Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 326239, 5 pages
http://dx.doi.org/10.1155/2013/326239

Research Article

Application of Variational Iteration Method with Energy Method for Nonlinear Equation Arisen from Packaging System

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Received 31 July 2013; Accepted 9 October 2013

Academic Editor: Oluwole Daniel Makinde

The variational iteration method (VIM) is widely applied for solving various kinds of nonlinear equations. Despite its simplicity and effectiveness, the accuracy of the method may depend on the iteration steps. However, the more iteration steps one makes, the more complex the results may become. To overcome this shortcoming, a new method combining the VIM with energy method (EM) is proposed and applied to study the nonlinear response of cubic packaging system. The analytical expressions of the important parameters such as the maximum displacement response, the maximum acceleration response of the system, the extended period of the shock, and the conditions for inner resonance of system were obtained. The results show that the maximum of the acceleration and the displacement and the extended period of the shock got by this method are very similar to the ones got by Runge-Kutta method. The result provides the new method for the dropping shock problem of nonlinear packaging system.

1. Introduction

Newton's damage boundary concept [1] is the foundation for the present packaging design. In this theory, the product packaging system was considered to be undamped single degree of freedom linear system. These may not be valid because of the complexity of products configuration and the diversity of cushioning material. In procedure of transportation and storage, the dropping shock might lead to serious damage of product. Wang [2, 3] and Wang et al. [4] developed the concept of the dropping damage boundary curve for linear and nonlinear packaging system to evaluate the damage of product as a result of drop impact because most packaged products are damaged as a result of shocks in transportation. In the dropping shock dynamic evaluation of the nonlinear packaging system including cubic and tangent nonlinear system and other systems, it is very important to obtain the dynamic response, the maximum displacement response, the maximum acceleration response of the system, and the extended period of the shock.

For the complexity of nonlinear systems, the numerical method is mainly used for the analysis of dropping shock characteristic [2–5]. Most recently, various analytical approaches for solving nonlinear differential equations were widely applied in the analysis of engineering practical problem [6–21], such as the homotopy perturbation method [6], parameter expansion method [7], energy balance method [8], and variational iteration method (VIM) [9–12]. The VIM has been widely applied in solving kinds of nonlinear equations. For given an initial approximate solution (may contain unknown parameters), a first-order approximate analytic solution is obtained, which is not limited by small parameter.

In this paper, the VIM is used to solve the dropping shock dynamic equation of the cubic nonlinear packaging system, and the first-order approximate solution was obtained. In order to improve the accuracy of the solutions, the new method combining the first-order approximate solution of the VIM with the energy method (EM) of packaging dynamics is developed, and the analytical expressions of the

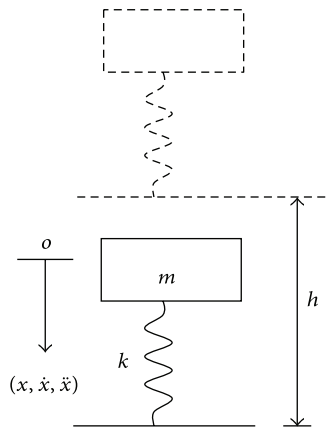

Figure 1: The dynamic model of cubic packaging system.

important parameters such as the maximum displacement response, the maximum acceleration response of the system, the extended period of the shock, and the conditions for inner resonance of system were obtained. The results show that the maximum of the acceleration and the displacement and the extended period of the shock got by this method are very similar to the ones got by Runge-Kutta (R-K) numerical method of order4. The result provides a new method for the dropping shock problem of the cubic nonlinear packaging system.

2. The First-Order Approximation Solution of the VIM

For the cubic nonlinear cushion packaging system [2], the dynamic model of the system is depicted in Figure 1; in the condition of no system damping, the dropping shock dynamic equation and the dropping shock initial conditions of system can be written as

$$\ddot{x} + \omega_0^2 x + kx^3 = 0, \tag{1}$$

$$x(0) = 0, \qquad \dot{x}(0) = \sqrt{2gh}, \tag{2}$$

where $k = \varepsilon\omega_0^2$, $\varepsilon = r/k_0$, $\omega_0 = \sqrt{k_0/m}$ is the frequency parameter, m is the mass of product, h is the dropping height of the packaging system, k_0 and r denote the linear elastic constant coefficient of the cushioning material and its incremental rate, respectively, and g is the gravity acceleration.

The VIM has been widely applied in solving many different kinds of nonlinear equations and is especially effective in solving nonlinear vibration problems with approximation. Assuming the initial solutions for (1) can be written as below

$$x_0 = A\sin(\alpha t), \tag{3}$$

where A and α are unknown parameters. By using the view of the VIM, we can construct the following iteration formula [9–12]:

$$x_{n+1} = x_n + \frac{1}{\omega_0}\int_0^t \sin\omega_0(s-t)\left[\ddot{x}_n(s) + \omega_0^2 x_n(s) + kx_n^3(s)\right]\mathrm{d}s. \tag{4}$$

The first-order iteration approximate solution was obtained as

$$x_1 = -\frac{3kA^3}{4(\omega_0^2-\alpha^2)}\sin(\alpha t) + \frac{kA^3}{4(\omega_0^2-9\alpha^2)}\sin(3\alpha t) + \left[\alpha\frac{A(\omega_0^2-\alpha^2)+(3/4)kA^3}{\omega_0(\omega_0^2-\alpha^2)} - \frac{3\alpha kA^3}{4\omega_0(\omega_0^2-9\alpha^2)}\right] \times \sin(\omega_0 t). \tag{5}$$

In order to ensure that no secular terms appear in the next iteration, the coefficient of $\sin(\omega_0 t)$ is equal to zero, from which the α can be determined. The frequency parameter can be obtained as

$$\alpha = \omega_0\sqrt{\frac{10 + 6\varepsilon A^2 + 8\sqrt{1 + (15/8)\varepsilon A^2 + (9/16)\varepsilon^2 A^4}}{18}}. \tag{6}$$

When ε is a small parameter, the approximation solution of the frequency can be expressed as

$$\alpha = \omega_0\sqrt{1 + \frac{3\varepsilon A^2}{4}}. \tag{7}$$

The extended period of the shock can be expressed as follows:

$$\tau = \frac{\pi}{\alpha}. \tag{8}$$

The first-order approximation solutions of the shock dynamic equation (1) can be written as follows:

$$x_1 = -\frac{3kA^3}{4(\omega_0^2-\alpha^2)}\sin(\alpha t) + \frac{kA^3}{4(\omega_0^2-9\alpha^2)}\sin(3\alpha t) \tag{9}$$

with α defined as (6). The velocity response and the acceleration response of the system can be expressed as

$$\dot{x}_1 = -\frac{3\alpha kA^3}{4(\omega_0^2-\alpha^2)}\cos(\alpha t) + \frac{3\alpha kA^3}{4(\omega_0^2-9\alpha^2)}\cos(3\alpha t), \tag{10}$$

$$\ddot{x}_1 = \frac{3\alpha^2 kA^3}{4(\omega_0^2-\alpha^2)}\sin(\alpha t) - \frac{9\alpha^2 kA^3}{4(\omega_0^2-9\alpha^2)}\sin(3\alpha t). \tag{11}$$

When $\alpha t = \pi/2$, substituting it into (9) and (11), respectively, the maximum displacement and maximum acceleration of the system can be written in the form

$$x_m = \left|\frac{3kA^3}{4(\omega_0^2-\alpha^2)} + \frac{kA^3}{4(\omega_0^2-9\alpha^2)}\right|, \tag{12}$$

$$\ddot{x}_m = \left|\frac{3\alpha^2 kA^3}{4(\omega_0^2-\alpha^2)} + \frac{9\alpha^2 kA^3}{4(\omega_0^2-9\alpha^2)}\right|. \tag{13}$$

To check the correctness of the solution of first-order approximation, substituting (2) into (10) and combining with (6) (here, $m = 10\,\text{kg}$, $k_0 = 600\,\text{Ncm}^{-1}$, $r = 72\,\text{Ncm}^{-3}$), the two key parameters A and α can be obtained as: $A = 3.2252\,\text{cm}$, $\alpha = 106.3257\,\text{s}^{-1}$. The extended period of the shock was obtained as

$$\tau = \frac{\pi}{\alpha} = 0.02955\,\text{s}. \tag{14}$$

When $\alpha t = \pi/2$, according to (9) and (11), the maximum displacement of the system is $x_m = 3.4772\,\text{cm}$ and the maximum acceleration of the system is $\ddot{x}_m = 45.9323\,\text{g}$, respectively.

For dynamics question of the dropping shock, the numerical solution to (1) can be obtained by applying the R-K method of order4. Then the extended period of the shock is $\tau = 0.02851\,\text{s}$, the maximum displacement of shock response is $x_m = 3.4026\,\text{cm}$, and the maximum acceleration of shock response is $\ddot{x}_m = 49.7949\,\text{g}$, respectively. Comparing of the first-order approximate solution with the numerical solution, the relative error of the extended period of the shock is 3.65% and the relative errors of the maximum displacement and the maximum acceleration of the first-order approximate solution are 2.2% and 7.75%, respectively. As shown in Figures 2 and 3, the dropping shock displacement response and acceleration response of the first-order approximation solution by VIM are compared with the R-K method of order4. These results show that the first-order approximate solution needs further discussion as regards how to satisfy the demand of engineering.

3. The Correction of the First-Order Approximate Solution

For the demand of engineering, it is necessary that the first-order approximate solution needs correction. The new method was suggested which integrates the VIM with the EM of packaging dynamic, and the new theoretical solution can be obtained for the nonlinear dropping shock. For that reason, W denotes the weight of the product, if there is no system damping; in the idea of energy method, the gravitational potential energy of the system turned into completely elastic potential energy of the system while the deformation of the cushion material achieved maximum x_m. Assume the dropping height of the packaging system be h; the gravitational potential energy of the system can be written as

$$U = Wh. \tag{15}$$

For the cubic nonlinear packaging system, from (1), the corresponding restoring force is expressed as

$$f(x) = k_0 x + r x^3. \tag{16}$$

By using the EM, we have

$$Wh = \int_0^{x_m} \left(k_0 x + r x^3\right) \mathrm{d}x. \tag{17}$$

The maximum displacement is obtained:

$$x_m = \sqrt{\frac{\sqrt{k_0^2 + 4Wrh} - k_0}{r}}. \tag{18}$$

In the condition of no system damping, the acceleration achieved the maximum while the replacement reached the maximum. From (1), the maximum of acceleration can be expressed as

$$\ddot{x}_m = \omega_0^2 \left(x_m + \varepsilon x_m^3\right). \tag{19}$$

We set (18) and (19) into (12) and (13), respectively, and the following relations were obtained:

$$\begin{aligned} x_m &= \left| \frac{3kA^3}{4\left(\omega_0^2 - \alpha^2\right)} + \frac{kA^3}{4\left(\omega_0^2 - 9\alpha^2\right)} \right| \\ &= \sqrt{\frac{\sqrt{k_0^2 + 4Wrh} - k_0}{r}}, \end{aligned} \tag{20}$$

$$\begin{aligned} \ddot{x}_m &= \left| \alpha^2 \frac{3kA^3}{4\left(\omega_0^2 - \alpha^2\right)} + \frac{9\alpha^2 kA^3}{4\left(\omega_0^2 - 9\alpha^2\right)} \right| \\ &= \omega_0^2 \left(x_m + \varepsilon x_m^3\right). \end{aligned} \tag{21}$$

Combining (20) with (21), the results are $A = 3.4014\,\text{cm}$ and $\alpha = 111.2367\,\text{s}^{-1}$, and they can be denoted as A_ν and α_ν. Substituting A_ν and α_ν into (9) and (11), then the correction solution of the first-order approximate by using VIM (denotes CVIM) is obtained as

$$\begin{aligned} x_1 &= -\frac{3kA_\nu^3}{4\left(\omega_0^2 - \alpha_\nu^2\right)} \sin(\alpha_\nu t) + \frac{kA_\nu^3}{4\left(\omega_0^2 - 9\alpha_\nu^2\right)} \sin(3\alpha_\nu t), \\ \ddot{x}_1 &= \frac{3\alpha_\nu^2 kA_\nu^3}{4\left(\omega_0^2 - \alpha_\nu^2\right)} \sin(\alpha_\nu t) - \frac{9\alpha_\nu^2 kA_\nu^3}{4\left(\omega_0^2 - 9\alpha_\nu^2\right)} \sin(3\alpha_\nu t). \end{aligned} \tag{22}$$

As $\alpha_\nu t = \pi/2$, from (22), the maximum displacement and the maximum acceleration response of the system, respectively, are $x_m = 3.4015\,\text{cm}$ and $\ddot{x}_m = 49.7367\,\text{g}$. The extended period of the shock is $\tau_\nu = \pi/\alpha_\nu = 0.02824\,\text{s}$. Compared with numerical solution, the relative errors of the extended period of the shock, the maximum displacement, and the maximum acceleration are less than 0.95%, 0.04%, and 0.13%, respectively. Additionally, the dropping shock displacement response and acceleration response of the CVIM are compared with the R-K method; see Figures 2 and 3; the displacement and acceleration response are very close to the R-K method. These results show good agreement.

4. Resonance

In a cushioning packaging system, any small vibration might lead to serious damage due to inner resonance. The inner resonance [13–16] is the key problem to optimal design. By

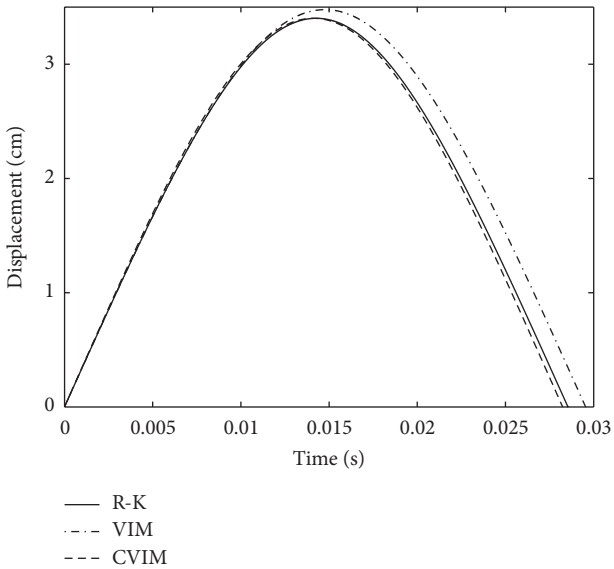

Figure 2: Comparison of the displacement response by the VIM and CVIM with the numerical simulation solved by the R-K method.

(5), the resonance can be expected when one of the following conditions is met:

$$\alpha = \omega_0 \sqrt{\frac{10 + 6\varepsilon A^2 + 8\sqrt{1 + (15/8)\,\varepsilon A^2 + (9/16)\,\varepsilon^2 A^4}}{18}},$$
$$\alpha = \omega_0,$$
$$\alpha = \frac{\omega_0}{3}. \tag{23}$$

These conditions should be avoided during the cushioning packaging design procedure.

5. Conclusions

In the dropping shock dynamic evaluation of the nonlinear packaging system, it is very important to obtain the maximum displacement response, the maximum acceleration response of the system, and the extended period of the shock.

The variational iteration method (VIM) is widely applied for solving various kinds of nonlinear equations. Despite its simplicity and effectiveness, the accuracy of the method may depend on the iteration steps. However, the more iteration steps one makes, the more complex the results may become. To overcome this shortcoming, a new method combining the VIM with energy method (EM) is proposed and applied to study the nonlinear response of cubic packaging system. The results show that the maximum of the acceleration and the displacement and the extended period of the time got by this method are very similar to the ones got by R-K numerical method of order4. The correction of the VIM has been shown to solve effectively, easily, and accurately the dropping shock problem of cubic nonlinear packaging system. The conditions for resonance, which should be avoided in the product packaging design procedure, can be obtained by the first-order iteration solution.

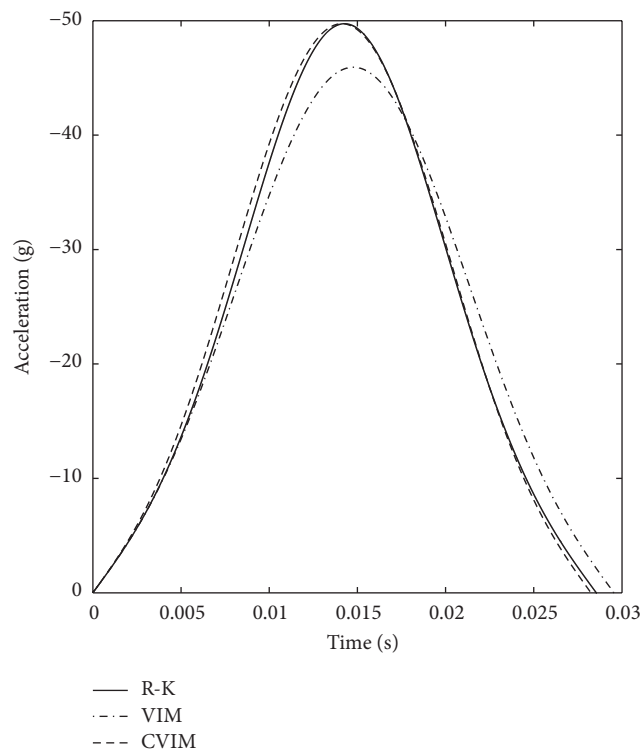

Figure 3: Comparison of the acceleration response by the VIM and CVIM with the numerical simulation solved by the R-K method.

Although the example given in this paper is the cubic nonlinear packaging system, this new method can be applicable to other dropping shock problems of nonlinear packaging system.

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