

Research Article

A Study of Nonlinear Fractional Differential Equations of Arbitrary Order with Riemann-Liouville Type Multistrip Boundary Conditions

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We develop the existence theory for nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type boundary conditions involving nonintersecting finite many strips of arbitrary length. Our results are based on some standard tools of fixed point theory. For the illustration of the results, some examples are also discussed.

1. Introduction

The subject of fractional calculus has recently developed into a hot topic for the researchers in view of its numerous applications in the field of physics, mechanics, chemistry, engineering, and so forth. One can find the systematic progress of the topic in the books ([1–6]). A significant characteristic of a fractional-order differential operator distinguishing it from the integer-order differential operator is that it is nonlocal in nature, that is, the future state of a dynamical system or process involving fractional derivative depends on its current state as well its past states. In fact, this feature of fractional-order operators has contributed towards the popularity of fractional-order models, which are recognized as more realistic and practical than the classical integer-order models. In other words, we can say that the memory and hereditary properties of various materials and processes can be described by differential equations of arbitrary order. There has been a rapid development in the theoretical aspects such as periodicity, asymptotic behavior, and numerical methods for fractional equations. For some recent work on the topic, see ([7–23]) and the references therein. In particular, Ahmad et al. [22] studied nonlinear fractional differential equations

and inclusions of arbitrary order with multistrip boundary conditions.

In this paper, we continue the study initiated in [22] and consider a boundary value problem of fractional differential equations of arbitrary order $q \in (n-1, n]$, $n \geq 2$ with finite many multistrip Riemann-Liouville type integral boundary conditions:

$$\begin{aligned} {}^c D^q x(t) &= f(t, x(t)), \quad t \in [0, T], \\ x(0) &= 0, \quad x'(0) = 0, \dots, x^{(n-2)}(0) = 0, \\ x(T) &= \sum_{i=1}^m \gamma_i \left[I^{\beta_i} x(\eta_i) - I^{\beta_i} x(\zeta_i) \right], \end{aligned} \quad (1)$$

where ${}^c D^q$ denotes the Caputo fractional derivative of order q , f is a given continuous function, I^{β_i} is the Riemann-Liouville fractional integral of order $\beta_i > 0$, $i = 1, 2, \dots, m$, $0 < \zeta_1 < \eta_1 < \zeta_2 < \eta_2 < \dots < \zeta_m < \eta_m < T$, and $\gamma_i \in \mathbb{R}$ are suitable chosen constants.

Regarding the motivation of the problem, we know that the strip conditions appear in the mathematical modeling of

certain real world problems, for instance, see [24, 25]. In [22], the authors considered the nonlocal strip conditions of the form:

$$x(1) = \sum_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} x(s) ds, \quad 0 < \zeta_i < \eta_i < 1, \quad (2)$$

$$i = 1, 2, \dots, (n-2).$$

In the problem (1), we have introduced Riemann-Liouville type multistrip integral boundary conditions which can be interpreted as the controller at the right-end of the interval under consideration is influenced by a discrete distribution of finite many nonintersecting sensors (strips) of arbitrary length expressed in terms of Riemann-Liouville type integral boundary conditions. For some engineering applications of strip conditions, see ([26–32]).

The main objective of the present study is to develop some existence results for the problem (1) by using standard techniques of fixed point theory. The paper is organized as follows. In Section 2 we discuss a linear variant of the problem (1), which plays a key role in developing the main results presented in Section 3. For the illustration of the theory, we have also included some examples.

2. Preliminary Result

Let us begin this section with some basic definitions of fractional calculus [2–4].

Definition 1. If $g(t) \in AC^n[a, b]$, then the Caputo derivative of fractional order q is defined as

$$\begin{aligned} {}^c D_{a^+}^q g(t) &= \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} g^{(n)}(s) ds \\ &= I_{a^+}^{n-q} D^n g(x), \quad n-1 < q < n, \quad n = [q] + 1, \end{aligned} \quad (3)$$

where $[q]$ denotes the integer part of the real number q . For details, see Theorem 2.1 ([4, page 92]). Here $AC^n[a, b]$ denote the space of real valued functions $g(t)$ which have continuous derivatives up to order $n-1$ on $[a, b]$ such that $g^{(n-1)}(t) \in AC[a, b]$.

Definition 2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, \quad (4)$$

provided the integral exists.

The following result associated with a linear variant of problem (1) plays a pivotal role in establishing the main results.

Lemma 3. For $h \in C[0, T]$, the fractional boundary value problem

$$\begin{aligned} {}^c D^q x(t) &= h(t), \quad t \in [0, T], \quad q \in (n-1, n] \\ x(0) &= 0, \quad x'(0) = 0, \dots, x^{(n-2)}(0) = 0, \end{aligned} \quad (5)$$

$$x(T) = \sum_{i=1}^m \gamma_i \left[I^{\beta_i} x(\eta_i) - I^{\beta_i} x(\zeta_i) \right],$$

has a unique solution $x(t) \in AC^n[0, T]$ given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds \\ &\quad - \frac{t^{n-1}}{\lambda \Gamma(q)} \int_0^T (T-s)^{q-1} h(s) ds \\ &\quad + \frac{t^{n-1}}{\lambda \Gamma(q)} \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \\ &\quad \times \left[\int_0^{\eta_i} \int_0^s (\eta_i-s)^{\beta_i-1} \right. \\ &\quad \times (s-u)^{q-1} h(u) du ds \\ &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i-s)^{\beta_i-1} (s-u)^{q-1} \right. \\ &\quad \left. \times h(u) du ds \right], \end{aligned} \quad (6)$$

where

$$\lambda = \left(T^{n-1} - \sum_{i=1}^m \gamma_i \frac{(\eta_i^{\beta_i+n-1} - \zeta_i^{\beta_i+n-1}) \Gamma(n)}{\Gamma(\beta_i + n)} \right) \neq 0. \quad (7)$$

Proof. The general solution of fractional differential equations in (5) can be written as

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds - c_0 - c_1 t - \dots - c_{n-1} t^{n-1}. \quad (8)$$

Using the given boundary conditions, it is found that $c_0 = 0$, $c_1 = 0, \dots, c_{n-2} = 0$. Applying the Riemann-Liouville integral operator I^{β_i} on (8), we get

$$\begin{aligned} I^{\beta_i} x(t) &= \frac{1}{\Gamma(\beta_i)} \int_0^t (t-s)^{\beta_i-1} \\ &\quad \times \left(\frac{1}{\Gamma(q)} \int_0^s (s-u)^{q-1} h(u) du - c_{n-1} s^{n-1} \right) ds \\ &= \frac{1}{\Gamma(\beta_i) \Gamma(q)} \int_0^t \int_0^s (t-s)^{\beta_i-1} (s-u)^{q-1} h(u) du ds \\ &\quad - c_{n-1} \frac{1}{\Gamma(\beta_i)} \int_0^t (t-s)^{\beta_i-1} s^{n-1} ds. \end{aligned} \quad (9)$$

Using the condition $x(T) = \sum_{i=1}^m \gamma_i [I^{\beta_i} x(\eta_i) - I^{\beta_i} x(\zeta_i)]$, together with the fact that

$$\frac{1}{\Gamma(\beta_i)} \int_0^t (t-s)^{\beta_i-1} s^{n-1} ds = \frac{t^{\beta_i+n-1} \Gamma(n)}{\Gamma(\beta_i+n)}, \quad (10)$$

we obtain

$$\begin{aligned} &\frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} h(s) ds - c_{n-1} T^{n-1} \\ &= \sum_{i=1}^m \frac{\gamma_i}{\Gamma(q) \Gamma(\beta_i)} \\ &\quad \times \left[\int_0^{\eta_i} \int_0^s (\eta_i-s)^{\beta_i-1} (s-u)^{q-1} h(u) du ds \right. \\ &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i-s)^{\beta_i-1} (s-u)^{q-1} h(u) du ds \right] \\ &\quad - c_{n-1} \sum_{i=1}^m \gamma_i \frac{(\eta_i^{\beta_i+n-1} - \zeta_i^{\beta_i+n-1}) \Gamma(n)}{\Gamma(\beta_i+n)}, \end{aligned} \quad (11)$$

which yields

$$\begin{aligned} c_{n-1} &= \frac{1}{\lambda \Gamma(q)} \int_0^T (T-s)^{q-1} h(s) ds \\ &\quad - \frac{1}{\lambda \Gamma(q)} \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \\ &\quad \times \left[\int_0^{\eta_i} \int_0^s (\eta_i-s)^{\beta_i-1} \right. \\ &\quad \times (s-u)^{q-1} h(u) du ds \\ &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i-s)^{\beta_i-1} (s-u)^{q-1} \right. \\ &\quad \times h(u) du ds \Big], \end{aligned} \quad (12)$$

where λ is given by (7). Substituting the values of $c_0, c_1, \dots, c_{n-2}, c_{n-1}$ in (8), we obtain (6). This completes the proof. \square

3. Main Results

Let $\mathcal{C} := C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions defined on $[0, T] \times \mathbb{R}$ endowed with a topology of uniform convergence with the norm $\|x\| = \sup_{t \in [0, T]} |x(t)|$.

By Lemma 3, we define an operator $\mathcal{P} : \mathcal{C} \rightarrow \mathcal{C}$ as

$$\begin{aligned} (\mathcal{P}x)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds \\ &\quad - \frac{t^{n-1}}{\lambda \Gamma(q)} \int_0^T (T-s)^{q-1} f(s, x(s)) ds \\ &\quad + \frac{t^{n-1}}{\lambda \Gamma(q)} \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \\ &\quad \times \left[\int_0^{\eta_i} \int_0^s (\eta_i-s)^{\beta_i-1} (s-u)^{q-1} \right. \\ &\quad \times f(u, x(u)) du ds \\ &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i-s)^{\beta_i-1} (s-u)^{q-1} \right. \\ &\quad \times f(u, x(u)) du ds \Big], \end{aligned} \quad (13)$$

$t \in [0, T].$

Observe that the problem (1) has a solution if and only if the associated fixed point problem $\mathcal{P}x = x$ has a fixed point.

In the first result we prove an existence and uniqueness result by means of Banach's contraction mapping principle. For the sake of convenience, we set

$$\begin{aligned} \Lambda &= \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda| \Gamma(q+1)} \\ &\quad + \frac{T^{n-1}}{|\lambda|} \sum_{i=1}^m \gamma_i \frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)}. \end{aligned} \quad (14)$$

Theorem 4. Suppose that $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and satisfies the following assumption:

(A₃)

$$\begin{aligned} |f(t, x) - f(t, y)| &\leq L|x - y|, \\ \forall t \in [0, 1], \quad L > 0, \quad x, y \in \mathbb{R}. \end{aligned} \quad (15)$$

Then the boundary value problem (1) has a unique solution provided

$$L < \frac{1}{\Lambda}, \quad (16)$$

where Λ is given by (14).

Proof. With $r \geq M\Lambda/(1 - L\Lambda)$, we define $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$, where $M = \sup_{t \in [0, T]} |f(t, 0)| < \infty$ and Λ is given by (14). Then we show that $\mathcal{P}B_r \subset B_r$. For $x \in B_r$, by means of the inequality $|f(s, x(s))| \leq |f(s, x(s)) - f(s, 0)| + |f(s, 0)| \leq L\|x\| + M \leq Lr + M$, it can easily be shown that

$$\|\mathcal{P}x\| = (Lr + M)\Lambda \leq r. \quad (17)$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0, T]$, we obtain

$$\begin{aligned} & \|(\mathcal{P}x) - (\mathcal{P}y)\| \\ & \leq \sup_{t \in [0, T]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s)) - f(s, y(s))| ds \right. \\ & \quad + \frac{t^{n-1}}{\lambda \Gamma(q)} \int_0^T (T-s)^{q-1} \\ & \quad \times |f(s, x(s)) - f(s, y(s))| ds \\ & \quad + \frac{t^{n-1}}{\lambda \Gamma(q)} \\ & \quad \times \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \\ & \quad \times \left[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} \right. \\ & \quad \times (s-u)^{q-1} du \\ & \quad \times |f(u, x(u)) - f(u, y(u))| ds \\ & \quad - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} (s-u)^{q-1} \\ & \quad \times |f(u, x(u)) \\ & \quad \left. \left. - f(u, y(u))| du ds \right] \right\} \\ & \leq L\Lambda \|x - y\|. \end{aligned} \quad (18)$$

Note that Λ depends only on the parameters involved in the problem. As $L\Lambda < 1$, therefore \mathcal{P} is a contraction. Hence, by Banach's contraction mapping principle, the problem (1) has a unique solution on $[0, T]$. \square

Example 5. Let us consider the following 4-strip nonlocal boundary value problem:

$$\begin{aligned} {}^c D^{9/2} x(t) &= f(t, x(t)), \quad t \in [0, 2], \\ x(0) &= 0, \quad x'(0) = 0, \quad x''(0) = 0, \quad x'''(0) = 0, \\ x(T) &= \sum_{i=1}^4 \gamma_i [I^{\beta_i} x(\eta_i) - I^{\beta_i} x(\zeta_i)], \end{aligned} \quad (19)$$

where $q = 9/2$, $n = 5$, $\zeta_1 = 1/4$, $\eta_1 = 1/2$, $\zeta_2 = 2/3$, $\eta_2 = 1$, $\zeta_3 = 5/4$, $\eta_3 = 4/3$, $\zeta_4 = 3/2$, $\eta_4 = 7/4$, $\gamma_1 = 5$, $\gamma_2 = 10$, $\gamma_3 = 15$, $\gamma_4 = 25$, $\beta_1 = 5/4$, $\beta_2 = 7/4$, $\beta_3 = 9/4$, $\beta_4 = 11/4$.

With the given values of the parameters involved, we find that

$$\begin{aligned} \lambda &= \left(T^{n-1} - \sum_{i=1}^m \gamma_i \frac{(\eta_i^{\beta_i+n-1} - \zeta_i^{\beta_i+n-1}) \Gamma(n)}{\Gamma(\beta_i + n)} \right) \\ &\approx 9.334784, \\ \Lambda &= \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda| \Gamma(q+1)} + \frac{T^{n-1}}{|\lambda|} \sum_{i=1}^m \gamma_i \frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q + \beta_i + 1)} \\ &\approx 1.406972. \end{aligned} \quad (20)$$

Let us choose

$$f(t, x(t)) = \frac{1}{\sqrt[3]{t+8}} (\tan^{-1} x) + \sqrt{4+3 \sin 2t}. \quad (21)$$

Clearly $L = 1/2$ as $|f(t, x) - f(t, y)| \leq (1/2)|x - y|$ and $L < 1/\Lambda$, where $\Lambda \approx 1.406972$. Therefore all the conditions of Theorem 4 hold and consequently there exists a unique solution for the problem (19) with $f(t, x(t))$ given by (21).

In case of the following unbounded nonlinear function:

$$f(t, x(t)) = \frac{x}{7} + \frac{1}{\sqrt[3]{t+8}} (\tan^{-1} x) + \sqrt{4+3 \sin 2t}, \quad (22)$$

we have $L = 9/14$ and $L < 1/\Lambda$ ($\Lambda \approx 1.406972$). As before, the problem (19) with $f(t, x(t))$ given by (22) has a unique solution.

In the second result we use the Leray-Schauder alternative.

Theorem 6 ((Leray-Schauder alternative) [33, page 4]). *Let X be a Banach space. Assume that $T : X \rightarrow X$ is completely continuous operator and the set*

$$V = \{u \in X \mid u = \mu T u, \quad 0 < \mu < 1\} \quad (23)$$

is bounded. Then T has a fixed point in X .

Theorem 7. *Assume that there exists a positive constant L_1 such that $|f(t, x)| \leq L_1$ for $t \in [0, T]$, $x \in \mathbb{R}$. Then the problem (1) has at least one solution.*

Proof. First of all, we show that the operator \mathcal{P} is completely continuous. Note that the operator \mathcal{P} is continuous in view

of the continuity of f . Let $\mathcal{B} \subset \mathcal{C}$ be a bounded set. By the assumption that $|f(t, x)| \leq L_1$, for $x \in \mathcal{B}$, we have

$$\begin{aligned}
 |(\mathcal{P}x)(t)| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s))| ds \\
 &+ \frac{t^{n-1}}{|\lambda| \Gamma(q)} \int_0^T (T-s)^{q-1} |f(s, x(s))| ds \\
 &+ \frac{t^{n-1}}{|\lambda| \Gamma(q)} \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \\
 &\quad \times \left[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} (s-u)^{q-1} \right. \\
 &\quad \times |f(u, x(u))| du ds \\
 &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} (s-u)^{q-1} \right. \\
 &\quad \times |f(u, x(u))| du ds \Big] \\
 &\leq L_1 \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{t^{n-1}}{|\lambda| \Gamma(q)} \int_0^T (T-s)^{q-1} ds \right. \\
 &\quad + \frac{t^{n-1}}{|\lambda| \Gamma(q)} \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \\
 &\quad \times \left[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} \right. \\
 &\quad \times (s-u)^{q-1} du ds \\
 &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} \right. \\
 &\quad \times (s-u)^{q-1} du ds \Big] \Big] \\
 &\leq L_1 \left\{ \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda| \Gamma(q+1)} \right. \\
 &\quad \left. + \frac{T^{n-1}}{|\lambda|} \sum_{i=1}^m \gamma_i \frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)} \right\} = L_2,
 \end{aligned} \tag{24}$$

which implies that $\|(\mathcal{P}x)\| \leq L_2$. Further, we find that

$$\begin{aligned}
 |(\mathcal{P}x)'(t)| &= \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} |f(s, x(s))| ds \\
 &+ \frac{(n-1)t^{n-2}}{|\lambda| \Gamma(q)} \int_0^T (T-s)^{q-1} |f(s, x(s))| ds
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{(n-1)t^{n-2}}{|\lambda| \Gamma(q)} \\
 &\times \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \left[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} (s-u)^{q-1} \right. \\
 &\quad \times |f(u, x(u))| du ds \\
 &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} (s-u)^{q-1} \right. \\
 &\quad \times |f(u, x(u))| du ds \Big] \\
 &\leq L_1 \left[\frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} ds \right. \\
 &\quad + \frac{(n-1)t^{n-2}}{|\lambda| \Gamma(q)} \int_0^T (T-s)^{q-1} ds \\
 &\quad + \frac{(n-1)t^{n-2}}{|\lambda| \Gamma(q)} \\
 &\quad \times \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \left(\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} \right. \\
 &\quad \times (s-u)^{q-1} \\
 &\quad \times |f(u, x(u))| du ds \\
 &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} \right. \\
 &\quad \times (s-u)^{q-1} \\
 &\quad \times |f(u, x(u))| \\
 &\quad \times du ds \Big) \Big] \\
 &\leq L_1 \left\{ \frac{T^{q-1}}{\Gamma(q)} + \frac{(n-1)T^{q+n-2}}{|\lambda| \Gamma(q+1)} \right. \\
 &\quad \left. + \frac{(n-1)T^{n-2}}{|\lambda|} \sum_{i=1}^m \gamma_i \frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)} \right\} = L_3.
 \end{aligned} \tag{25}$$

Hence, for $t_1, t_2 \in [0, T]$, we have

$$|(\mathcal{P}x)(t_2) - (\mathcal{P}x)(t_1)| \leq \int_{t_1}^{t_2} |(\mathcal{P}x)'(s)| ds \leq L_3 (t_2 - t_1). \tag{26}$$

This implies that \mathcal{P} is equicontinuous on $[0, T]$. Thus, by the Arzelà-Ascoli theorem, the operator $\mathcal{P} : \mathcal{C} \rightarrow \mathcal{C}$ is completely continuous.

Next, we consider the set

$$V = \{x \in \mathcal{C} \mid x = \mu \mathcal{P}x, \ 0 < \mu < 1\}, \tag{27}$$

and show that the set V is bounded. Let $x \in V$, then $x = \mu \mathcal{P}x$, $0 < \mu < 1$. For any $t \in [0, T]$, we have

$$\begin{aligned}
 |x(t)| &= \mu |(\mathcal{P}x)(t)| \\
 &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s))| ds \\
 &\quad + \frac{t^{n-1}}{|\lambda| \Gamma(q)} \int_0^T (T-s)^{q-1} |f(s, x(s))| ds \\
 &\quad + \frac{t^{n-1}}{|\lambda| \Gamma(q)} \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \\
 &\quad \times \left[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} (s-u)^{q-1} \right. \\
 &\quad \times |f(u, x(u))| du ds \\
 &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} (s-u)^{q-1} \right. \\
 &\quad \times |f(u, x(u))| du ds \Big] \\
 &\leq L_1 \left\{ \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda| \Gamma(q+1)} \right. \\
 &\quad \left. + \frac{T^{n-1}}{|\lambda|} \sum_{i=1}^m \gamma_i \frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)} \right\} = M_1.
 \end{aligned} \tag{28}$$

Thus, $\|x\| \leq M_1$ for any $t \in [0, T]$. So, the set V is bounded. Thus, by the conclusion of Theorem 6, the operator \mathcal{P} has at least one fixed point, which implies that (1) has at least one solution. \square

Example 8. Consider the boundary value problem of Example 5 with

$$f(t, x(t)) = \frac{3e^{\sqrt{(2-|x(t))|^3}} [\cos 4t + 2 \ln(1 + 4 \sin^2 x(t))]}{\sqrt{(10 + \cos x(t))}}. \tag{29}$$

Observe that $|f(t, x)| \leq L_1$ with $L_1 = e^{2\sqrt{2}}(1 + \ln 25)$. Thus the hypothesis of Theorem 7 is satisfied. Hence by the conclusion of Theorem 7, the problem (19) with $f(t, x(t))$ given by (29) has at least one solution.

In the next we prove one more existence result for problem (1), based on the following known result.

Theorem 9 (see [34]). *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$ and let $T : \bar{\Omega} \rightarrow X$ be a completely continuous operator such that*

$$\|Tu\| \leq \|u\|, \quad \forall u \in \partial\Omega. \tag{30}$$

Then T has a fixed point in $\bar{\Omega}$.

Theorem 10. *Let there exist a small positive number τ such that $|f(t, x)| \leq \nu|x|$ for $0 < |x| < \tau$, with $0 < \nu \leq 1/\Lambda$, where Λ is given by (14). Then the problem (1) has at least one solution.*

Proof. Let us define $\mathcal{B}_\tau = \{x \in \mathcal{C} \mid \|x\| < \tau\}$ and take $x \in \mathcal{C}$ such that $\|x\| = \tau$, that is, $x \in \partial\mathcal{B}_\tau$. As before, it can be shown that \mathcal{P} is completely continuous and

$$\begin{aligned}
 \|\mathcal{P}x\| &\leq \sup_{t \in [0, t]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s))| ds \right. \\
 &\quad + \frac{t^{n-1}}{|\lambda| \Gamma(q)} \\
 &\quad \times \int_0^T (T-s)^{q-1} |f(s, x(s))| ds \\
 &\quad + \frac{t^{n-1}}{|\lambda| \Gamma(q)} \\
 &\quad \times \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \\
 &\quad \times \left[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} (s-u)^{q-1} \right. \\
 &\quad \times |f(u, x(u))| du ds \\
 &\quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} (s-u)^{q-1} \right. \\
 &\quad \times |f(u, x(u))| du ds \Big] \Big\} \\
 &\leq \Lambda \nu \|x\|,
 \end{aligned} \tag{31}$$

which in view of the given condition ($\nu\Lambda \leq 1$), gives $\|\mathcal{P}x\| \leq \|x\|$, $x \in \partial\mathcal{B}_\tau$. Therefore, by Theorem 9, the operator \mathcal{P} has at least one fixed point, which in turn implies that the problem (1) has at least one solution. \square

Example 11. Consider the boundary value problem of Example 5 and let us consider

$$\begin{aligned}
 f(t, x(t)) &= x(b^5 + x^4(t))^{1/5} + 2(1 + \cos(t^4 + 3))^5 \\
 &\quad \times (1 - \cos x(t)), \quad x \neq 0, \quad b > 0.
 \end{aligned} \tag{32}$$

For sufficiently small x (ignoring x^2 and higher powers of x), we have

$$\begin{aligned}
 &\left| x(b^5 + x^4(t))^{1/5} + 2(1 + \cos(t^4 + 3))^5 (1 - \cos x(t)) \right| \\
 &\leq b|x|.
 \end{aligned} \tag{33}$$

Choosing $b \leq 1/\Lambda$, all the assumptions of Theorem 10 hold. Therefore, the conclusion of Theorem 10 implies that the

problem (19) with $f(t, x(t))$ given by (32) has at least one solution.

Our final existence result is based on Leray-Schauder nonlinear alternative.

Lemma 12 ((Nonlinear alternative for single valued maps) [33, page 135]). *Let E be a Banach space, C a closed, convex subset of E , U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \rightarrow C$ is a continuous, compact (i.e., $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

- (i) F has a fixed point in \overline{U} , or
- (ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.

Theorem 13. *Assume that*

- (A₁) *there exist a function $\sigma \in C([0, 1], \mathbb{R}^+)$, and a non-decreasing function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $|f(t, x)| \leq \sigma(t)\psi(\|x\|)$, for all $(t, x) \in [0, T] \times \mathbb{R}$;*
- (A₂) *there exists a constant $M > 0$ such that*

$$\frac{M}{\psi(M) \Lambda \|\sigma\|} > 1. \quad (34)$$

Then the boundary value problem (1) has at least one solution on $[0, T]$.

Proof. Consider the operator $\mathcal{P} : \mathcal{C} \rightarrow \mathcal{C}$ defined by (13). We show that \mathcal{P} maps bounded sets into bounded sets in $C([0, T], \mathbb{R})$. For a positive number r , let $B_r = \{x \in C([0, T], \mathbb{R}) : \|x\| \leq r\}$ be a bounded set in $C([0, T], \mathbb{R})$. Then

$$\begin{aligned} \|\mathcal{P}x\| \leq & \sup_{t \in [0, T]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s))| ds \right. \\ & + \frac{t^{n-1}}{|\lambda| \Gamma(q)} \int_0^T (T-s)^{q-1} |f(s, x(s))| ds \\ & + \frac{t^{n-1}}{|\lambda| \Gamma(q)} \\ & \times \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \\ & \times \left[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} \right. \\ & \times (s-u)^{q-1} \\ & \times |f(u, x(u))| du ds \\ & - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} \\ & \times (s-u)^{q-1} \\ & \times |f(u, x(u))| \\ & \times du ds \left. \right\} \end{aligned}$$

$$\begin{aligned} \leq & \psi(r) \left\{ \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda| \Gamma(q+1)} \right. \\ & \left. + \frac{T^{n-1}}{|\lambda|} \sum_{i=1}^m \gamma_i \frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)} \right\} \|\sigma\|. \end{aligned} \quad (35)$$

Next we show that F maps bounded sets into equicontinuous sets of $C([0, 1], \mathbb{R})$. Let $t', t'' \in [0, 1]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set of $C([0, 1], \mathbb{R})$. Then we obtain

$$\begin{aligned} & |(\mathcal{P}x)(t'') - (\mathcal{P}x)(t')| \\ &= \left| \frac{1}{\Gamma(q)} \int_0^{t''} (t''-s)^{q-1} f(s, x(s)) ds \right. \\ & \quad - \frac{1}{\Gamma(q)} \int_0^{t'} (t'-s)^{q-1} f(s, x(s)) ds \\ & \quad - \frac{[(t'')^{n-1} - (t')^{n-1}]}{\lambda \Gamma(q)} \int_0^T (T-s)^{q-1} f(s, x(s)) ds \\ & \quad + \frac{[(t'')^{n-1} - (t')^{n-1}]}{\lambda \Gamma(q)} \\ & \quad \times \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \left[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} (s-u)^{q-1} \right. \\ & \quad \times |f(u, x(u))| du ds \\ & \quad - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} (s-u)^{q-1} \\ & \quad \times |f(u, x(u))| du ds \left. \right] \Big| \\ &\leq \frac{1}{\Gamma(q)} \int_0^{t'} |(t''-s)^{q-1} - (t'-s)^{q-1}| \psi(r) \sigma(s) ds \\ & \quad + \frac{1}{\Gamma(q)} \int_{t'}^{t''} |t''-s|^{q-1} \psi(r) \sigma(s) ds \\ & \quad + \frac{|(t'')^{n-1} - (t')^{n-1}|}{|\lambda| \Gamma(q)} \int_0^T |T-s|^{q-1} \psi(r) \sigma(s) ds \\ & \quad + \frac{|(t'')^{n-1} - (t')^{n-1}|}{|\lambda| \Gamma(q)} \\ & \quad \times \sum_{i=1}^m \frac{\gamma_i}{\Gamma(\beta_i)} \\ & \quad \times \left[\int_0^{\eta_i} \int_0^s (\eta_i - s)^{\beta_i-1} (s-u)^{q-1} du \psi(r) \sigma(s) ds \right. \\ & \quad \left. - \int_0^{\zeta_i} \int_0^s (\zeta_i - s)^{\beta_i-1} (s-u)^{q-1} \psi(r) \sigma(s) du ds \right]. \end{aligned} \quad (36)$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t'' - t' \rightarrow 0$. As $\mathcal{P} : C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R})$ satisfies the above assumptions, therefore it follows by the Arzelà-Ascoli theorem that \mathcal{P} is completely continuous.

Let x be a solution. Then, for $t \in [0, T]$, and following the similar computations as before, we find that

$$\begin{aligned} |x(t)| &= |\mu(\mathcal{P}x)(t)| \\ &\leq \psi(r) \left\{ \frac{T^q}{\Gamma(q+1)} + \frac{T^{q+n-1}}{|\lambda| \Gamma(q+1)} \right. \\ &\quad \left. + \frac{T^{n-1}}{|\lambda|} \sum_{i=1}^m \gamma_i \frac{\eta_i^{q+\beta_i} - \zeta_i^{q+\beta_i}}{\Gamma(q+\beta_i+1)} \right\} \|\sigma\|. \end{aligned} \quad (37)$$

In consequence, we have

$$\frac{\|x\|}{\psi(\|x\|) \Lambda \|\sigma\|} \leq 1. \quad (38)$$

Thus, by (A_2) , there exists M such that $\|x\| \neq M$. Let us set

$$V = \{x \in C([0, T], \mathbb{R}) : \|x\| < M + 1\}. \quad (39)$$

Note that the operator $\mathcal{P} : \bar{V} \rightarrow C([0, T], \mathbb{R})$ is continuous and completely continuous. From the choice of V , there is no $x \in \partial V$ such that $x = \mu \mathcal{P}(x)$ for some $\mu \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 12), we deduce that \mathcal{P} has a fixed point $x \in \bar{V}$ which is a solution of the problem (1). This completes the proof. \square

Example 14. Consider the boundary value problem of Example 5 with

$$f(t, x(t)) = \frac{1}{\sqrt{t+4}} \left(1 + \frac{|x|}{1+|x|} \right) \leq \sigma(t) \psi(\|x\|). \quad (40)$$

Then $\sigma(t) = 1/\sqrt{t+4}$ and $\psi(\|x\|) = 2$. Using $\|\sigma\| = 1/2$, $\Lambda \approx 1.406972$, we find by the condition (A_2) that $M > \Lambda$. Thus all the assumptions of Theorem 13 are satisfied. Hence, it follows by Theorem 13 that the problem (19) with $f(t, x(t))$ defined by (40) has at least one solution.

If we choose an unbounded nonlinearity as follows:

$$f(t, x(t)) = \frac{1}{\sqrt{t+4}} \left(1 + \frac{|x|}{1+|x|} + \frac{|x|}{2} \right). \quad (41)$$

Then $f(t, x(t)) \leq \sigma(t) \psi(\|x\|)$ with $\sigma(t) = 1/\sqrt{t+4}$ and $\psi(\|x\|) = 2 + \|x\|/2$. Using the earlier arguments, with $\|\sigma\| = 1/2$, $\Lambda \approx 1.406972$, we find that $M > M_1$, $M_1 \approx 2.170392$. Hence the problem (19) with $f(t, x(t))$ given by (41) has at least one solution.

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