t\right\|^2_{\mathscr{C}_h}\right) \\
&\le 4\left(l^2 \sup_{s\in[0,t]} E|z(s)|^2 + \|z_0\|^2_{\mathscr{C}_h}\right. \\
&\qquad \left. + l^2 \sup_{s\in[0,t]} E\left|\widetilde{\phi}(s)\right|^2 + \left\|\widetilde{\phi}_0\right\|^2_{\mathscr{C}_h}\right) \\
&\le 4l^2\left(q + M^2E|\phi(0)|^2_{\mathbb{H}}\right) + 4\|\phi\|^2_{\mathscr{C}_h}.
\end{aligned} \tag{32}$$

Consider the map $\Pi : \mathscr{C}_b^0 \to \mathscr{C}_b^0$ defined by

$$(\Pi z)(t) = \begin{cases} 0, & t \in (-\infty, 0]; \\ S_\alpha(t)G(0,\phi) - G\left(t,\widetilde{\phi}_t + z_t\right) \\ \quad - \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s) \\ \qquad \times G\left(s,\widetilde{\phi}_s + z_s\right)ds \\ \quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)Bu_{\widetilde{\phi}+z}(s)\,ds \\ \quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s) \\ \qquad \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right)dW(\tau)\right]ds, & t \in J. \end{cases} \tag{33}$$

A similar argument as (26), we can show that Π is well defined on B_q for each $q > 0$. Note that the operator Q with a fixed point is equivalent to show that the operator Π has fixed point. To this end, we decompose Π as $\Pi = \Pi_1 + \Pi_2$, where the operators Π_1 and Π_2 are defined on B_q, respectively, by

$$\begin{aligned}
(\Pi_1 z)(t) &= S_\alpha(t)G(0,\phi) - G\left(t,\widetilde{\phi}_t + z_t\right) \\
&\quad - \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)G\left(s,\widetilde{\phi}_s + z_s\right)ds \\
&\quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)Bu_{\widetilde{\phi}+z}(s)\,ds, \\
(\Pi_2 z)(t) &= \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s) \\
&\qquad \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right)dW(\tau)\right]ds.
\end{aligned} \tag{34}$$

Thus, Theorem 9 follows from the next theorem. □

Theorem 10. *Assume that the assumptions of Theorem 9 hold. Then, $\Pi = \Pi_1 + \Pi_2$ satisfies all the conditions in Lemma 4.*

Proof. The proof is followed by the several steps.

Step 1. There exists a positive number q such that $\Pi(B_q) \subset B_q$. If it is not true, then for each positive number q, there exists a

function $z^q(\cdot) \in B_q$, but $\Pi(z^q) \notin B_q$; that is, $E|(\Pi z^q)(t)|_{\mathbb{H}}^2 > q$ for some $t = t(q) \in J$. An elementary calculation can show that

$$\begin{aligned}
q &\le E\left|\Pi\left(z^q\right)(t)\right|_{\mathbb{H}}^2 \\
&\le 5E\left|S_\alpha(t)\, G(0,\phi)\right|_{\mathbb{H}}^2 + 5E\left|G\left(t,\widetilde{\phi}_t + z_t^q\right)\right|_{\mathbb{H}}^2 \\
&\quad + 5E\left|\int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s)\, G\left(s,\widetilde{\phi}_s + z_s^q\right) ds\right|_{\mathbb{H}}^2 \\
&\quad + 5E\left|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \right. \\
&\qquad \left. \times \left[\int_0^s \sigma\left(s,\widetilde{\phi}_\tau + z_\tau^q\right) dW(\tau)\right] ds\right|_{\mathbb{H}}^2 \\
&\quad + 5E\left|\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\, B u_{\widetilde{\phi}+z^q}(s)\, ds\right|_{\mathbb{H}}^2 \\
&= 5\sum_{i=1}^{5} I_i.
\end{aligned} \tag{35}$$

We estimate each term on the right hand side of (35). By virtue of Lemma 8, assumptions (A_0)-(A_1), and (32), we obtain

$$\begin{aligned}
&I_1 + I_2 \\
&\quad \le M^2\left\|A^{-\beta}\right\|^2 E\left|A^\beta G(0,\phi)\right|_{\mathbb{H}}^2 \\
&\qquad + \left\|A^{-\beta}\right\|^2 E\left|A^\beta G\left(t,\widetilde{\phi}_t + z_t^q\right)\right|_{\mathbb{H}}^2 \\
&\quad \le M_0\left[M^2\left(1 + \|\phi\|_{\mathscr{C}_h}^2\right) + \left(1 + \left\|\widetilde{\phi}_t + z_t^q\right\|_{\mathscr{C}_h}^2\right)\right] \\
&\quad \le M_0\left[M^2\left(1 + \|\phi\|_{\mathscr{C}_h}^2\right)\right. \\
&\qquad \left. + \left(1 + 4l^2\left(q + M^2 E|\phi(0)|_{\mathbb{H}}^2\right) + 4\|\phi\|_{\mathscr{C}_h}^2\right)\right];
\end{aligned} \tag{36}$$

in view of Lemma 7 and Hölder inequality, we have

$$\begin{aligned}
I_3 &\le E\left[\int_0^t \left|(t-s)^{\alpha-1} A^{1-\beta} T_\alpha(t-s)\right.\right. \\
&\qquad \left.\left. \times\ A^\beta G\left(s,\widetilde{\phi}_s + z_s^q\right)\right|_{\mathbb{H}} ds\right]^2 \\
&\le \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} E \\
&\quad \times \left[\int_0^t \left|(t-s)^{\alpha\beta-1} A^\beta G\left(s,\widetilde{\phi}_s + z_s^q\right)\right|_{\mathbb{H}} ds\right]^2 \\
&\le \frac{\alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \int_0^t (t-s)^{2(\alpha\beta-1)} ds \\
&\quad \times \int_0^t M_G\left(1 + \left|\widetilde{\phi}_s + z_s^q\right|_{\mathscr{C}_h}^2\right) ds \\
&\le \frac{M_G \alpha^2 C_{1-\beta}^2 \Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)} \frac{b^{2\alpha\beta-1}}{2\alpha\beta-1} \\
&\quad \times \int_0^t \left(1 + \left|\widetilde{\phi}_s + z_s^q\right|_{\mathscr{C}_h}^2\right) ds \\
&\le M_G P\left(1 + 4l^2\left(q + M^2 E|\phi(0)|_{\mathbb{H}}^2\right) + 4\|\phi\|_{\mathscr{C}_h}^2\right),
\end{aligned} \tag{37}$$

where M_0 and P are defined in (18) and (19), respectively.

Applying Burkhölder-Davis-Gundy's inequality and assumptions (A_2), we get

$$\begin{aligned}
I_4 &\le E\left[\int_0^t \left|(t-s)^{\alpha-1} T_\alpha(t-s)\right.\right. \\
&\qquad \left.\left. \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau^q\right) dW(\tau)\right]\right|_{\mathbb{H}} ds\right]^2 \\
&\le \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \int_0^t (t-s)^{2(\alpha-1)} ds \\
&\quad \times \int_0^t E\left|\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau^q\right) dW(\tau)\right|_{\mathbb{H}}^2 ds \\
&\le \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha-1}}{2\alpha-1} \\
&\quad \times \int_0^t \operatorname{Tr} Q \int_0^s E\left|\sigma\left(s,\widetilde{\phi}_\tau + z_\tau^q\right)\right|_{\mathscr{L}_2^0}^2 d\tau\, ds \\
&\le \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{b^{2\alpha-1}}{2\alpha-1} \operatorname{Tr} Q \\
&\quad \times \int_0^t m(s)\, \Xi\left(\left\|\widetilde{\phi}_\tau + z_\tau^q\right\|_{\mathscr{C}_h}^2\right) ds \\
&\le \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{\operatorname{Tr} Q b^{2\alpha-1}}{2\alpha-1} \Xi \\
&\quad \times \left(4l^2\left(q + M^2 E|\phi(0)|_{\mathbb{H}}^2\right) + 4\|\phi\|_{\mathscr{C}_h}^2\right) \int_0^t m(s)\, ds.
\end{aligned} \tag{38}$$

On the other hand, in view of (24) and (A_3), we have

$$\begin{aligned}
&E\left|B u_{z^q+\widetilde{\phi}}(t)\right|_{\mathbb{H}}^2 \\
&\quad \le N_1 N_2 E\left|\left\{x^* - S_\alpha(b)\left[\phi(0) + G(0,\phi)\right]\right.\right. \\
&\qquad + G\left(b,\widetilde{\phi}_b + z_b^q\right) \\
&\qquad + \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s)\, G\left(s,\widetilde{\phi}_s + z_s^q\right) ds
\end{aligned}$$

$$-\int_0^b (b-s)^{\alpha-1} T_\alpha (b-s) \times \left[\int_0^s \sigma\left(s,\tau,\tilde{\phi}_\tau + z_\tau^q\right) dW(\tau)\right] ds\Big\}(t)\Big|_{\mathbb{H}}^2; \tag{39}$$

thus, by the same procedure as (36)–(38), it follows that

$$\begin{aligned} I_5 &\le \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \int_0^t (t-s)^{2(\alpha-1)} ds \int_0^t E\left|Bu_{z^q+\tilde{\phi}}(s)\right|_{\mathbb{H}}^2 ds \\ &\le N\Big\{C_0 + 5N_1N_2 \\ &\quad\times\Big[4M_G\left\|A^{-\beta}\right\|^2 l^2 q + 4l^2 M_G P q \\ &\quad + 2Nb^{-1}\operatorname{Tr} Q\Xi \\ &\quad\times\left(4l^2\left(q + M^2 E|\phi(0)|_{\mathbb{H}}^2\right) + 4\|\phi\|_{\mathscr{C}_h}^2\right) \\ &\quad\times\int_0^b m(s)\,ds\Big]\Big\}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} C_0 = 5N_1N_2\Big[&E|x^*|_{\mathbb{H}}^2 + 2M^2E|\phi(0)|_{\mathbb{H}}^2 \\ &+ 2M^2M_0\left(1+\|\phi\|_{\mathscr{C}_h}^2\right) \\ &+ M_0\left(1+4l^2M^2E|\phi(0)|_{\mathbb{H}}^2 + 4\|\phi\|_{\mathscr{C}_h}^2\right) \\ &+ M_GP\left(1+4l^2M^2E|\phi(0)|_{\mathbb{H}}^2 + 4\|\phi\|_{\mathscr{C}_h}^2\right)\Big]. \end{aligned} \tag{41}$$

M_0 and P are defined in (18) and (19), respectively.

Combining these estimates (35) to (40) yields

$$\begin{aligned} q &\le E\left|\Pi(z^q)(t)\right|_{\mathbb{H}}^2 \\ &\le L_0 + 20M_0l^2(1+5N_1N_2N)q \\ &\quad + 20l^2M_GP(1+5N_1N_2N)q \\ &\quad + 5\left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{\operatorname{Tr} Qb^{2\alpha-1}}{2\alpha-1} \\ &\quad\times N\Xi\left(4l^2\left(q+M^2E|\phi(0)|_{\mathbb{H}}^2\right) + 4\|\phi\|_{\mathscr{C}_h}^2\right)\int_0^b m(s)\,ds, \end{aligned} \tag{42}$$

where

$$\begin{aligned} L_0 = 5N &+ 5M^2M_0\left(1+\|\phi\|_{\mathscr{C}_h}^2\right)C_0 \\ &+ 5M_0\left(1+4l^2M^2E|\phi(0)|_{\mathbb{H}}^2 + 4\|\phi\|_{\mathscr{C}_h}^2\right) \\ &+ 5M_GP\left(1+4l^2M^2E|\phi(0)|_{\mathbb{H}}^2 + 4\|\phi\|_{\mathscr{C}_h}^2\right). \end{aligned} \tag{43}$$

Dividing both sides of (42) by q and taking $q \to \infty$, we obtain that

$$\begin{aligned} &20(1+5N_1N_2N)l^2 \\ &\quad\times\left(M_0 + M_GP + Nb^{-1}\operatorname{Tr} Q\Lambda\int_0^b m(s)\,ds\right) \ge 1, \end{aligned} \tag{44}$$

which is a contradiction by assumption (A_5). Thus, for some positive number q, $\Pi(B_q) \subset B_q$.

Step 2. Π_1 is a contractive mapping. Let $x, v \in B_q$. From the assumptions on G and σ, it is easy to verify that the following inequality holds:

$$\begin{aligned} &E\left|u_{x+\tilde{\phi}}(s) - u_{v+\tilde{\phi}}(s)\right|_{\mathbb{H}}^2 \\ &\le 3N_2\Big[\left\|A^{-\beta}\right\|^2 M_G\|x_b - v_b\|_{\mathscr{C}_h}^2 \\ &\quad + \frac{\alpha^2C_{1-\beta}^2\Gamma^2(1+\beta)}{\Gamma^2(1+\alpha\beta)}\frac{b^{2\alpha\beta-1}}{2\alpha\beta-1}\int_0^b M_G\|x_s - v_s\|_{\mathscr{C}_h}^2 ds \\ &\quad + \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2\frac{b^{2\alpha-1}}{2\alpha-1}\int_0^b M_\sigma\operatorname{Tr} Q\|x_s - v_s\|_{\mathscr{C}_h}^2 ds\Big]. \end{aligned} \tag{45}$$

Thus, by the assumptions, we have

$$\begin{aligned} &E\left|(\Pi_1x)(t) - (\Pi_1v)(t)\right|_{\mathbb{H}}^2 \\ &\le 3E\left|G\left(t,\tilde{\phi}_t + x_t\right) - G\left(t,\tilde{\phi}_t + v_t\right)\right|_{\mathbb{H}}^2 \\ &\quad + 3E\Big|\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s) \\ &\qquad\times\left[G\left(s,\tilde{\phi}_s + x_s\right) - G\left(s,\tilde{\phi}_s + v_s\right)\right]ds\Big|_{\mathbb{H}}^2 \\ &\quad + 3E\left|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)B\left(u_{\tilde{\phi}+x} - u_{\tilde{\phi}+v}\right)ds\right|_{\mathbb{H}}^2 \\ &\le 3M_0l^2(1+3N_1N_2N)\sup_{0\le s\le b}E|x(s) - v(s)|_{\mathbb{H}}^2 \\ &\quad + 3PM_Gl^2(1+3N_1N_2N)\sup_{0\le s\le b}E|x(s) - v(s)|_{\mathbb{H}}^2 \\ &\quad + 9N_1N_2N^2\operatorname{Tr} QM_\sigma l^2\sup_{0\le s\le b}E|x(s) - v(s)|_{\mathbb{H}}^2, \end{aligned} \tag{46}$$

where we have used the fact that $u_0 = v_0 = 0$. Hence,

$$\|\Pi_1x - \Pi_1v\|_b^2 \le L\|u - v\|_b^2; \tag{47}$$

so, Π_1 is a contraction by (23).

Step 3. We show that the operator Π_2 is compact. Let $q > 0$ be such that $\Pi_2(B_q) \subset B_q$. The proof will be divided into the following claims.

Claim 1. Π_2 maps bounded sets into bounded sets in B_q.

Indeed, it is enough to show that there exists a positive constant Δ such that for each $t \in J, z \in B_q$, we have $\| \Pi_2 z\|_b^2 \leq \Delta$. If $z \in B_q$, from (32), it follows that

$$\left\|z_t + \widetilde{\phi}_t\right\|_{\mathscr{C}_h}^2 \leq 4l^2 \left(q + M^2 E\left|\phi(0)\right|_{\mathbb{H}}^2\right) + 4\|\phi\|_{\mathscr{C}_h}^2 := q'. \quad (48)$$

By the similar argument as before, we get

$$\begin{aligned} E\left|\Pi_2 z(t)\right|_{\mathbb{H}}^2 &\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \int_0^t (t-s)^{2(\alpha-1)} ds \\ &\quad \times \int_0^t E\left|\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right) dW(\tau)\right|_{\mathbb{H}}^2 ds \\ &\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{\operatorname{Tr} Q b^{2\alpha-1}}{2\alpha-1} \Xi\left(q'\right) \int_0^t m(s)\, ds \\ &\leq \left(\frac{M\alpha}{\Gamma(1+\alpha)}\right)^2 \frac{\operatorname{Tr} Q b^{2\alpha-1}}{2\alpha-1} \Xi\left(q'\right) \int_0^b m(s)\, ds \\ &:= \Delta. \end{aligned} \quad (49)$$

Therefore, for each $z \in B_q$, it holds that $\| \Pi_2 z\|_b^2 \leq \Delta$.

Claim 2. The set $\{\Pi_2 z, z \in B_q\}$ is an equicontinuous family of functions on J. Let $0 < \epsilon < t < b$ and $\delta > 0$ such that $\| T_\alpha(s_1) - T_\alpha(s_2) \| < \epsilon$, for every $s_1, s_2 \in J$ with $|s_1 - s_2| < \delta$. For $z \in B_q$, $0 < |h| < \delta$, $t + h \in J$, we have

$$\begin{aligned} &E\left|\Pi_2 z(t+h) - \Pi_2 z(t)\right|_{\mathbb{H}}^2 \\ &\quad = E\left|\int_0^{t+h} (t+h-s)^{\alpha-1} T_\alpha(t+h-s) \right. \\ &\qquad \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right) dW(\tau)\right] ds \\ &\qquad \left. - \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \right. \\ &\qquad \left. \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right) dW(\tau)\right] ds\right|_{\mathbb{H}}^2 \\ &\quad \leq 3E\left|\int_t^{t+h} (t+h-s)^{\alpha-1} T_\alpha(t+h-s) \right. \\ &\qquad \left. \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right) dW(\tau)\right] ds\right|_{\mathbb{H}}^2 \\ &\qquad + 3E\left|\int_0^t \left[(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] T_\alpha(t+h-s) \right. \\ &\qquad \left. \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right) dW(\tau)\right] ds\right|_{\mathbb{H}}^2 \\ &\qquad + 3E\left|\int_0^t (t-s)^{\alpha-1} \left[T_\alpha(t+h-s) - T_\alpha(t-s)\right] \right. \\ &\qquad \left. \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right) dW(\tau)\right] ds\right|_{\mathbb{H}}^2. \end{aligned} \quad (50)$$

Applying assumption (A_2), Hölder inequality, together with Lemma 7, we get

$$\begin{aligned} &E\left|\Pi_2 z(t+h) - \Pi_2 z(t)\right|_{\mathbb{H}}^2 \\ &\quad \leq 3Z(\alpha) \int_t^{t+h} (t+h-s)^{2(\alpha-1)} ds \\ &\qquad \times \int_t^{t+h} E\left|\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right) dW(\tau)\right|_{\mathbb{H}}^2 ds \\ &\qquad + 3Z(\alpha) \int_0^t \left[(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right]^2 ds \\ &\qquad \times \int_0^t E\left|\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right) dW(\tau)\right|_{\mathbb{H}}^2 ds \\ &\qquad + 3\epsilon \int_0^t (t-s)^{2(\alpha-1)} ds \\ &\qquad \times \int_0^t E\left|\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right) dW(\tau)\right|_{\mathbb{H}}^2 ds \\ &\quad \leq 3Z(\alpha) \frac{h^{2\alpha-1}}{2\alpha-1} \int_t^{t+h} \operatorname{Tr} Q M_\sigma m(s)\, \Xi\left(q'\right) ds \\ &\qquad + 3Z(\alpha) \int_0^t \left[(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\right]^2 ds \\ &\qquad \times \int_0^t \operatorname{Tr} Q M_\sigma m(s)\, \Xi\left(q'\right) ds \\ &\qquad + 3\epsilon \int_0^t (t-s)^{2(\alpha-1)} ds \int_0^t \operatorname{Tr} Q M_\sigma m(s)\, \Xi\left(q'\right) ds, \end{aligned} \quad (51)$$

with $Z(\alpha) = (M\alpha/\Gamma(1+\alpha))^2$. Therefore, for ϵ sufficiently small, the right hand side of (51) tends to zero as $h \to 0$. Thus, the set $\{\Pi_2 z, z \in B_q\}$ is equicontinuous by assumption (A_1).

Claim 3. Π_2 maps B_q into a precompact set in B_q. Let $0 < t \leq b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $\delta > 0$, define an operator $\Pi_2^{\epsilon,\delta}$ on B_q by

$$\begin{aligned} &\left(\Pi_2^{\epsilon,\delta} z\right)(t) \\ &\quad = \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{\alpha-1} \eta_\alpha(\theta)\, S\left((t-s)^\alpha \theta\right) \\ &\qquad \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_\tau + z_\tau\right) dW(\tau)\right] d\theta\, ds \end{aligned}$$

$$= S(\epsilon^{\alpha}\delta)\,\alpha \int_0^{t-\epsilon}\int_{\delta}^{\infty} \theta(t-s)^{\alpha-1}\eta_{\alpha}(\theta)\, S \times \left((t-s)^{\alpha}\theta - \epsilon^{\alpha}\delta\right) \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_{\tau}+z_{\tau}\right) dW(\tau)\right] d\theta\, ds. \tag{52}$$

Since $S(t)\sigma(s,\tau,\widetilde{\phi}_{\tau}+z_{\tau})$ belongs to the compact set $\mho$, the set $\{\Pi_2^{\epsilon,\delta}z)(t), z \in B_q\}$ is precompact in $\mathbb{H}$ for every $\epsilon \in (0,t)$, $\delta > 0$. Moreover, for each $z \in B_q$, we have

$$\begin{aligned} &E\left|(\Pi_2 z)(t) - \left(\Pi_2^{\epsilon,\delta} z\right)(t)\right|_{\mathbb{H}}^2 \\ &\le 2\alpha^2 E\left|\int_0^t\int_0^{\delta} \theta(t-s)^{\alpha-1}\eta_{\alpha}(\theta)\, S\left((t-s)^{\alpha}\theta\right) \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_{\tau}+z_{\tau}\right) dW(\tau)\right] d\theta\, ds\right|_{\mathbb{H}}^2 \\ &\quad + 2\alpha^2 E\left|\int_{t-\epsilon}^t\int_{\delta}^{\infty} \theta(t-s)^{\alpha-1}\eta_{\alpha}(\theta)\, S\left((t-s)^{\alpha}\theta\right) \times \left[\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_{\tau}+z_{\tau}\right) dW(\tau)\right] d\theta\, ds\right|_{\mathbb{H}}^2 \\ &= J_1 + J_2. \end{aligned} \tag{53}$$

An application of Hölder inequality together with Burkhölder-Davis-Gundy's inequality, it follows that

$$\begin{aligned} J_1 &\le 2\alpha^2 M^2 \int_0^t (t-s)^{2(\alpha-1)} E \times \left|\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_{\tau}+z_{\tau}\right) dW(\tau)\right|_{\mathbb{H}}^2 ds \times \int_0^t \left|\int_0^{\delta} \theta\eta_{\alpha}(\theta)\, d\theta\right|^2 ds \\ &\le 2\alpha^2 M^2 b \int_0^t (t-s)^{2(\alpha-1)} \operatorname{Tr} Q \times \int_0^s E\left|\sigma\left(s,\tau,\widetilde{\phi}_{\tau}+z_{\tau}\right)\right|_{\mathscr{L}_2^0}^2 d\tau\, ds \left|\int_0^{\delta} \theta\eta_{\alpha}(\theta) d\theta\right|^2 \\ &\le 2\alpha^2 M^2 b \int_0^t \operatorname{Tr} Q\Xi\left(q'\right) m(s)\,(t-s)^{2(\alpha-1)} ds \times \left|\int_0^{\delta} \theta\eta_{\alpha}(\theta)\, d\theta\right|^2, \\ J_2 &\le 2\alpha^2 M^2 \int_{t-\epsilon}^t (t-s)^{2(\alpha-1)} E \times \left|\int_0^s \sigma\left(s,\tau,\widetilde{\phi}_{\tau}+z_{\tau}\right) dW(\tau)\right|_{\mathbb{H}}^2 ds \times \int_{t-\epsilon}^t \left|\int_0^{\infty} \theta\eta_{\alpha}(\theta)\, d\theta\right|^2 ds \\ &\le \frac{2\alpha^2 M^2 \epsilon}{\Gamma^2(1+\alpha)} \int_{t-\epsilon}^t \operatorname{Tr} Q\Xi\left(q'\right) m(s)\,(t-s)^{2(\alpha-1)} ds. \end{aligned} \tag{54}$$

From (54), we deduce that for each $z \in B_q$,

$$E\left|(\Pi_2 z)(t) - \left(\Pi_2^{\epsilon,\delta} z\right)(t)\right|_{\mathbb{H}}^2 \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0^+,\ \delta \longrightarrow 0^+. \tag{55}$$

Therefore, there are relatively compact sets arbitrary close to the set $\{\Pi_2(t), z \in B_q\}$; hence, the set $\{(\Pi_2 z)(t), z \in B_q\}$ is also precompact in B_q.

Thus, from Arzelá-Ascoli theorem together with assumptions on f and σ, we conclude that Π_2 is a compact operator. Therefore, Π is a condensing map on B_q. This completes the proof. □

Remark 11. In order to describe various real-world problems in physical and engineering sciences subject to abrupt changes at certain instants during the evolution process, impulsive differential equations have been used to model the system. The technique used here can be extended to establish the controllability of neutral fractional stochastic integro-differential systems with impulsive effect and infinite delay. The controllability result can be obtained by suitably introducing the impulsive effects defined in [19].

4. Conclusions

In this paper, we have studied the controllability of fractional neutral stochastic integro-differential systems with infinite delay in an abstract space. Through fractional calculus and Sadovskii's fixed point principle, we have investigated the sufficient conditions for the controllability of the system considered.

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