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Research Article

Numerical and Analytical Study for Fourth-Order Integro-Differential Equations Using a Pseudospectral Method

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A numerical method for solving fourth-order integro-differential equations is presented. This method is based on replacement of the unknown function by a truncated series of well-known shifted Chebyshev expansion of functions. An approximate formula of the integer derivative is introduced. The introduced method converts the proposed equation by means of collocation points to system of algebraic equations with shifted Chebyshev coefficients. Thus, by solving this system of equations, the shifted Chebyshev coefficients are obtained. Special attention is given to study the convergence analysis and derive an upper bound of the error of the presented approximate formula. Numerical results are performed in order to illustrate the usefulness and show the efficiency and the accuracy of the present work.

1. Introduction

The integro-differential equation (IDE) is an equation that involves both integrals and derivatives of an unknown function. Mathematical modeling of real-life problems usually results in functional equations, like ordinary or partial differential equations, and integral and integro-differential equations, stochastic equations. Many mathematical formulations of physical phenomena contain integro-differential equations; these equations arise in many fields like physics, astronomy, potential theory, fluid dynamics, biological models, and chemical kinetics. Integro-differential equations; are usually difficult to solve analytically; so, it is required to obtain an efficient approximate solution [1–5]. Recently, several numerical methods to solve IDEs have been given such as variational iteration method [6, 7], homotopy perturbation method [8, 9], spline functions expansion [10, 11], and collocation method [12–15].

Chebyshev polynomials are well-known family of orthogonal polynomials on the interval $[-1, 1]$ that have many applications [4, 6, 8, 13]. They are widely used because of their good properties in the approximation of functions. However, with our best knowledge, very little work was done to adapt these polynomials to the solution of integro-differential equations. Orthogonal polynomials have a great variety and wealth of properties. Some of these properties take a very concise form in the case of the Chebyshev polynomials, making Chebyshev polynomials of leading importance among orthogonal polynomials. The Chebyshev polynomials belong to an exclusive band of orthogonal polynomials, known as Jacobi polynomials, which correspond to weight functions of the form $(1-x)^{\alpha}(1+x)^{\beta}$ and which are solutions of Sturm-Liouville equations [16].

In this work, we derive an approximate formula of the integral derivative $y^{(n)}(x)$ and derive an upper bound of the error of this formula, and then we use this formula to solve a class of two-point boundary value problems (BVPs) for the fourth-order integro-differential equations as

$$y^{(iv)}(x) = f(x) + \gamma y(x) + \int_0^x \left[p(t)\, y(t) + q(t)\, \Theta\left(y(t)\right)\right] dt, \qquad 0 \le x,\ t \le 1, \tag{1}$$

under the boundary and initial conditions

$$y(0)=\alpha_0,\qquad y''(0)=\alpha_1,$$
$$y(1)=\beta_0,\qquad y''(1)=\beta_1, \tag{2}$$

where $f(x)$, $p(x)$, and $q(x)$ are known functions and γ, α_0, α_1, β_0, and β_1 are suitable constants. Several numerical methods to solve the fourth-order integro-differential equations have been given such as Chebyshev cardinal functions [17], variational iteration method [7], and others.

2. Some Basic Properties and Derivation of an Approximate Formula of the Derivative for Chebyshev Polynomials Expansion

The Chebyshev polynomial of the first kind is a polynomial in z of degree n, defined by the relation

$$T_n(z)=\cos n\theta,\quad \text{when } z=\cos\theta. \tag{3}$$

The Chebyshev polynomials of degree $n>0$ of the first kind have precisely n zeros and $n+1$ local extrema in the interval $[-1,1]$. The zeros of $T_n(z)$ are denoted by

$$z_k=\cos\frac{(k-1/2)\pi}{n},\quad k=1,2,\ldots,n. \tag{4}$$

The Chebyshev polynomials can be determined with the aid of the following recurrence formula [18]:

$$T_{n+1}(z)=2zT_n(z)-T_{n-1}(z),$$
$$T_0(z)=1,\qquad T_1(z)=z,\quad n=1,2,\ldots. \tag{5}$$

The analytic form of the Chebyshev polynomials $T_n(z)$ of degree n is given by

$$T_n(z)=n\sum_{i=0}^{[n/2]}(-1)^i2^{n-2i-1}\frac{(n-i-1)!}{(i)!\,(n-2i)!}z^{n-2i}, \tag{6}$$

where $[n/2]$ denotes the integer part of $n/2$. The orthogonality condition is

$$\int_{-1}^{1}\frac{T_i(z)T_j(z)}{\sqrt{1-z^2}}dx=\begin{cases}\pi, & \text{for } i=j=0;\\ \dfrac{\pi}{2}, & \text{for } i=j\neq 0;\\ 0, & \text{for } i\neq j.\end{cases} \tag{7}$$

In order to use these polynomials on the interval $[0,1]$, we define the so called shifted Chebyshev polynomials by introducing the change of variable $z=2x-1$. The shifted Chebyshev polynomials are denoted by $T_n^*(x)$ and defined as $T_n^*(x)=T_n(2x-1)=T_{2n}(\sqrt{x})$.

The function $y(x)$, which belongs to the space of square integrable in $[0,1]$, may be expressed in terms of shifted Chebyshev polynomials as

$$y(x)=\sum_{i=0}^{\infty}c_iT_i^*(x), \tag{8}$$

where the coefficients c_i are given by

$$c_0=\frac{1}{\pi}\int_0^1\frac{y(x)T_0^*(x)}{\sqrt{x-x^2}}dx,\qquad c_i=\frac{2}{\pi}\int_0^1\frac{y(x)T_i^*(x)}{\sqrt{x-x^2}}dx,\quad i=1,2,\ldots. \tag{9}$$

In practice, only the first $(m+1)$ terms of shifted Chebyshev polynomials are considered. Then, we have that

$$y_m(x)=\sum_{i=0}^{m}c_iT_i^*(x). \tag{10}$$

Lemma 1. *The analytic form of the shifted Chebyshev polynomials $T_n^*(x)$ of degree n is given by*

$$T_n^*(x)=n\sum_{k=0}^{n}(-1)^{n-k}\frac{2^{2k}(n+k-1)!}{(2k)!\,(n-k)!}x^k,\quad n=1,2,\ldots. \tag{11}$$

Proof. Since we have $T_n^*(x)=T_{2n}(\sqrt{x})$, then by substituting in (6), we can obtain that

$$T_n^*(x)=2n\sum_{i=0}^{n}(-1)^i\frac{2^{2n-2i-1}(2n-i-1)!}{(i)!\,(2n-2i)!}x^{n-i},\quad n=1,2,\ldots. \tag{12}$$

Now, we put $k=n-i$ in (12) we obtain the desired result (11). □

The main approximate formula of the derivative of $y_m(x)$, and is given in the following theorem.

Theorem 2. *Let $y(x)$ be approximated by shifted Chebyshev polynomials as* (10)*, and also suppose that r is integer; then,*

$$D^r(y_m(x))=\sum_{i=r}^{m}\sum_{k=r}^{i}c_i\lambda_{i,k,r}x^{k-r}, \tag{13}$$

where $\lambda_{i,k,r}$ is given by

$$\lambda_{i,k,r}=(-1)^{i-k}\frac{2^{2k}i(i+k-1)!k!}{(i-k)!\,(2k)!\,(k-r)!}. \tag{14}$$

Proof. Since the differential operator D^r is linear, we can obtain that

$$D^r(y_m(x))=\sum_{i=0}^{m}c_iD^r(T_i^*(x)). \tag{15}$$

Since $D^rc=0$, c is a constant, and

$$D^rx^n=\begin{cases}0, & \text{for } n\in N,\ n<r,\\ \dfrac{n!}{(n-r)!}x^{n-r}, & \text{for } n\in N,\ n\geq r.\end{cases} \tag{16}$$

Then, we have that

$$D^rT_i^*(x)=0,\quad i=0,1,\ldots,r-1, \tag{17}$$

and for $i = r,\ r+1,\ldots,m$, and by using (16), we get that

$$D^r T_i^*(x) = i\sum_{k=r}^{i}(-1)^{i-k}\frac{2^{2k}(i+k-1)!}{(i-k)!\,(2k)!}D^r x^k = i\sum_{k=r}^{i}(-1)^{i-k}\frac{2^{2k}(i+k-1)!k!}{(i-k)!\,(2k)!\,(k-r)!}x^{k-r}. \tag{18}$$

A combination of (17), (18), and (14) leads to the desired result and completes the proof of the theorem. □

3. Error Analysis

In this section, special attention is given to study the convergence analysis and evaluate the upper bound of the error of the proposed formula.

Theorem 3 (Chebyshev truncation theorem; see [18]). *The error in approximating $y(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If*

$$y_m(x) = \sum_{k=0}^{m} c_k T_k(x), \tag{19}$$

then

$$E_T(m) \equiv \left|y(x) - y_m(x)\right| \le \sum_{k=m+1}^{\infty}\left|c_k\right|, \tag{20}$$

for all $y(x)$, all m, and all $x \in [-1,1]$.

Theorem 4. *The derivative of order r for the shifted Chebyshev polynomials can be expressed in terms of the shifted Chebyshev polynomials themselves in the following form:*

$$D^r\left(T_i^*(x)\right) = \sum_{k=r}^{i}\sum_{j=0}^{k-r}\Theta_{i,j,k}T_j^*(x), \tag{21}$$

where

$$\Theta_{i,j,k} = \frac{(-1)^{i-k}\,2i\,(i+k-1)!\Gamma(k-r+1/2)}{h_j\Gamma(k+1/2)\,(i-k)!\,(k-r-j)!\,(k+j-r)!}, \quad h_0 = 2,\ h_j = 1,\ j = 0,1,\ldots. \tag{22}$$

Proof. We use the properties of the shifted Chebyshev polynomials [18] and expand x^{k-r} in (18) in the following form:

$$x^{k-r} = \sum_{j=0}^{k-r} c_{kj}T_j^*(x), \tag{23}$$

where c_{kj} can be obtained using (9), and $y(x) = x^{k-r}$; then,

$$c_{kj} = \frac{2}{h_j\pi}\int_0^1 \frac{x^{k-r}T_j^*(x)}{\sqrt{x-x^2}}dx, \quad h_0 = 2,\ h_j = 1,\ j = 1,2,\ldots. \tag{24}$$

At $j = 0$, we find that $c_{k0} = (1/\pi)\int_0^1 (x^{k-r}T_0^*(x)/\sqrt{x-x^2})dx = (1/\sqrt{\pi})(\Gamma(k-r+1/2)/(k-r)!)$; also, at any j and using the formula (10), we can find that

$$c_{kj} = \frac{j}{\sqrt{\pi}}\sum_{l=0}^{j}(-1)^{j-l}\frac{(j+l-1)!2^{2l+1}\Gamma(k+l-r+1/2)}{(j-l)!\,(2l)!\,(k+l-r)!}, \quad j = 1,2,3,\ldots, \tag{25}$$

employing (18) and (23) gives

$$D^r\left(T_i^*(x)\right) = \sum_{k=r}^{i}\sum_{j=0}^{k-r}\Theta_{i,j,k}T_j^*(x), \quad i = r,\ r+1,\ldots, \tag{26}$$

where

$$\Theta_{i,j,k} = \begin{cases} i\,\dfrac{(-1)^{i-k}(i+k-1)!2^{2k}k!\Gamma(k-r+1/2)}{(i-k)!\,(2k)!\sqrt{\pi}\,(\Gamma(k+1-r))^2}, & j = 0; \\ \dfrac{(-1)^{i-k}ij\,(i+k-1)!2^{2k+1}\,k!}{\sqrt{\pi}\,(k-r)!\,(i-k)!\,(2k)!} \\ \quad\times\displaystyle\sum_{l=0}^{j}\dfrac{(-1)^{j-l}(j+l-1)!2^{2l}\Gamma(k+l-r+1/2)}{(j-l)!\,(2l)!\,(k+l-r)!}, & j = 1,2,3,\ldots. \end{cases} \tag{27}$$

After some lengthy manipulation, $\Theta_{i,j,k}$ can be put in the following form:

$$\Theta_{i,j,k} = \frac{(-1)^{i-k}2i\,(i+k-1)!\Gamma(k-r+1/2)}{h_j\Gamma(k+1/2)\,(i-k)!\,(k-r-j)!\,(k+j-r)!}, \quad j = 0,1,\ldots, \tag{28}$$

and this completes the proof of the theorem. □

Theorem 5. *The error $|E_T(m)| = |D^r y(x) - D^r y_m(x)|$ in approximating $D^r y(x)$ by $D^r y_m(x)$ is bounded by*

$$\left|E_T(m)\right| \le \left|\sum_{i=m+1}^{\infty} c_i\left(\sum_{k=r}^{i}\sum_{j=0}^{k-r}\Theta_{i,j,k}\right)\right|. \tag{29}$$

Proof. A combination of (8), (10), and (21) leads to

$$\left|E_T(m)\right| = \left|D^r y(x) - D^r y_m(x)\right| = \left|\sum_{i=m+1}^{\infty} c_i\left(\sum_{k=r}^{i}\sum_{j=0}^{k-r}\Theta_{i,j,k}T_j^*(x)\right)\right|, \tag{30}$$

but $|T_j^*(x)| \le 1$; so, we can obtain that

$$\left|E_T(m)\right| \le \left|\sum_{i=m+1}^{\infty} c_i\left(\sum_{k=r}^{i}\sum_{j=0}^{k-r}\Theta_{i,j,k}\right)\right|, \tag{31}$$

and subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds complete the proof of the theorem. □

4. Procedure Solution for the Fourth-Order Integro-Differential Equation

In this section, we will present the proposed method to solve numerically the fourth-order integro-differential equation of the form in (1). The unknown function $y(x)$ may be expanded by finite series of shifted Chebyshev polynomials as in the following approximation:

$$y_m(x) = \sum_{n=0}^{m} c_n T_n^*(x), \tag{32}$$

and approximated formula of its derivatives can be defined in Theorem 2. From (1), (32), and Theorem 2, we have that

$$\begin{aligned}\sum_{i=r}^{m}\sum_{k=r}^{i} c_i \lambda_{i,k,r}\, x^{k-r} &= f(x) + \gamma \sum_{n=0}^{m} c_n T_n^*(x) \\ &\quad + \int_0^x \left[p(t)\left(\sum_{n=0}^{m} c_n T_n^*(t)\right) \right. \\ &\quad \left. + q(t)\,\Theta\left(\sum_{n=0}^{m} c_n T_n^*(t)\right)\right] dt.\end{aligned} \tag{33}$$

We now collocate (33) at $(m-1+r)$ points x_s, $s = 0, 1, \ldots, m-r$ as

$$\begin{aligned}\sum_{i=4}^{m}\sum_{k=4}^{i} c_i \lambda_{i,k,4}\, x_s^{k-4} &= f(x_s) + \gamma \sum_{n=0}^{m} c_n T_n^*(x_s) \\ &\quad + \int_0^{x_s} \left[p(t)\left(\sum_{n=0}^{m} c_n T_n^*(t)\right) \right. \\ &\quad \left. + q(t)\,\Theta\left(\sum_{n=0}^{m} c_n T_n^*(t)\right)\right] dt.\end{aligned} \tag{34}$$

For suitable collocation points, we use roots of shifted Chebyshev polynomial $T_{m+1-r}^*(x)$. The integral terms in (34) can be found using composite trapezoidal integration technique as

$$\begin{aligned}&\int_0^{x_s} \left[p(t)\left(\sum_{n=0}^{m} c_n T_n^*(t)\right) + q(t)\,\Theta\left(\sum_{n=0}^{m} c_n T_n^*(t)\right)\right] dt \\ &\quad \cong \frac{h_s}{2}\left(\Omega(t_0) + \Omega(t_L) + 2\sum_{k=1}^{L-1}\Omega(t_k)\right),\end{aligned} \tag{35}$$

where $\Omega(t) = p(t)\sum_{n=0}^{m} c_n T_n^*(t) + q(t)\Theta(\sum_{n=0}^{m} c_n T_n^*(t))$, $h_s = x_s/L$, for an arbitrary integer L, $t_{j+1} = t_j + h_s$, $s = 0, 1, \ldots, m-r$, and $j = 0, 1, \ldots, L$. So, by using (34) and (35), we obtain

$$\begin{aligned}\sum_{i=4}^{m}\sum_{k=4}^{i} c_i \lambda_{i,k,r} x_s^{k-r} &= f(x_s) + \gamma \sum_{n=0}^{m} c_n T_n^*(x_s) \\ &\quad + \frac{h_s}{2}\left(\Omega(t_0) + \Omega(t_L) + 2\sum_{k=1}^{L-1}\Omega(t_k)\right).\end{aligned} \tag{36}$$

Also, by substituting (32) in the boundary conditions (2), we can obtain r equations as follows:

$$\begin{aligned}&\sum_{i=0}^{m}(-1)^i c_i = \alpha_0, \qquad \sum_{i=0}^{m} c_i = \beta_0, \\ &\sum_{i=2}^{m} c_i T_i^{*\prime\prime}(0) = \alpha_1, \qquad \sum_{i=2}^{m} c_i T_i^{*\prime\prime}(1) = \beta_1.\end{aligned} \tag{37}$$

Equation (36), together with r equations of the boundary conditions (37), give $(m+1)$ of system of algebraic equations which can be solved, for the unknowns c_n, $n = 0, 1, \ldots, m$, using conjugate gradient method or Newton iteration method.

5. Numerical Results

In this section, to verify the validity and the accuracy and support our theoretical discussion of the proposed method, we give some computations results of numerical examples.

Example 6. Consider the nonlinear fourth-order integro-differential equation as in (1) and (2) with $f(x) = 1$, $\gamma = 0$, $p(t) = 0$, $q(t) = e^{-t}$, and $\Theta(y) = y^2(x)$; then, the integro-differential equation will be

$$y^{(iv)}(x) = 1 + \int_0^x e^{-t} y^2(t)\, dt, \quad 0 \le x \le 1, \tag{38}$$

subject to the boundary conditions

$$y(0) = y''(0) = 1, \qquad y(1) = y''(1) = e. \tag{39}$$

The exact solution of this problem is $y(x) = e^x$ [7].

We apply the suggested method with $m = 5$ and approximate the solution $y(x)$ as follows:

$$y_5(x) = \sum_{n=0}^{5} c_n T_n^*(x). \tag{40}$$

From (38), (40), and Theorem 2, we have that

$$\sum_{i=4}^{5}\sum_{k=4}^{i} c_i \lambda_{i,k,4} x^{k-4} = 1 + \int_0^x e^{-t}\left(\sum_{n=0}^{5} c_n T_n^*(t)\right)^2 dt. \tag{41}$$

We now collocate (41) at points, x_s, $s = 0, 1$ as

$$\sum_{i=4}^{5}\sum_{k=4}^{i} c_i\, \lambda_{i,k,4} x_s^{k-4} = 1 + \int_0^{x_s} e^{-t}\left(\sum_{n=0}^{5} c_n T_n^*(t)\right)^2 dt. \tag{42}$$

For suitable collocation points we use roots of shifted Chebyshev polynomial $T_2^*(x)$. The integral terms in (42) can be found using composite trapezoidal integration technique as

$$\int_0^{x_s} e^{-t}\left(\sum_{n=0}^{5} c_n T_n^*(t)\right)^2 dt = \frac{h_s}{2}\left(\Omega(t_0)+\Omega(t_L)+2\sum_{k=1}^{L-1}\Omega(t_k)\right), \tag{43}$$

where $\Omega(t) = e^{-t}(\sum_{n=0}^{5} c_n T_n^*(t))^2$, $h_s = x_s/L$, for an arbitrary integer L, $t_{j+1} = t_j + h_s$, $s = 0, 1$, and $j = 0, 1, \ldots, L$. So, by using (43) and (42), we obtain

$$\sum_{i=4}^{5}\sum_{k=4}^{i} c_i\ \lambda_{i,k,4}\ x_s^{k-4} = 1 + \frac{h_s}{2}\left(\Omega(t_0)+\Omega(t_L)+2\sum_{k=1}^{L-1}\Omega(t_k)\right). \tag{44}$$

Also, by substituting (40) in the boundary conditions (39), we can obtain four equations as follows:

$$\begin{gathered} c_0 - c_1 + c_2 - c_3 + c_4 - c_5 = 1, \\ c_0 + c_1 + c_2 + c_3 + c_4 + c_5 = e, \\ l_0c_0 + l_1c_1 + l_2c_2 + l_3c_3 + l_4c_4 + l_5c_5 = 1, \\ s_0c_0 + s_1c_1 + s_2c_2 + s_3c_3 + s_4c_4 + s_5c_5 = e, \end{gathered} \tag{45}$$

where $l_i = T_i^{*\prime\prime}(0)$ and $s_i = T_i^{*\prime\prime}(1)$.

Equation (44), together with four equations of the boundary conditions (45), represent, a nonlinear system of six algebraic equations in the coefficients c_n; by solving it using the Newton iteration method, we obtain

$$\begin{aligned} c_0 &= 1.75379, & c_1 &= 0.85039, & c_2 &= 0.10478, \\ c_3 &= 0.00872, & c_4 &= 0.00057, & c_5 &= 0.00003. \end{aligned} \tag{46}$$

The behavior of the approximate solution using the proposed method with $m = 5$, the approximate solution using variational iteration method (VIM), and the exact solution are presented in Figure 1. Table 1 shows the behavior of the absolute error between exact solution and approximate solution using the presented method at $m = 6$ and $m = 8$. From Figure 1 and Table 1, it is clear that the proposed method can be considered as an efficient method to solve the nonlinear integro-differential equations. Table 1 indicates that as m increases the errors decrease more rapidly; hence, for better results, using number m is recommended. Also, we can conclude that the obtained approximated solution is in excellent agreement with the exact solution.

Example 7. Consider the linear fourth-order integro-differential equation as in (1) and (2) with $f(x) = x+(x+3)e^x$, $\gamma = 1$, $p(t) = -1$, $h(t) = 0$, and $\Theta(y) = y(x)$; then, the integro-differential equation will be

$$y^{(iv)}(x) = x + (x+3)e^x + y(x) - \int_0^x y(t)\,dt, \quad 0 \le x \le 1, \tag{47}$$

subject to the boundary conditions

$$\begin{aligned} y(0) &= 1, & (1) &= 1 + e, \\ y''(0) &= 2, & y''(1) &= 3e. \end{aligned} \tag{48}$$

The exact solution of this problem is $y(x) = 1 + xe^x$ [17].

We apply the suggested method with $m = 5$ and approximate the solution $y(x)$ as follows:

$$y(x) \cong \sum_{n=0}^{5} c_n T_n^*(x). \tag{49}$$

By the same procedure in the previous example, we have

$$\sum_{i=4}^{5}\sum_{k=4}^{i} c_i\lambda_{i,k,4}x_s^{k-4} = f(x_s) + \sum_{n=0}^{5} c_n T_n^*(x_s) + \frac{h_s}{2} \times\left(\Omega(t_0)+\Omega(t_L)+2\sum_{k=1}^{L-1}\Omega(t_k)\right), \quad s = 0, 1, 2, \tag{50}$$

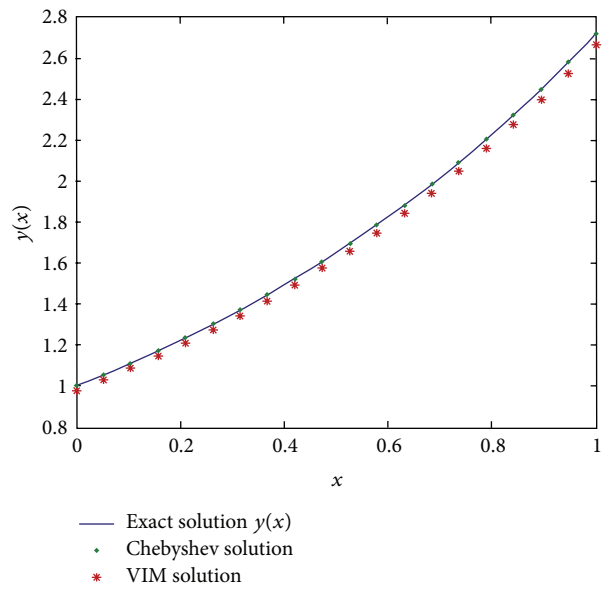

Figure 1: The behavior of the exact solution, the approximate solution using VIM, and the approximate solution using the proposed method at $m = 5$.

Table 1: The behavior of the absolute error between the exact solution and approximate solution at $m = 6$ and $m = 8$.

x	$\lvert y_{\text{ex.}} - y_{\text{ap.}}\rvert$ at $m = 6$	$\lvert y_{\text{ex.}} - y_{\text{ap.}}\rvert$ at $m = 8$
0.0	$2.2548e-10$	$2.0254e-10$
0.2	$2.3654e-04$	$1.2548e-06$
0.4	$3.5687e-04$	$3.2541e-06$
0.6	$0.1587e-04$	$5.2548e-06$
0.8	$9.2450e-04$	$7.2581e-06$
1.0	$1.2589e-10$	$2.2548e-10$

Table 2: The behaviour of the absolute error between the exact solution and approximate solution at $m = 7$ and $m = 9$.

x	$\lvert y_{\text{ex.}} - y_{\text{ap.}}\rvert$ at $m = 7$	$\lvert y_{\text{ex.}} - y_{\text{ap.}}\rvert$ at $m = 9$
0.0	$1.2587e-08$	$5.1236e-09$
0.2	$6.2548e-03$	$2.2258e-05$
0.4	$2.0254e-03$	$9.2154e-05$
0.6	$1.3654e-03$	$2.0054e-05$
0.8	$0.2540e-03$	$2.3690e-05$
1.0	$6.0254e-08$	$5.2478e-09$

where $\Omega(t) = \sum_{n=0}^{5} c_n T_n^*(t)$, and the nodes $t_{j+1} = t_j + h_s$, $j = 0, 1, \ldots, L$, $t_0 = 0$, and $h_s = x_s/L$. We can write the initialboundary conditions in the form

$$\begin{aligned} c_0 - c_1 + c_2 - c_3 + c_4 - c_5 + c_6 &= 1, \\ c_0 + c_1 + c_2 + c_3 + c_4 + c_5 + c_6 &= 1 + e, \\ l_0c_0 + l_1c_1 + l_2c_2 + l_3c_3 + l_4c_4 + l_5c_5 + l_6c_6 &= 2, \\ s_0c_0 + s_1c_1 + s_2c_2 + s_3c_3 + s_4c_4 + s_5c_5 + s_6c_6 &= 3e. \end{aligned} \tag{51}$$

By using (50) and (51), we obtain a linear system of seven algebraic equations in the coefficients c_n; by solving it using the conjugate gradient method, we obtain

$$\begin{aligned} c_0 &= 2.09189, \quad c_1 = 1.32820, \quad c_2 = 0.26461, \\ c_3 &= 0.03079, \quad c_4 = 0.00264, \quad c_5 = 0.00015. \end{aligned} \tag{52}$$

The behavior of the approximate solution using the proposed method with $m = 6$, the approximate solution using variational iteration method (VIM) and the exact solution are presented in Figure 2. Table 2 shows the behaviour of the absolute error between exact solution and approximate solution using the presented method at $m = 7$ and $m = 9$. From this figure, it is clear that the proposed method can be considered as an efficient method to solve the linear integro-differential equations. Also, we can conclude that the obtained approximate solution is in excellent agreement with the exact solution.

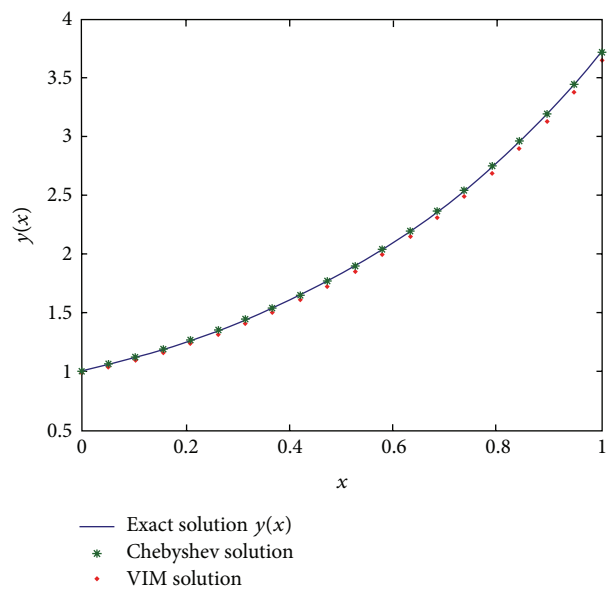

Figure 2: The behavior of the exact solution, the approximate solution using VIM, and the approximate solution using the proposed method at $m = 6$.

6. Conclusion and Discussion

Integro-differential equations are usually difficult to solve analytically; so, it is required to obtain the approximate solution. In this paper, we proposed the pseudospectral method using shifted Chebyshev method for solving the integro-differential equations. The Chebyshev method is useful for acquiring both the general solution and particular solution as demonstrated in examples. Special attention is given to study the convergence analysis and derive an upper bound of the error of the derived approximate formula. From our obtained results, we can conclude that the proposed method gives solutions in excellent agreement with the exact solution and better than the other methods. An interesting feature of this method is that when an integral system has linearly independent polynomial solution of degree m or less than m, the method can be used for finding the analytical solution. All computations are done using MATLAB 8.

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