Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 359701, 8 pages
http://dx.doi.org/10.1155/2013/359701

Research Article

State-Dependent Utilities and Incomplete Markets

Jaime A. Londoño

Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

Correspondence should be addressed to Jaime A. Londoño; jaime.a.londono@gmail.com

Received 23 January 2013; Accepted 1 May 2013

Academic Editor: Guangchen Wang

The problem of optimal consumption and investment for an agent that does not influence the market is solved. The optimization criteria are based on a state-dependent utility functional as proposed in Londoño (2009). The proposed solution is given in any market without state-tame arbitrage opportunities, includes several utilities structures, and includes *incomplete markets* where there are multiple state variables. The solutions obtained for optimal wealths consumptions, and portfolios are explicit and easily computable; the main condition for the result to hold is that the income process of each agent is hedgeable, requiring a natural condition on employer and employee to agree on a contract whose risk can be managed by both parties. In this paper we also developed a theory of markets when the processes are generalization of Brownian flows on manifolds, since this framework shows to be the natural one whenever the problem of intertemporal equilibrium is addressed.

1. Introduction

The problem of optimal consumption and investment for a "small investor" whose actions do not influence market prices in *complete markets* and where consumers have dependent utility structures has been studied in Londoño [1]. The modern treatment of this problem when the asset prices follow Itô processes started with the seminal works of Merton [2, 3]. Using a "martingale" approach, Cox and Huang [4] and Karatzas et al. [5] solved the problem in more general settings in the case of complete markets. Analytical and numerical problems with those earlier solutions and lack of agreement with empirical data motivated alternative treatments; see Londoño [1] for some literature related with problems associated with the standard models as well as some reviews of other approaches.

In incomplete markets, there are even more inconveniences associated with the theory of optimal consumption and investment. General results have been derived in Karatzas and Žitković [6], Hugonnier and Kramkov [7], and Cvitanić et al. [8]. The solutions obtained by them are very limited, and almost nontrivial cases have been solved explicitly. We notice that even though the utilities structures studied by some of them allow state dependence, they are not able to handle the case presented in this paper since they ask for *bounded* (with bounds that depend on time and value of consumption) utility random fields that do not include the ones considered here; see Karatzas and Žitković [6, Definition 3.1 and Proposition 3.5. (item 1)]. In the case presented in this paper the utility random fields are usually considered as unbounded (in the state variable) when the time and the consumption are fixed; see Remark 8. In case of hedgeable (insurable) random endowments and incomplete markets the solution is well known in the case of Markovian markets with non state variables. For instance Merton [3] stated that, in computing the optimal decision rules, the individual capitalizes the lifetime flow of wage income at the market (risk-free) rate of interest and then treats the capitalized value as an addition to the current stock of wealth. Similar results are obtained in more general Markovian markets, as is discussed in Karatzas and Shreve [9]. This result is even true for semimartingale complete markets as is pointed out in Karatzas and Žitković [6].

Whenever state variables are introduced, the solutions provided to complete markets are no longer available even in the presence of hedgeable income, or even with no income. Computable solutions are not known in general settings, since hedgeability of the income structure does not necessarily allow for the optimal portfolio to be hedgeable in the standard models (see Karatzas and shreve [9, Sections 3.6 and 4.4]). However, current models of equilibrium allow for the existence of state variables that model the dynamics of the relevant variables of the economy as in Merton [10], Breeden [11], Cox et al. [12], or Londoño [13] to cite just a few.

The main result of this paper is the solution to the problem of optimal consumption and investment for an agent that does not influence the market. It is assumed that the utility maximization criteria used are based on a state-dependent utility functional as proposed in Londoño [1]. For the optimization problem of this paper, it is assumed that consumers are endowed with an initial capital and a hedgeable random endowment where the underlying market is not necessarily assumed to be complete.

Here, we extend further the approach presented in Londoño [1]. In this model utilities reflect the level of consumption satisfaction of flows of cash in future times as they are valued by the market when the economic agents are making their consumption and investment decisions. The utilities used in this paper were introduced in Londoño [1] and are equivalent to state-dependent utilities in standard settings, where dependence on the state is through the state price density process (see (7) and Remark 8). The main assumption of the theorem, besides the interpretation of the utility functionals, is that the income of the agent should be hedgeable; in plain English it is required that the employer and employee agree on a labor contract that allows each party to hedge any risk associated with the this contract, and in this way any economic agent would be able to cover the compromises of this contract in exchange for a fair price. In the context of this paper it is not needed to be able to cover all the financial liabilities in a given economy (for instance it is not needed that any "reasonable" derivative could be hedge).

The solutions of the optimal consumption and portfolio problem are obtained in a very general setting which includes several functional forms for utilities and considers general restrictions on allowable wealth that are used in the current literature. We obtained *simple* and *computable* solutions that are optimal consumption and investment strategies in all studied cases. In our model it is always true that when the endowment is hedgeable, the problem becomes equivalent to one where the entire endowment process is replaced by its present value, in the form of an augmented initial wealth; see Corollary 9.

The theoretical framework proposed in this paper is one of stochastic flows on manifolds; this is an extension of the framework proposed in Londoño [14]. In the spirit of Merton [10], Breeden [11], and Cox et al. [12] we assume a Markovian setting for the "state variables" that includes not only the price processes but also additional variables that describe the evolution of the economy. This setting proved to be the right framework to study equilibrium problems, since equilibrium defines restrictions on the variables that make them take values on manifolds; see Londoño [13].

This paper also depends on the valuation and arbitrage theory presented in Londoño [14]; this is an extension of the theory of state tameness (Londoño [15]) that provided a unified framework for valuation of financial instruments, of both European and American types, with an algebraic appealing character and economic justification. In Londoño [14, 15] the conditions presented for valuation of financial instruments of American type are the weakest possible. Additional characteristics of the framework developed in Londoño [14, 15] are weak conditions on the coefficients on the volatility matrix of the price process and a development of a theory of valuation of contingent claims with random expiration date.

To the best of our knowledge this theory of arbitrage and valuation is the most general existing setting in the case of (continuous) semimartingales driven by Brownian filtrations with continuous coefficients. For a review of the state of the art on valuation and arbitrage theory we suggest the reader to look at Londoño [15] and the references therein.

2. The Model

First we introduce some notation which will be frequently used in this paper. Let $\mathbb{D} \subset [s,T] \times \mathbb{R}^k$ be a measurable set for $0 \le s < T$, with section $\mathbb{D}(t) = \{x \in \mathbb{R}^k \mid (t,x) \in \mathbb{D}\}$ for $s \le t \le T$. We assume that for each t, $\mathbb{D}(t)$ has a differential structure, which at this point is not necessary to specify. Examples of this differential structure are sets whose sections $\mathbb{D}(t)$ are the solution sets $\{x \mid \varphi(t,x) = 0\}$ of a function $\varphi : [s,T] \times U \to \mathbb{R}^p$ for some $p < k$, where $U \subset \mathbb{R}^k$ is an open set, and φ is a continuous function, that is, differentiable in the spatial variable, and whose partial derivatives are continuous (also in the time variable). Moreover, in order to define a manifold structure it is customary to impose that the differential $D\varphi(t,\cdot)$ has maximal rank for each t. Other examples are sets whose sections $\mathbb{D}(t)$ are integral manifolds defined by $k-p$ $(1 \le p < k)$ continuous vector fields $X_1^s,\ldots,X_{k-p}^s : [s,T] \times U \to \mathbb{R}^k$, where U is an open set as above, and $X_1^s(t,\cdot),\ldots,X_{k-p}^s(t,\cdot)$ are continuously differentiable vector fields that are linearly independent at each space point. It is well known that these two conditions above imply a differential structure on each of the specified subsets; see Warner [16]. In this paper it is always assumed that the degree of "smoothness" of the differential structure is sufficient for every definition to make sense. By this we mean that the transition functions, the functions defining the solution sets or the vector fields, are sufficiently differentiable.

Let m be a nonnegative integer, and let $\mathbb{D}$ be a set defined as above. We say that $f \in C^{m,\delta}(\mathbb{D} : \mathbb{R}^n)$ if f is a continuous function $f \in C(\mathbb{D}, \mathbb{R}^n)$, and there exist an open set $\mathbb{U}$ (relative to the topology of $[0,T] \times \mathbb{R}^k$) with $\mathbb{D} \subset \mathbb{U} \subset [0,T] \times \mathbb{R}^k$ and a continuous extension $\tilde{f}$ of f such that $\tilde{f}(t,\cdot) \in C^{m,\delta}(\mathbb{U}(t) : \mathbb{R}^n)$, where this last space is the Fréchet space of m-times continuous differentiable functions whose mth-order derivatives are δ-Hölder continuous with seminorms $\|\cdot\|_{m,\delta:K}$ defined in Kunita [10, Section 3.1] with $\int_s^T \|\tilde{f}(t,\cdot)\|_{m,\delta:K}\, dt < \infty$, where $K \subset \mathbb{U}$ is a compact set, $0 \le \delta \le 1$. In case $m = 0$ (or $\delta = 0$), we denote $C^{m,\delta}(\mathbb{D} : \mathbb{R}^n)$ simply by $C^{\delta}(\mathbb{D} : \mathbb{R}^n)$ ($C^m(\mathbb{D} : \mathbb{R}^n)$), and whenever clear we denote the above spaces simply as $C^{m,\delta}$, C^{δ}, and C^m, respectively. We denote by $C^{m,0+}$ the set $\cup_{\delta>0} C^{m,\delta}$, and $C^{0+} = C^{0,0+}$. Let $\mathbb{D}' \subset [s,T] \times \mathbb{R}^k$ be a measurable set with sections $\mathbb{D}'(t)$; we say that a function f belongs to the class

$C^{m,\delta}(\mathbb{D} : \mathbb{D}')$, where $\delta > 0$ or $\delta = 0+$, if $f(t,\cdot)$ takes values in $\mathbb{D}'(t)$ and $f \in C^{m,\delta}(\mathbb{D} : \mathbb{R}^n)$. We assume in the following definitions that we have a differential structure sufficiently smooth according to the space defined. We notice that the definition of Hölder continuity is made with respect to the Euclidean distance derived from the fact that these sets are subsets of an Euclidean space.

Definitions of consistent processes that we review below are natural generalizations (of processes defined on embeddings) of the definitions made in Londoño [15].

We assume a d-dimensional Brownian motion $\{W(t), \mathcal{F}_t;\ 0 \le t \le T\}$ starting at 0 and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathcal{F} = \mathcal{F}_T$ and $\{\mathcal{F}_t, 0 \le t \le T\}$ is the $\mathbf{P}$ augmentation by the null sets of the natural filtration $\mathcal{F}_t^W = \sigma(W(s), 0 \le s \le t)$. Let $(\mathcal{F}_{s,t}) = \{\mathcal{F}_{s,t}, 0 \le s \le t \le T\}$ be the two-parameter filtration, where $\mathcal{F}_{s,t}$ is the smallest sub-σ-field containing all null sets and $\sigma(W_s(u) \mid s \le u \le t)$, where $W_s(u) \equiv W(u) - W(s)$. For each $0 \le s \le T$ we also define the σ-field $\mathcal{P}_s$ of progressive measurable sets after time s as the σ-field of sets $P \in \mathcal{B}([s,T]) \otimes \mathcal{F}_{s,T}$, and the product σ-field, such that $\chi_P(t,\omega)$, $t \ge s$, is a $\mathcal{F}_{s,t}$ progressive measurable (in t) process, where χ is the indicator function. We denote by μ_s the measure on $\mathcal{P}_s$ defined by $\mu_s(P) = \mathbf{E}\int_s^T \chi_P(t,\omega)\, dt$.

Let $\varphi_{s,t}(x,\omega)$, $0 \le s \le t \le T$, $x \in \mathbb{D}(s)$, be a $\mathbb{R}^n$-valued random field on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathbb{D}$ has a differential structure sufficiently smooth for the following definitions to make sense, and let $\mathbb{D}' \subset [0,T] \times \mathbb{R}^n$ be a measurable set. We say that $\varphi_{s,t}$ is a *continuous* $C^{m,\delta}(\mathbb{D} : \mathbb{D}')$-*semimartingale* if $\varphi_s : t \to \varphi_{s,t}(\cdot)$ is a continuous random field with values in $C^{m,\delta}(\mathbb{D} \cap ([s,T] \times \mathbb{R}^k) : \mathbb{D}' \cap ([s,T] \times \mathbb{R}^n))$, that is, a continuous $(\mathcal{F}_{s,\cdot})$ semimartingale process. In addition we assume that $\varphi_{s,t}(x)$ can be decomposed as $\varphi_{s,t}(x) = \varphi_{s,t}^{\mathrm{loc}}(x) + \varphi_{s,t}^{\mathrm{fv}}(x)$, where $\varphi_{s,\cdot}^{\mathrm{loc}}(\cdot)$ is a continuous $C^{m,\delta}(\mathbb{D} \cap ([s,T] \times \mathbb{R}^k) : \mathbb{D}' \cap ([s,T] \times \mathbb{R}^n))$-local-martingale, and $\varphi_{s,\cdot}^{\mathrm{fv}}(\cdot)$ is a continuous $C^{m,\delta}(\mathbb{D} \cap ([s,T] \times \mathbb{R}^k) : \mathbb{D}' \cap ([s,T] \times \mathbb{R}^n))$-process of bounded variation for each $0 \le s \le T$. A pair (a,b), where $a_{s,t}(x,y)$ and $b_{s,t}(x)$ are measurable random fields $\mathcal{F}_{s,t}$-progressive measurable in t for all $x, y \in \mathbb{D}(s)$, $0 \le s \le T$, is said to be the *local characteristics of* φ if $(a_{s,\cdot}(x,y), b_{s,\cdot}(x))$ is the local characteristic of $\varphi_s \equiv \varphi_{s,\cdot}(\cdot)$ for any $s \le T$, (see Kunita [17]). In addition, a pair (σ, b), where $\sigma_{s,t}(x)$ is a measurable random field, that is, $(\mathcal{F}_{s,t})$-progressive measurable in t with values in the set $L(\mathbb{R}^d : \mathbb{R}^n)$ of real-valued matrices with size $n \times d$, and b is as above, is said to be the *diffusion and drift processes of* φ if

$$\varphi_{s,t}^{\mathrm{loc}}(x)(\omega) = \int_s^t \sigma_{s,u}(x)\, dW_s(u) \tag{1}$$

for all x, s, t, and ω. If $b_{s,\cdot}(\cdot)$ and $\sigma_{s,\cdot}(\cdot)$ are processes of class $C^{m,\delta}$ for all $0 \le s \le T$, we will say that φ has *diffusion and drift of class* $C^{m,\delta}$.

Let $\varphi_{s,t}(x)$ and $\psi_{s,t}(x)$ be continuous $C(\mathbb{D} : \mathbb{R}^n)$ and $C(\mathbb{D} : \mathbb{D})$ semimartingales, respectively, where in addition, it is assumed that $\psi(s,s,x) = x$ for all $x \in \mathbb{D}(s)$. We say that the process φ is a ψ-*consistent semi-martingale process* if for each $0 \le s \le s' \le T$ there exists a set $N_{s,s'} \in \mathcal{P}_{s'}$ with $\mu_{s'}(N_{s,s'}) = 0$, such that $\varphi_{s,t}(x) = \varphi_{s',t}(\psi_{s,s'}(x))$ for all $(t,\omega) \notin N_{s,s'}$ and all $x \in \mathbb{D}(s)$. We say that the process φ is a *consistent semi-martingale process* if φ is a φ-consistent process.

A few words should be said about the existence of solutions of stochastic differential equations on manifolds, and although our approach is surely not the more general, we believe that it is the simplest since it does not require any technicalities of differential geometry. The simplicity of our approach is due to our definitions of functions of type $C^{m,\delta}$ on a manifold. Assume that $b : \mathbb{D} \to \mathbb{R}^n$ and $\sigma : \mathbb{D} \to L(\mathbb{R}^n : \mathbb{R}^d)$ are functions in $C^{m,\delta}$, where $m = 0$ and $\delta = 1$ or $m \ge 1$ and $\delta > 0$. Let $\varphi_{s,t}(x)$ be the local (maximal) solution of

$$\begin{aligned}\varphi_{s,t}(x) &= \tilde{b}(t, \varphi_{s,t}(x))\, dt \\ &\quad + \tilde{\sigma}(t, \varphi_{s,t}(x))\, dW_s(t), \quad \varphi_{s,s}(x) = x\end{aligned} \tag{2}$$

for $x \in \mathbb{U}(s)$, where $\tilde{b}$ and $\tilde{\sigma}$ are the extensions of b and σ to some open set $\mathbb{U}$. We assume that $\varphi_{s,t}(x)$ is the local solution of class $C^{m,\epsilon}$ for any $0 \le \epsilon < \delta$, see Kunita [17, Theorems 4.7.1 and 4.7.2]. If for any $x \in \mathbb{D}(s)$, $\varphi_{s,t}(x)$ has a nonexplosive solution with values in $\mathbb{D}$, we will say that $\varphi_{s,t}(x)$, $x \in \mathbb{D}(s)$ is the solution to the stochastic differential equation on $\mathbb{D}$. It is straightforward to see that this defines uniquely a process that does not depend on the extensions $\tilde{b}$ and $\tilde{\sigma}$ that are used.

Next, we describe a financial market. We assume an m-dimensional Itô process Θ of two parameters with values in $C^{2,0+}(\mathbb{K} : \mathbb{K})$ (for $m > 1$), where $\mathbb{K}(s) \subset \mathbb{R}_+ \times \mathbb{R}^{m-1}$ (where $\mathbb{R}_+ = (0,\infty)$) for each $0 \le s \le T$, and $\mathbb{K} \subset [0,T] \times \mathbb{R}_+ \times \mathbb{R}^{m-1}$ is some measurable set with a differential structure as discussed above. We assume an $n{+}m$-dimensional Itô process (P, Θ) of two parameters with values in $C^{0+}(\mathbb{D} : \mathbb{D})$, where $\mathbb{D}$ is a measurable set with sections $\mathbb{D}(s) \subset \mathbb{R}_+^n \times \mathbb{K}(s)$ for each s, where $\mathbb{R}_+^n = (\mathbb{R}_+)^n$ and with a differential structure as discussed above. We assume that for each initial condition $\vartheta = (\vartheta_0, \ldots, \vartheta_{m-1})^\top \in \mathbb{K}(s)$,

$$\begin{aligned}\frac{d\Theta_{s,t}^0}{\Theta_{s,t}^0} &= \left(r(t, \Theta_{s,t})\, dt + \left\|\theta(t, \Theta_{s,t})\right\|^2\right) dt \\ &\quad + \sum_{1 \le j \le d} \theta^j(t, \Theta_{s,t})\, dW_s^j(t), \quad \Theta_{s,s}^0(\vartheta) = \vartheta_0,\end{aligned} \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm, and

$$\begin{gathered}\Theta_{s,t}^i = \rho^i(t, \Theta_{s,t})\, dt + \sum_{1 \le j \le d} \varrho^{i,j}(t, \Theta_{s,t})\, dW_s^j(t) \\ \Theta_{s,s}^i(\vartheta) = \vartheta_i, \quad i = 1, \ldots, m-1\end{gathered} \tag{4}$$

for some continuous functions $r, \vartheta^j, \rho^i, \varrho^{i,j}$ for $j = 1, \ldots, d$ of class $C^{2,0+}$, and $i = 1, \ldots, m-1$ for which global solutions of these stochastic differential equations exit. We also assume that for each $(p_1, \ldots, p_n, \vartheta_0, \ldots, \vartheta_{m-1})^\top = (p^\top, \vartheta^\top)^\top \in \mathbb{D}(s)$,

$$\begin{gathered}\frac{dP_{s,t}^i}{P_{s,t}^i} = b^i(t, P_{s,t}, \Theta_{s,t})\, dt + \sum_{1 \le j \le d} \sigma^{i,j}(t, P_{s,t}, \Theta_{s,t})\, dW_s^j(t) \\ P_{s,s}^i(p, \vartheta) = p_i, \quad i = 0, \ldots, n,\end{gathered} \tag{5}$$

where, for $1 \le i \le n$, $P^i_{s,t} = \pi^i \circ P_{s,t}$ is *the price of shares outstanding for the i-stock* and where π^i denotes the projection on the ith-component for $1 \le i \le n$. We assume that $\sigma^{i,j}, b^i$ are continuous functions of class $C^{2,0+}$ for which global solutions of the set of differential equations above exist and are unique.

We point out that in a free of (state) arbitrage opportunities and (state) complete market, $\vartheta_0/\Theta^0_{s,t}$ is the process (with two parameters) that discounts the flow of money in every future state t, $s \le t \le T$ to bring the value of the flow to current time s; for instance see Londoño [15] and the definition of $H_{s,t}$ below.

The process of bounded variation $B_{s,t} \equiv B_{s,t}(\vartheta)$, whose evolution is given by the stochastic differential equation

$$dB_{s,t} = r(t, \Theta_{s,t}) B_{s,t} dt, \quad B_{s,s} = 1, \quad \text{for } s \le t \le T, \tag{6}$$

will be called the *bond price process*.

We define the *state price density process* $H_{s,t} \equiv H_{s,t}(\vartheta)$ to be the continuous $C^{0+}(\mathbb{K} : \mathbb{R}_+)$ process given by

$$H_{s,t} = B^{-1}_{s,t} Z_{s,t}, \quad s \le t \le T, \tag{7}$$

where $Z_{s,t} \equiv Z_{s,t}(\vartheta)$ is the process defined as

$$Z_{s,t} = \exp\left\{-\int_s^t \theta^\top(u, \Theta_{s,u})\, dW(u) - \frac{1}{2}\int_s^t \left\|\theta(u, \Theta_{s,u})\right\|^2 du\right\} \tag{8}$$

for $s \le t \le T$, and $B^{-1}_{s,t} = 1/B_{s,t}$. We point out that

$$H_{s,t} = \frac{\vartheta_0}{\Theta^0_{s,t}} \tag{9}$$

easily follows the Itô's lemma. Throughout this paper we will assume that $\theta(t, \vartheta) \in \ker^\perp(\sigma(t, p, \vartheta))$, where $\ker^\perp(\sigma(\cdot))$ denotes the orthogonal complement of the kernel of $\sigma(\cdot)$ and

$$b(t, p, \vartheta) + \delta(t, p, \vartheta) - r(t, \vartheta)\mathbf{1}_n = \sigma(t, p, \vartheta)\theta(t, \vartheta) \tag{10}$$

for all $(p^\top, \vartheta^\top)^\top \in \mathbb{D}(s)$ and t, where $\mathbf{1}_n^\top = (1, \ldots, 1) \in \mathbb{R}^n$. This latter assumption implies that there are no state-tame arbitrage opportunities (see Londoño [15]). In addition to the above condition we assume in this paper that the market satisfies the following condition that we call smooth market condition. We notice that since $\theta(t, \vartheta) \in \ker^\perp(\sigma(t, p, \vartheta)) = \operatorname{Im}(\sigma^\top(t, p, \vartheta))$, the existence of a measurable function κ with the property expressed in (11) follows for any financial market, and therefore the condition below is indeed a weak condition on the smoothness of the mentioned property.

Condition 1 (smooth market condition). There exists a $C^{2,0+}$-matrix-valued function κ defined on $\mathbb{D}$ with the property that

$$\sigma^\top(t, p, \theta)\kappa(t, p, \vartheta) = \theta(t, \vartheta). \tag{11}$$

We point out that we require no dependence of prices on the evolution of the functions r and θ. We say that $b_{s,t} \equiv b(t, P_{s,t}, \Theta_{s,t})$ is the *return process*, $\sigma_{s,t} \equiv \sigma(t, P_{s,t}, \Theta_{s,t})$ is the *volatility process*, $\delta_{s,t} \equiv \delta(t, P_{s,t}, \Theta_{s,t})$ is the *process of dividends*, $r_{s,t} \equiv r(t, \Theta_{s,t})$ is the *interest rate process*, and $\theta_{s,t} \equiv \theta(t, \Theta_{s,t})$ is the *market price of risk process*.

For a structure as above, we say that $\mathscr{M} = (P, \Theta, \mathbb{D}, \mathbb{K}, b, \sigma, \delta, \theta, \rho, \varrho, r, p^0, \vartheta^0)$ is a *financial market with terminal time T and initial time 0, feasible set of values $\mathbb{D}$, feasible set of state values $\mathbb{K}$, vector of returns $b(t, \cdot) = (b_1(t, \cdot) \ldots, b_n(t, \cdot))^\top$, matrix of volatility coefficients $\sigma(t, \cdot) = (\sigma^{i,j}(t, \cdot))$, vector of dividends $\delta(t, \cdot) = (\delta_1(t, \cdot), \ldots, \delta_n(t, \cdot))^\top$, market price of risk $\theta(t, \cdot) = (\theta_1(t, \cdot), \ldots, \theta_d(t, \cdot))^\top$, interest rate $r(t, \cdot)$, drift of the state process $\rho(t, \cdot) = (\rho^1(t, \cdot), \ldots, \rho^m(t, \cdot))^\top$, diffusion matrix for the state process $\varrho(t, \cdot) = (\varrho^{i,j}(t, \cdot))$, vector of initial prices $p^0 \in \mathbb{R}^n_+$*, and initial state variables ϑ^0.

Next, we review and extend some definitions from Londoño [1] that are needed to describe equilibrium.

Definition 1. Assume a measurable set $\mathbb{X} \subset [0, T] \times \mathbb{R}^{n+m+1}$ with some differential structure as discussed in Section 2 with (nonempty) section $\mathbb{X}(s) \subset \mathbb{R} \times \mathbb{D}(s)$ for each $0 \le s \le T$. Assume a family of continuous Itô's processes

$$X = \left\{X_{s,t}(x, p, \vartheta); (x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s),\ 0 \le s \le t \le T\right\}, \tag{12}$$

such that (X, P, Θ) is a consistent process of class $C^{0+}(\mathbb{X} : \mathbb{X})$. We say that X is a *wealth evolution structure*; we will denote this by writing $X \in \mathscr{X}(\mathscr{M})$. For a detailed description of consistent processes and related ones see Londoño [1, 14]. One says that $\mathbb{X}$ is *a feasible set of values for* (X, P, Θ).

Let $(\pi^0, \pi) = \{(\pi^0_{s,t}(x, p, \vartheta), \ldots, \pi^n_{s,t}(x, p, \vartheta));\ (x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s), 0 \le s \le t \le T\}$ be a (X, P, Θ)-consistent progressive measurable process of class C^{0+} with $\pi_0 + (\pi)^\top \mathbf{1}_n = X$. Assume that $c = \{c_{s,t}(x, p, \vartheta) : (x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s), 0 \le s \le t \le T\}$ is a nonnegative consistent (X, P, Θ) progressive measurable process of class C^{0+} and that $Q = \{Q_{s,t}(p, \vartheta) : (p^\top, \vartheta^\top)^\top \in \mathbb{D}(s), 0 \le s \le t \le T\}$ is a nonnegative consistent (P, Θ) progressive measurable process of class C^{0+}. It is assumed that

$$\begin{aligned} B^{-1}_{s,t}(\vartheta) X_{s,t}(x, p, \vartheta) &= x + \int_s^t B^{-1}_{s,u}(\vartheta)\left(Q_{s,u}(p, \vartheta) - c_{s,u}(x, p, \vartheta)\right) du \\ &\quad + \int_s^t B^{-1}_{s,u}(\vartheta)(\pi)^\top_{s,u}(x, p, \vartheta)\,\sigma_{s,u}(p, \vartheta)\, dW_s(u) \end{aligned} \tag{13}$$

$$\int_s^t B^{-1}_{s,u}(\vartheta)(\pi)^\top_{s,u}(x, p, \vartheta) \times \left(b_{s,u}(p, \vartheta) + \delta_{s,u}(p, \vartheta) - r_{s,u}(\vartheta)\mathbf{1}_n\right) du$$

for all $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s)$ and $0 \le s \le t \le T$. In addition it is assumed that

$$\begin{aligned} &\mathbf{E}\left[\int_s^T H_{s,t}(\vartheta)\, c_{s,t}(x, p, \vartheta)\, dt\right] < \infty, \\ &\mathbf{E}\left[\int_s^T H_{s,t}(\vartheta)\, Q_{s,t}(p, \vartheta)\, dt\right] < \infty \end{aligned} \tag{14}$$

for all $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s)$ and $0 \le s \le t \le T$. We say that $((\pi_0, \pi), c, Q)$ as above is a portfolio evolution structure with rate of consumption $c_{s,t}$ and rate of endowment Q. We say (X, c, Q) as above is a hedgeable rate of consumption and endowment evolution structure with feasible set of values $\mathbb{X}$. A (P, Θ)-consistent process Q (without any additional structure) that satisfies (14) is called a rate of endowment evolution process. If a wealth process $(X, 0, Q)$ is a hedgeable rate of consumption and endowment evolution structure with feasible set of values $\mathbb{X}$, we just say that (X, Q) is a hedgeable endowment evolution structure.

A subsistence random field L for the market $\mathscr{M}$ is a (P, Θ)-consistent process with drift and diffusion of class C^{0+}, where $L_{s,\cdot}(p, \vartheta) H_{s,\cdot}(\vartheta)$ is uniformly bounded below for all $(p^\top, \vartheta^\top)^\top \in \mathbb{D}(s)$ (where the bound might depend on p, ϑ, and s) such that

$$\mathbf{E}\left[H_{s,t}(\vartheta) L_{s,t}(p, \vartheta)\right] < \infty \tag{15}$$

for all t. We will say that the couple (π, c) of portfolio on stocks and rate of consumption is admissible for (L, Q) and write $(\pi, c) \in \mathscr{A}(L, Q)$ if for any $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s)$ with $x \ge L_{s,s}(p, \vartheta)$

$$X_{s,t}(x, p, \vartheta) \ge L_{s,t}(p, \vartheta) \quad \forall t. \tag{16}$$

If there does not exist a couple of portfolio on stocks and rate of consumption admissible for (L, Q), we say that the class cited above is empty, and we would denote this by $\mathscr{A}(L, Q) = \emptyset$.

3. Consumption and Portfolio Optimization

Throughout this paper we are mainly interested in portfolio evolution structures that are obtained as the result of the optimal behavior of consumers as it is explained below. Next, we review definitions and properties of the type of utilities that are used in this paper.

Definition 2. Consider a function $U : (0, \infty) \mapsto \mathbb{R}$ is continuous, strictly increasing, strictly concave, and continuously differentiable with $U'(\infty) = \lim_{x\to\infty} U'(x) = 0$ and $U'(0+) \triangleq \lim_{x\downarrow 0} U'(x) = \infty$. Such a function will be called a utility function.

Classic examples of utility functions are $U_\alpha(x) = x^\alpha/\alpha$ for some $\alpha \in (0, 1)$, $0 \le x < \infty$, and $U(x) = \log(x)$. For every utility function $U(\cdot)$, we will denote by $I(\cdot)$ the inverse of the derivative $U'(\cdot)$; both of these functions are continuous, strictly decreasing and map $(0, \infty)$ onto itself with $I(0+) = U'(0+) = \lim_{x\to 0^+} U'(x) = \infty$, $I(\infty) = \lim_{x\to\infty} I(x) = U'(\infty) = 0$. We extend U by $U(0) = U(0^+)$ (we keep the same notation to the extension to $[0, \infty)$ of U hoping that it will be clear to the reader). It is a well-known result that

$$\max_{0<x<\infty} \left(U(x) - xy\right) = U\left(I(y)\right) - yI(y), \quad 0 < y < \infty. \tag{17}$$

Definition 3. Consider a continuous function $U_1 : [0, T] \times (0, \infty) \mapsto \mathbb{R}$, such that $U_1(t, \cdot)$ is a utility function in the sense of Definition 2 for all $t \in [0, T]$. It follows that $I_1(t, x) \triangleq (\partial U_1(t, x)/\partial x)^{-1}$, the inverse of the derivative of U, is a continuous function. Similarly, if a utility function $U_2 : (0, \infty) \mapsto \mathbb{R}$ is given, then $I_2(x) \triangleq (\partial U_2(x)/\partial x)^{-1}$ is continuous. Let one denote

$$\mathscr{X}(t, y) \triangleq I_2(y) + \int_t^T I_1(t', y)\, dt'. \tag{18}$$

We call a couple of functions U_1 and U_2 a state preference structure.

Under the conditions outlined in the previous definition, it is easy to see that $\mathscr{X} : [0, T] \times (0, \infty) \to (0, \infty)$ is a continuous function with the property that for each t, $\mathscr{X}(t, \cdot)$ maps $(0, \infty)$ onto itself, strictly decreasing with $\mathscr{X}(t, 0+) = \lim_{y\downarrow 0} \mathscr{X}(t, y) = \infty$ and $\mathscr{X}(t, \infty) = \lim_{y\to\infty} \mathscr{X}(t, y) = 0$.

We extend U_1 and U_2 by defining $U_1(t, 0) = U(t, 0^+)$ for all $0 \le t \le T$ and $U_2(0) = U_2(0^+)$, and we keep the same notation to the extension of U_1 to $[0, T] \times [0, \infty)$ and the extension of U_2 to $[0, \infty)$.

We point out that $\mathscr{X}^{-1}$, defined for each t as $\mathscr{X}^{-1}(t, \cdot)$, the inverse of $\mathscr{X}(t, \cdot)$, shares the same properties stated above about $\mathscr{X}$. The discussion of those utility functions defined above, is given in Londoño [1].

For $s \le t$, define $\alpha(s, t) = \mathscr{X}(s, \mathscr{X}^{-1}(t, \cdot))$. Then $\alpha(s, t) = \alpha(s, t') \circ \alpha(t', t)$ for all s, t, and t' in $[0, T]$, where $\circ$ denotes standard composition of functions. We also observe that if $\alpha^I(s, t) \triangleq I_1(s, \mathscr{X}^{-1}(t, \cdot))$, then $\alpha^I(s, t) \circ \alpha(t, s) = \alpha^I(s, s)$. Some examples discussed in Londoño [1] include power utility structures $U_1(t, x) = x^\alpha h(t)$ and $U_2(x) = cx^\alpha$ with $\alpha \in (0, 1)$ and $c \ge 0$, where $h : [0, T] \to (0, \infty)$ is any continuous positive function. Logarithmic utility structures are also included, where h is as above, $U_1(t, x) = h(t)\log(x)$, and $U_2(x) = c\log(x)$ with $c \ge 0$. Finally in the cited paper "separable preference structures" are introduced, where h is as above, $U_1(t, x) = h(t)u(x/h(t))$, and $U_2(x) = cu(x/c)$, where $u(\cdot)$ is a utility function and $c > 0$.

Before we state the main result of this paper, we first require an additional definition.

Definition 4. Assume a rate of endowment evolution process Q of class $C^{2,0+}$ with current value of future endowments L defined as

$$\begin{aligned} L_{s,t}(p, \vartheta) &= \frac{-1}{H_{s,t}(\vartheta)} \mathbf{E}\left[\int_t^T H_{s,u}(\vartheta) Q_{s,u}(p, \vartheta)\, du \mid \mathscr{F}_{s,t}\right] \\ &= -\mathbf{E}\left[\int_t^T H_{t,u}(\Theta_{s,t}(\vartheta)) Q_{t,u}(P_{s,t}(p, \vartheta), \Theta_{s,t}(p, \vartheta))\, du\right] \\ &= -\Pi(t, P_{s,t}(p, \vartheta), \Theta_{s,t}(\vartheta)), \end{aligned} \tag{19}$$

where it is assumed that

$$\Pi(t, p, \vartheta) \triangleq -\mathbf{E}\left[\int_t^T H_{t,u}(\vartheta) Q_{t,u}(p, \vartheta)\, du\right] \tag{20}$$

is a function of class $C^{2,0+}$. If (L, Q) is a hedgeable endowment evolution structure with portfolio π_Q with feasible set of values $\mathbb{X}_Q$ defined as

$$\mathbb{X}_Q = \left\{(s, x, p, \vartheta) \mid x = -\Pi(s, p, \vartheta), \left(p^\top, \vartheta^\top\right)^\top \in \mathbb{D}(s)\right\}, \tag{21}$$

one will say that Q is a hedgeable income structure.

Remark 5. We point out that unless the market is state complete (see Londoño, [15, Theorem 3.1] and Londoño [14]), in which case the volatility matrix has maximal range (in the sense that its rank is equal to n), there might be an infinity number of portfolios that hedge the given income structure. If the market is not necessarily complete but has no state-tame arbitrage opportunities, it satisfies (10) therefore by definition of wealth associated to a portfolio

$$\begin{aligned} & B_{s,t}^{-1}(\vartheta) L_{s,t}(p, \vartheta) \\ & \quad = x + \int_s^t B_{s,u}^{-1}(\vartheta) Q_{s,u}(p, \vartheta)\, du \\ & \qquad + \int_s^t B_{s,u}^{-1}(\vartheta) \left(\sigma_{s,u}^\top(p, \vartheta) \pi_{s,u}(x, p, \vartheta)\right)^\top dW_s(u) \\ & \qquad + \int_s^t B_{s,u}^{-1}(\vartheta) \left(\sigma_{s,u}^\top(p, \vartheta) \pi_{s,u}(x, p, \vartheta)\right)^\top \theta_{s,u}(\vartheta)\, du \end{aligned} \tag{22}$$

for all $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}_Q(s)$ and $0 \le s \le t \le T$, where $L_{s,t}$ is the process in Definition 4. It follows that the same wealth is hedged by any portfolio $\pi_{s,t} + \kappa_{s,t}$, where $\kappa_{s,t} \in \ker(\sigma_{s,t}^\top) \neq \emptyset$.

The following theorem extends the theory of optimal consumption and investment of Londoño [1] to incomplete markets which is the main result of this paper.

Theorem 6. *Assume that $\mathscr{M} = (P, \Theta, \mathbb{D}, \mathbb{K}, b, \sigma, \delta, \theta, \rho, \varrho, r, p^0, \vartheta^0)$ is a financial market that satisfies the smooth market condition (Condition 1) and* (10)*. Also, assume that (U_1, U_2) is a state preference structure for a consumer with hedgeable income structure Q (with hedging portfolio on the stocks $\pi_{s,t}^Q$).*

Denote by $\alpha_{s,t}$ and α_t^I the functions that correspond to the state preference structure (U_1, U_2) as defined after Definition 3. Define

$$\mathbb{X} = \left\{\left(s, x, p^\top, \vartheta^\top\right)^\top \mid x > \Pi(s, p, \vartheta), \left(p^\top, \vartheta^\top\right)^\top \in \mathbb{D}(s)\right\}. \tag{23}$$

Let ξ be defined as

$$\begin{aligned} \xi_{s,t}(x, p, \vartheta) \triangleq{} & \Pi\left(t, P_{s,t}(p, \vartheta), \Theta_{s,t}(\vartheta)\right) \\ & + H_{s,t}^{-1}(\vartheta)\, \alpha_{t,s}\left(x - \Pi(s, p, \vartheta)\right), \end{aligned} \tag{24}$$

and let c be defined as

$$c_{s,t}(x, p, \vartheta) \triangleq H_{s,t}^{-1}(\vartheta) \left(\alpha_t^I \circ \alpha_{t,s}\right)\left(x - \Pi(s, p, \vartheta)\right. \tag{25}$$

for any $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s)$. If ξ is a wealth evolution structure with feasible set of values $\mathbb{X}$, then it is a hedgeable cumulative consumption and endowment structure with portfolio $(\pi, c) \in \mathscr{A}(L, Q)$ which is an optimal solution for the problem of optimal consumption and investment. The optimality is in the sense that

$$\begin{aligned} & \mathbf{E}\left[\int_0^T U_1\left(t, H_{0,t}(\vartheta)\, c_{0,t}(x, p, \vartheta)\right) dt \right. \\ & \qquad \left. + U_2\left(H_{0,T}(\vartheta)\, \xi_{0,T}(x, p, \vartheta)\right)\right] \\ & \quad \ge \mathbf{E}\left[\int_0^T U_1\left(t, H_{0,t}(\vartheta)\, \tilde{c}_{0,t}(x, p, \vartheta)\right) dt \right. \\ & \qquad \left. + U_2\left(H_{0,T}(\vartheta)\, \tilde{\xi}_{0,T}(x, p, \vartheta)\right)\right] \end{aligned} \tag{26}$$

for all $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s)$, where $(\tilde{\xi}, \tilde{c}, Q)$ is any hedgeable cumulative consumption and endowment structure with $(\tilde{\pi}, \tilde{c}) \in \mathscr{A}(L, Q)$, and

$$\begin{aligned} & \mathbf{E}\left[\int_0^T U_1^-\left(t, H_{0,t}(\vartheta)\, \tilde{c}_{0,t}(x, p, \vartheta)\right) dt \right. \\ & \qquad \left. + U_2^-\left(H_{0,T}(\vartheta)\, \tilde{\xi}_{0,T}(x, p, \vartheta)\right)\right] < \infty, \end{aligned} \tag{27}$$

where $U_1^-(t, x) = -(U_1(t, x) \wedge 0)$ and $U_2^-(x) = -(U_2(x) \wedge 0)$. An optimal portfolio on stocks $\pi_{s,t}(x, p, \vartheta)$ is

$$\frac{\alpha_{t,s}\left(x - \Pi(s, p, \vartheta)\right)}{H_{s,t}} \kappa\left(t, P_{s,t}, \Theta_{s,t}\right) - \pi_{s,t}^Q\left(-\Pi(s, p, \vartheta), p, \vartheta\right), \tag{28}$$

where κ is the function defined by (11)*. Moreover, $\pi_{s,t} + \eta_{s,t}$ is an optimal portfolio for any (X, P, Θ)-consistent process $\eta_{s,t}(x, p, \vartheta) \in \ker \sigma_{s,t}^\top(x, p, \vartheta)$ for $(x, p, \vartheta) \in \mathbb{X}(s)$.*

Proof. Define $\mathbb{X}_Q$ as the set given in Definition 4, and let π_Q be the portfolio that hedges the income process. It follows that

$$\begin{aligned} & H_{s,t} L_{s,t} - \int_s^t H_{s,u} Q_{s,u}\, du \\ & \quad = -\mathbf{E}\left[\int_s^T H_{s,u} Q_{s,u}\, du \mid \mathscr{F}_{s,t}\right] \\ & \quad = -\mathbf{E}\left[\int_s^T H_{s,u} Q_{s,u}\, du\right] \\ & \qquad + \int_s^t H_{s,u}\left[\sigma_{s,u}^\top \pi_Q - L_{s,u} \theta_{s,u}\right]^\top dW_s(u), \end{aligned} \tag{29}$$

and on the other hand let us define $\pi_{s,t}^C(x - \Pi(s, p, \vartheta), p, \vartheta)$ by the process that satisfies

$$\sigma_{s,t}^\top \pi_{s,t}^C = \frac{\alpha_{t,s}\left(x - \Pi(s, p, \vartheta)\right)}{H_{s,t}} \theta_{s,t}. \tag{30}$$

The existence of this process follows by Condition 1. Therefore

$$\begin{aligned}
&H_{s,t}H_{s,t}^{-1}\alpha_{t,s}\left(x-\Pi\left(s,p,\vartheta\right)\right)\\
&\qquad+\int_s^t H_{s,u}H_{s,u}^{-1}\alpha_u^I\circ\alpha_{u,s}\left(x-\Pi\left(s,p,\vartheta\right)\right)du\\
&=\alpha_{t,s}\left(x-\Pi\left(s,p,\vartheta\right)\right)\\
&\qquad+\int_s^t \alpha_u^I\circ\alpha_{u,s}\left(x-\Pi\left(s,p,\vartheta\right)\right)du\\
&=\left(x-\Pi\left(s,p,\vartheta\right)\right)\\
&=\left(x-\Pi\left(s,p,\vartheta\right)\right)\\
&\qquad+\int_s^t H_{s,u}\left(\sigma_{s,u}^\top\pi_{s,u}^C\right.\\
&\qquad\qquad\left.-H_{s,u}^{-1}\alpha_{u,s}\left(x-\Pi\left(s,p,\vartheta\right)\right)\theta_{s,u}\right)^\top dW_s\left(u\right).
\end{aligned}\tag{31}$$

Then

$$\begin{aligned}
&H_{s,t}\xi_{s,t}+\int_s^t H_{s,u}\left(c_{s,u}-Q_{s,u}\right)du\\
&=x+\int_s^t H_{s,u}\Big(\sigma_{s,u}^\top(\pi_{s,u}^C-\pi_{s,u}^Q)-\xi_{s,u}\theta_{s.u}\Big)^\top dW_s\left(u\right).
\end{aligned}\tag{32}$$

It follows as an application of Itô's rule that $\pi_{s,t}^C-\pi_{s,t}^Q$ is a portfolio that hedges $\xi_{s,t}$, and therefore (ξ,c,Q) is a hedgeable rate of consumption and endowment evolution structure. The proof that the consumption and endowment structure (ξ,c,Q) is optimal in the sense of the theorem is identical to the proof of [14, Theorem 2]. □

Remark 7. It is clear from the proof of Theorem 6 that even if a sufficient number of stocks are added in order to make a complete market, then the consumption, wealth, and portfolio processes obtained in the above mentioned theorem are still optimal solutions.

Remark 8. We notice that the optimization problem described above is equivalent to the standard optimization problem for (state-dependent) utility random fields when the random field is defined as follows:

$$\begin{aligned}
U_1\left(t,c,\omega\right)&=U_1\left(t,cH_{0,t}\left(\omega\right)\right),\quad \omega\in\Omega,\ c>0,\\
U_2\left(x,\omega\right)&=U_2\left(xH_{0,T}\left(\omega\right)\right),\quad \omega\in\Omega,\ x>0,
\end{aligned}\tag{33}$$

where the definition of utility random fields is the one presented by Karatzas and Žitković [6, Definition 3.1]. However we notice that Karatzas and Žitković [6] require the random fields to be bounded, and this is not the case for our random fields; therefore their results do not include ours.

As a result of the previous theorem, from a point of view of consumption of a consumer, he behaves as he sells the value of his income at the beginning and optimizes the consumption assuming no income at all.

Corollary 9. *Assume the conditions and notation of Theorem 6, and assume that the "augmented wealth"*

$$\begin{aligned}
&\xi'\left(x-\Pi\left(s,p,\vartheta\right),p,\vartheta\right)\\
&\qquad=\xi_{s,t}\left(x,p,\vartheta\right)-\Pi\left(t,P_{s,t}\left(p,\vartheta\right),\Theta_{s,t}\left(\vartheta\right)\right),
\end{aligned}\tag{34}$$

where $\xi_{s,t}$ is the optimal wealth obtained in Theorem 6, is a wealth evolution structure with feasible set of values

$$\mathbb{Y}=\left\{\left(s,y,p^\top,\vartheta^\top\right)^\top\mid y>0,\left(p^\top,\vartheta^\top\right)^\top\in\mathbb{D}\left(s\right)\right\}.\tag{35}$$

Then, for any $x>\Pi(s,p,\vartheta)$ and $(p^\top,\vartheta^\top)^\top\in\mathbb{D}(s)$ and initial wealth $y=x-\Pi(s,p,\vartheta)$, the process of optimal hedgeable cumulative consumption and endowment structure (with no income) $(\xi',c,0)$ has the same optimal consumption as the one obtained in Theorem 6 (when it is assumed an income process $Q_{s,t}$).

Similar results to Theorem 6 can be obtained from the problems of maximizing the expected utility from discounted consumption (alone) or from the problem of maximizing the expected utility from final wealth (alone) as was discussed in Londoño [14, Theorems 3 and 4]. The proof follows on the lines of Theorem 6.

Acknowledgment

The authors would like to thank Professor Darrell Duffie for the detailed reading of this paper, positive and encouraging comments, and valuable suggestions that led to a substantial improvement of the paper. All remaining errors are the author's.

References

[1] J. A. Londoño, "State-dependent utility," *Journal of Applied Probability*, vol. 46, no. 1, pp. 55–70, 2009.

[2] R. C. Merton, "Lifetime portfolio selection under uncertainty: the continuos time case," *Review of Economics and Statistics*, vol. 51, no. 3, pp. 247–257, 1969.

[3] R. C. Merton, "Optimum consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, vol. 3, no. 4, pp. 373–413, 1971.

[4] J. C. Cox and C.-f. Huang, "Optimal consumption and portfolio policies when asset prices follow a diffusion process," *Journal of Economic Theory*, vol. 49, no. 1, pp. 33–83, 1989.

[5] I. Karatzas, J. P. Lehoczky, and S. E. Shreve, "Optimal portfolio and consumption decisions for a "small investor" on a finite horizon," *SIAM Journal on Control and Optimization*, vol. 25, no. 6, pp. 1557–1586, 1987.

[6] I. Karatzas and G. Žitković, "Optimal consumption from investment and random endowment in incomplete semimartingale markets," *The Annals of Probability*, vol. 31, no. 4, pp. 1821–1858, 2003.

[7] J. Hugonnier and D. Kramkov, "Optimal investment with random endowments in incomplete markets," *The Annals of Applied Probability*, vol. 14, no. 2, pp. 845–864, 2004.

[8] J. Cvitanić, W. Schachermayer, and H. Wang, "Utility maximization in incomplete markets with random endowment," *Finance and Stochastics*, vol. 5, no. 2, pp. 259–272, 2001.

[9] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, vol. 39 of *Applications of Mathematics*, Springer, New York, NY, USA, 1998.

[10] R. C. Merton, "An intertemporal capital asset pricing model," *Econometrica*, vol. 41, pp. 867–887, 1973.

[11] D. T. Breeden, "An intertemporal asset pricing model with stochastic consumption and investment opportunities," *Journal of Financial Economics*, vol. 7, no. 3, pp. 265–296, 1979.

[12] J. C. Cox, J. E. Ingersoll Jr., and S. A. Ross, "An intertemporal general equilibrium model of asset prices," *Econometrica*, vol. 53, no. 2, pp. 363–384, 1985.

[13] J. A. Londoño, *A Sensitive Intertemporal Equilibrium for Relative Well-Being*, Universidad Nacional de Colombia, Sede Bogota, Colombia, 2013.

[14] J. A. Londoño, "A more general valuation and arbitrage theory for Itô processes," *Stochastic Analysis and Applications*, vol. 26, no. 4, pp. 809–831, 2008.

[15] J. A. Londoño, "State tameness: a new approach for credit constrains," *Electronic Communications in Probability*, vol. 9, pp. 1–13, 2004.

[16] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, vol. 94 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1983.

[17] H. Kunita, *Stochastic Flows and Stochastic Differential Equations*, vol. 24 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.