# Stability and $l_{1}$-Gain Control of Positive Switched Systems with Time-Varying Delays via Delta Operator Approach 

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#### Abstract

This paper investigates the problems of stability and $l_{1}$-gain controller design for positive switched systems with time-varying delays via delta operator approach. The purpose is to design a switching signal and a state feedback controller such that the resulting closedloop system is exponentially stable with $l_{1}$-gain performance. Based on the average dwell time approach, a sufficient condition for the existence of an $l_{1}$-gain controller for the considered system is established by constructing an appropriate copositive type LyapunovKrasovskii functional in delta domain. Moreover, the obtained conditions can unify some previously suggested relevant methods in the literature of both continuous- and discrete-time systems into the delta operator framework. Finally, a numerical example is presented to explicitly demonstrate the effectiveness and feasibility of the proposed method.


## 1. Introduction

Positive systems mean that their states and outputs are nonnegative whenever the initial conditions and inputs are nonnegative [1, 2]. A positive switched system consists of a family of positive subsystems and a switching signal, coordinating the operation of various subsystems to specify when and how the switching takes place among the subsystems. Recently, due to the broad applications in communication systems [3, 4], formation flying [5], viral mutation dynamics under drug treatment [2], and systems theories [6-10], positive systems have been highlighted and investigated by many researchers [11-14]. It has been shown that a linear copositive Lyapunov functional is powerful for the analysis and synthesis of positive systems [15-17].

The delta operator, a novel method with good finite word length performance under fast sampling rates, has drawn considerable interest in the past three decades. As we know, the standard shift operator was mostly adopted in the study of control theories for discrete-time systems. However, the dynamic response of a discrete system does not converge smoothly to its continuous counterpart when the sampling period tends to zero; namely, data are taken at high sampling rates. Until Goodwin et al. proposed a delta operator method
in [18] to take the place of the traditional shift operator, the above problem is avoided. It was shown that delta operator requires smaller word length when implemented in fixedpoint digital control processors than shift operator does [19]. The delta operator model can be regarded as a useful approach to deal with discrete-time systems under high sampling rates through the analysis methods of continuoustime systems [20-23]. Based on significant early investigations such as [24-26] studying the basic properties and performance of delta operator model, numerical properties and practical applications of delta operator model have been extensively investigated [27-29]. The delta operator is defined by

$$
\delta x(t)= \begin{cases}\frac{d x(t)}{d t}, & T=0  \tag{1}\\ \frac{(x(t+T)-x(t))}{T}, & T \neq 0,\end{cases}
$$

where $T$ is a sampling period. When $T \rightarrow 0$, the delta operator model will approach the continuous system before discretization and reflect a quasicontinuous performance.

In real engineering, time delays are involved in many fields, such as mechanics, medicine, chemistry, biology,
physics, economics, engineering, and control theory [30-33]. The existence of time delay may give rise to the deterioration of system performance and instability. Many results have been reported for time-delay systems [34-39].

In addition, exogenous disturbances are commonly unavoidable in practical process, and the output will be inevitably affected by the disturbance in a system. Because of the peculiar nonnegative property of positive systems, the $l_{1}$ gain (or $L_{1}$-gain) index [39] can characterize the disturbance rejection property, by means of which we can limit the effect of disturbance in a prescribed level. Some results on $l_{1}$-gain (or $L_{1}$-gain) analysis and control for positive systems have been reported in the literature [39, 40]. However, few results on the issue of $l_{1}$-gain performance for positive switched systems via delta operator approach are proposed, which motivates the current research.

In this paper, we focus our attention on investigating the stability and $l_{1}$-gain controller design for positive switched systems with time-varying delays via delta operator approach. The main contributions of this paper are fourfold. (1) The positive switched systems via delta operator approach are investigated for the first time. (2) By applying the average dwell time approach, sufficient conditions of exponential stability for positive switched delta operator systems are derived. Moreover, the results obtained can be applied to both continuous-time systems and discrete-time systems. (3) $l_{1}$-gain performance analysis of the underlying system is developed. (4) A state feedback controller design scheme is proposed such that the corresponding closed-loop system is exponentially stable with an $l_{1}$-gain performance.

The remainder of the paper is as follows. The problem formulation and some necessary lemmas are provided in Section 2. In Section 3, the issues of stability, $l_{1}$-gain performance analysis, and control of the underlying system are developed. A numerical example is presented to demonstrate the feasibility of the obtained results in Section 4. In Section 5, concluding remarks are given.

Notations. $A \succeq 0(\leq, \succ, \prec)$ means that all entries of matrix $A$ are nonnegative (nonpositive, positive, and negative); $A \succ$ $B(A \succeq B)$ means that $A-B \succ 0(A-B \succeq 0) ; A^{T}$ means the transpose of matrix $A ; R\left(R_{+}\right)$is the set of all real (positive real) numbers; $R^{n}\left(R_{+}^{n}\right)$ is an $n$-dimensional real (positive real) vector space; $R^{m \times n}$ is the set of all $m \times n$-dimensional real matrices; $Z_{+}$refers to the set of all positive integers; the vector 1 -norm is denoted by $\|x\|=\sum_{k=1}^{n}\left|x_{k}\right|$, where $x_{k}$ is the $k$ th element of $x \in R^{n} ; 1_{n} \in R^{n}$ denotes a column vector with $n$ rows containing only 1 entry; $l_{1}\left[k_{0}, \infty\right)$ is the space of absolute summable sequence on $\left[k_{0}, \infty\right)$; that is, we say $z:\left[k_{0}, \infty\right) \rightarrow R^{p}$ is in $l_{1}\left[k_{0}, \infty\right)$ if $\sum_{k=k_{0}}^{\infty}\|z(k)\|<\infty$.

## 2. Problem Formulation

Consider the following switched delta operator system with time-varying delays:

$$
\delta x(k)=A_{\sigma(k)} x(k)+A_{d \sigma(k)} x\left(k-d_{k}\right)+D_{\sigma(k)} w(k),
$$

$$
\begin{gather*}
x\left(k_{0}+\theta\right)=\varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0 \\
z(k)=C_{\sigma(k)} x(k)+E_{\sigma(k)} w(k) \tag{2}
\end{gather*}
$$

where $x(k) \in R^{n}$ denotes the state; $z(k) \in R^{l}$ is the controlled output; and $w(k) \in R^{w}$ is the disturbance input, which belongs to $l_{1}\left[k_{0}, \infty\right) . k$ means the time $t=k T$ and $T>0$ is the sampling period; $k_{0}$ is the initial time. $\sigma(k):\left[k_{0}, \infty\right) \rightarrow$ $\underline{m}=\{1,2, \ldots, m\}$ is the switching signal with $m$ representing the number of subsystems. $A_{p}, A_{d p}, C_{p}, D_{p}$, and $E_{p}$ are constant matrices with appropriate dimensions. $d_{k}$ denotes the time-varying discrete delay which satisfies $0 \leq \underline{d} \leq d_{k} \leq$ $\bar{d}$ for known integers $\underline{d}$ and $\bar{d} ;\{\varphi(\theta), \theta=-\bar{d},-\bar{d}+1, \ldots, 0\}$ is a given discrete vector-valued initial condition. The switch is assumed to only occur at the sampling time in this paper.

Remark 1. To illustrate the main advantage of delta operator systems directly, we consider a typical continuous system without time delays as follows:

$$
\begin{align*}
& \dot{x}(t)=A x(t)+D w(t) \\
& z(t)=C x(t)+E w(t) \tag{3}
\end{align*}
$$

Using the traditional shift operator approach to discretize the system, the following discrete form in $z$-domain can be obtained $(k=0,1,2, \ldots)$ :

$$
\begin{gather*}
x((k+1) T)=A_{z} x(k T)+D_{z} w(k T), \\
z(k T)=C_{z} x(k T)+E_{z} w(k T) \tag{4}
\end{gather*}
$$

where $A_{z}=e^{A T}, D_{z}=\left(\int_{0}^{T} e^{A t} d t\right) D, C_{z}=C$, and $E_{z}=E$. When $T \rightarrow 0, \lim _{T \rightarrow 0} A_{z}=I$ and $\lim _{T \rightarrow 0} D_{z}=0$. The movement of the system poles towards stable boundary makes the system defective with the increase in the sampling rates. However, by utilizing the delta operator approach, we can obtain the following system expressed in delta domain:

$$
\begin{align*}
\delta x(k T) & =A_{\delta} x(k T)+D_{\delta} w(k T) \\
z(k T) & =C_{\delta} x(k T)+E_{\delta} w(k T) \tag{5}
\end{align*}
$$

where $A_{\delta}=\left(A_{z}-I\right) / T, D_{\delta}=D_{z} / T, C_{\delta}=C$, and $E_{\delta}=E$. When $T \rightarrow 0, \lim _{T \rightarrow 0} A_{\delta}=A$ and $\lim _{T \rightarrow 0} D_{z}=D$. It can be seen that the system matrices are the same as those of the original continuous system, alleviating the problems encountered with fast sampling.

Remark 2. Since a delta operator system can be regarded as a quasicontinuous system when $T \rightarrow 0$, the term $\delta x(k)$ can be utilized like $\dot{x}(t)$ in normal continuous-time systems.

Definition 3. System (2) is said to be positive if, for any initial conditions $\varphi(\theta) \succeq 0, \theta=-\bar{d},-\bar{d}+1, \ldots, 0$, any inputs $w(k) \succeq 0$, and any switching signals $\sigma(k)$, the corresponding trajectories $x(k) \geq 0$ and $z(k) \geq 0$ hold for all $k \geq k_{0}$.

Remark 4. Definition 3 follows the general positivity definition of a positive system, which means that the state and
output are nonnegative whenever the initial condition and input are nonnegative $[1,2]$.

Lemma 5. System (2) is positive if and only if $\left(I+T A_{p}\right) \succeq 0$, $A_{d p} \succeq 0, D_{p} \succeq 0, C_{p} \succeq 0$, and $E_{p} \succeq 0$, for all $p \in \underline{m}$.

Proof. From the definition of delta operator $\delta$, the discrete form of system (2) can be obtained as follows:

$$
\begin{align*}
& x(k+1)=\left(I+T A_{\sigma(k)}\right) x(k)+T A_{d \sigma(k)} x\left(k-d_{k}\right) \\
&+T D_{\sigma(k)} w(k), \\
& x\left(k_{0}+\right.\theta)=\varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0,  \tag{6}\\
& z(k)=C_{\sigma(k)} x(k)+E_{\sigma(k)} w(k) .
\end{align*}
$$

Combining Lemma 2 in [41] and Lemma 1 in [42], one can obtain the remaining proof easily.

Remark 6. When $T \rightarrow 0$, system (2) degenerates to a general continuous-time positive switched system as follows:

$$
\begin{gather*}
\dot{x}(t)=A_{\sigma(t)} x(t)+A_{d \sigma(t)} x(t-d(t))+D_{\sigma(t)} w(t), \\
x\left(t_{0}+\theta\right)=\varphi(\theta), \quad \theta \in\left[-d_{2}, 0\right]  \tag{7}\\
z(t)=C_{\sigma(t)} x(t)+E_{\sigma(t)} w(t),
\end{gather*}
$$

where $d(t)$ denotes the time-varying delay which is everywhere time differentiable and satisfies $0 \leq d_{1} \leq d(t) \leq d_{2}$ and $\dot{d}(t) \leq d_{d}<1$ for known constants $d_{1}, d_{2}$, and $d_{d}$. Then according to [39], system (7) is positive if and only if $A_{p}$ are Metzler matrices, and $A_{d p} \geq 0, C_{p} \succeq 0, D_{p} \succeq 0$, and $E_{p} \succeq 0$, for all $p \in \underline{m}$.

Remark 7. In the light of Lemma 2.1 of [43], it is clear that the $p$ th subsystem in system (2) is positive if and only if ( $I+$ $\left.T A_{p}\right) \succeq 0, \mathrm{~A}_{d p} \succeq 0, D_{p} \succeq 0, C_{p} \succeq 0$, and $E_{p} \succeq 0$, for all $p \in$ $\underline{m}$. Thus we can have an equivalent expression of Lemma 5: system (2) is positive under any switching signals if and only if it consists of a family of positive subsystems.

Definition 8 (see [44]). System (2) with $w(k)=0$ is said to be exponentially stable under $\sigma(k)$ if, for constants $\alpha>0$ and $\beta>0$, the solution $x(k)$ satisfies

$$
\begin{equation*}
\|x(k)\| \leq \alpha\left\|x\left(k_{0}\right)\right\|_{c} e^{-\beta\left(k-k_{0}\right)}, \quad \forall k \geq k_{0} \tag{8}
\end{equation*}
$$

where $\left\|x\left(k_{0}\right)\right\|_{c}=\sup _{-\bar{d} \leq \theta \leq 0}\left\|x\left(k_{0}+\theta\right)\right\|$.
Definition 9 (see [45]). For any switching signal $\sigma(k)$ and any $k_{2}>k_{1} \geq 0$, let $N_{\sigma}\left(k_{1}, k_{2}\right)$ denote the number of switches of $\sigma(k)$ over the interval [ $k_{1}, k_{2}$ ). For given $\tau_{a}>0$ and $N_{0} \geq 0$, if the inequality

$$
\begin{equation*}
N_{\sigma}\left(k_{1}, k_{2}\right) \leq N_{0}+\frac{k_{2}-k_{1}}{\tau_{a}} \tag{9}
\end{equation*}
$$

holds, then the positive constant $\tau_{a}$ is called an average dwell time and $N_{0}$ is called a chattering bound.

Without loss of generality, one chooses $N_{0}=0$ in this paper.

Definition 10. For $0<\alpha<1 / T$ and $\gamma>0$, system (2) is said to have a prescribed $l_{1}$-gain performance level $\gamma$ if there exists a switching signal $\sigma(k)$ such that the following conditions are satisfied:
(a) system (2) is exponentially stable when $w(k)=0$;
(b) under zero initial condition, that is, $\varphi(\theta)=0, \theta=-\bar{d}$, $-\bar{d}+1, \ldots, 0$, system (2) satisfies

$$
\begin{array}{r}
\sum_{k=k_{0}}^{\infty}(1-T \alpha)^{\left(k-k_{0}\right)}\|z(k)\| \leq \gamma \sum_{k=k_{0}}^{\infty}\|w(k)\|  \tag{10}\\
\forall w(k) \in l_{1}\left[k_{0}, \infty\right), \quad w(k) \neq 0
\end{array}
$$

Remark 11. In Definition 10, as proposed in [39], $l_{1}$-gain performance index $\gamma$ characterizes system's suppression to exogenous disturbances. The smaller the value of $\gamma$ is, the better the performance of the system is, that is, the lesser the effect of the disturbance input on the control output is.

The purposes of this paper are (1) to find a class of switching signals $\sigma(k)$ under which system (2) is exponentially stable and possesses an $l_{1}$-gain performance and (2) to determine a class of switching signals and a state feedback controller $u(k)=K_{\sigma(k)} x(k)$ for the following positive switched delta operator system with time-varying delays:

$$
\begin{align*}
\delta x(k)= & A_{\sigma(k)} x(k)+A_{d \sigma(k)} x\left(k-d_{k}\right) \\
& +B_{\sigma(k)} u(k)+D_{\sigma(k)} w(k),  \tag{11}\\
x\left(k_{0}+\theta\right)= & \varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0, \\
z(k)= & C_{\sigma(k)} x(k)+E_{\sigma(k)} w(k)
\end{align*}
$$

such that the resulting closed-loop system is exponentially stable with an $l_{1}$-gain performance.

## 3. Main Results

This section will focus on the problems of stability analysis and $l_{1}$-gain controller design for positive switched delta operator systems with time-varying delays.
3.1. Stability Analysis. First, we consider the following switched positive delta operator system:

$$
\begin{gather*}
\delta x(k)=A_{\sigma(k)} x(k)+A_{d \sigma(k)} x\left(k-d_{k}\right), \\
x\left(k_{0}+\theta\right)=\varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0 \tag{12}
\end{gather*}
$$

where $I+T A_{p} \succeq 0, A_{d p} \succeq 0$ for $p \in \underline{m}$, and $d_{k}$ is defined the same as system (2).

Sufficient conditions of exponential stability of system (12) are provided in the following theorem.

Theorem 12. Given a positive constant $0<\alpha<1 / T$, if there exist $v_{p}, v_{p}, \vartheta_{p} \in R_{+}^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
A_{p}^{T} v_{p}+\alpha v_{p}+(1-T \alpha)(\bar{d}-\underline{d}+1) v_{p}+(1-T \alpha) \vartheta_{p} \leq 0, \\
A_{d p}^{T} v_{p}-(1-T \alpha)^{\bar{d}+1} v_{p} \leq 0, \tag{13}
\end{gather*}
$$

where $v_{p}=\left[v_{p 1}, v_{p 2}, \ldots, v_{p n}\right]^{T}, v_{p}=\left[v_{p 1}, v_{p 2}, \ldots, v_{p n}\right]^{T}$, and $\vartheta_{p}=\left[\vartheta_{p 1}, \vartheta_{p 2}, \ldots, \vartheta_{p n}\right]^{T}$, then system (12) is exponentially stable for any switching signals $\sigma(k)$ with average dwell time

$$
\begin{equation*}
\tau_{a}>\tau_{a}^{*}=-\frac{\ln \mu}{\ln (1-T \alpha)} \tag{14}
\end{equation*}
$$

where $\mu \geq 1$ satisfies

$$
\begin{equation*}
v_{p} \leq \mu v_{q}, \quad v_{p} \preceq \mu v_{q}, \quad \vartheta_{p} \preceq \mu \vartheta_{q}, \quad \forall p, q \in \underline{m} . \tag{15}
\end{equation*}
$$

Furthermore, the state decay of system (12) is given by

$$
\begin{equation*}
\|x(k)\| \leq a b^{\left(k-k_{0}\right)}\left\|x\left(k_{0}\right)\right\|_{c}, \tag{16}
\end{equation*}
$$

where

$$
\begin{gather*}
\begin{array}{c}
a=\frac{\varepsilon_{2}}{\varepsilon_{1}}+\frac{T \varepsilon_{3} \bar{d}}{\varepsilon_{1}}+\frac{0.5 T \varepsilon_{3}(\bar{d}+\underline{d}-1)(\bar{d}-\underline{d})}{\varepsilon_{1}}+\frac{T \varepsilon_{4} \bar{d}}{\varepsilon_{1}}, \\
b=\mu^{1 / \tau_{a}}(1-T \alpha), \\
\varepsilon_{1}
\end{array}=\min _{(r, p) \in \underline{n} \times \underline{m}}\left\{v_{p r}\right\}, \\
\varepsilon_{2}=\max _{(r, p) \in \underline{n} \times \underline{m}}\left\{v_{p r}\right\}, \\
\varepsilon_{3}=\max _{(r, p) \in \underline{n} \times \underline{m}}\left\{v_{p r}\right\}, \\
\varepsilon_{4}=\max _{(r, p) \in \underline{n} \times \underline{m}}\left\{\vartheta_{p r}\right\},  \tag{17}\\
\left\|x\left(k_{0}\right)\right\|_{c}=\sup _{-\bar{d} \leq \theta \leq 0}\left\|x\left(k_{0}+\theta\right)\right\|, \quad \underline{n}=\{1,2, \ldots, n\} .
\end{gather*}
$$

Proof. Choose the following piecewise copositive type Lyapunov functional for the $p$ th subsystem in system (12):

$$
\begin{align*}
V_{p}(k, x(k))= & V_{p 1}(k, x(k))+V_{p 2}(k, x(k))+V_{p 3}(k, x(k)) \\
& +V_{p 4}(k, x(k)), \tag{18}
\end{align*}
$$

where

$$
\begin{gather*}
V_{p 1}(k, x(k))=x^{T}(k) v_{p} \\
V_{p 2}(k, x(k))=T \sum_{s=k-d_{k}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p} \\
V_{p 3}(k, x(k))=T \sum_{l=-\bar{d}+1}^{-d} \sum_{s=k+l}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p} \\
V_{p 4}(k, x(k))=T \sum_{s=k-\bar{d}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) \vartheta_{p}, \quad \forall p \in \underline{m} . \tag{19}
\end{gather*}
$$

For simplicity, $V_{p}(k, x(k))$ is written as $V_{p}(k)$ (correspondingly, $V(k, x(k))$ is written as $V(k))$ in the later section of the paper.

The Lyapunov function in delta domain has the following form:

$$
-\sum_{s=k+1-\bar{d}}^{k-d}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p}
$$

$$
\delta V_{p 4}(k, x(k))=\frac{1}{T}\left[V_{p 4}(k+1)-V_{p 4}(k)\right]
$$

$$
=\frac{1}{T}\left[T \sum_{s=k+1-\bar{d}}^{(k+1)-1}(1-T \alpha)^{k+1-s} x^{T}(s) \vartheta_{p}\right.
$$

$$
\left.-T \sum_{s=k-\bar{d}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) \vartheta_{p}\right]
$$

$$
\begin{aligned}
& \delta V_{p 1}(k, x(k))=\delta\left(x^{T}(k) v_{p}\right) \\
& =\left(\delta x^{T}(k)\right) v_{p} \\
& =x^{T}(k) A_{p}^{T} v_{p}+x^{T}\left(k-d_{k}\right) A_{d p}^{T} v_{p}, \\
& \delta V_{p 2}(k, x(k))=\frac{1}{T}\left[V_{p 2}(k+1)-V_{p 2}(k)\right] \\
& =\frac{1}{T}\left[T \sum_{s=k+1-d_{k+1}}^{(k+1)-1}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p}\right. \\
& \left.-T \sum_{s=k-d_{k}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p}\right] \\
& \leq-T \alpha \sum_{s=k-d_{k}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p} \\
& +(1-T \alpha) x^{T}(k) v_{p} \\
& -(1-T \alpha)^{\bar{d}+1} x^{T}\left(k-d_{k}\right) v_{p} \\
& +\sum_{s=k+1-\bar{d}}^{k-d}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p} \text {, } \\
& \delta V_{p 3}(k, x(k))=\frac{1}{T}\left[V_{p 3}(k+1)-V_{p 3}(k)\right] \\
& =\frac{1}{T}\left[T \sum_{l=-\bar{d}+1}^{-d} \sum_{s=k+1+l}^{(k+1)-1}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p}\right. \\
& \left.-T \sum_{l=-\bar{d}+1}^{-d} \sum_{s=k+l}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p}\right] \\
& =-T \alpha \sum_{l=-\bar{d}+1}^{-d} \sum_{s=k+l}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p} \\
& +(1-T \alpha)(\bar{d}-\underline{d}) x^{T}(k) v_{p}
\end{aligned}
$$

$$
\begin{align*}
= & -T \alpha \sum_{s=k-\bar{d}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) \vartheta_{p} \\
& +(1-T \alpha) x^{T}(k) \vartheta_{p} \\
& -(1-T \alpha)^{\bar{d}+1} x^{T}(k-\bar{d}) \vartheta_{p} \tag{20}
\end{align*}
$$

According to (20), we have

$$
\left.\begin{array}{l}
\delta V_{p}(k, x(k))+\alpha V_{p}(k, x(k)) \\
\leq x^{T}(k) A_{p}^{T} v_{p}+x^{T}\left(k-d_{k}\right) A_{d p}^{T} v_{p}+\alpha x^{T}(k) v_{p} \\
\quad+(1-T \alpha) x^{T}(k) v_{p}-(1-T \alpha)^{\bar{d}+1} x^{T}\left(k-d_{k}\right) v_{p} \\
\quad+\sum_{s=k+1-\bar{d}}^{k-d}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p} \\
\quad+(1-T \alpha)(\bar{d}-\underline{d}) x^{T}(k) v_{p} \\
\quad-\sum_{s=k+1-\bar{d}}^{k-\underline{d}}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p}  \tag{21}\\
\quad+(1-T \alpha) x^{T}(k) \vartheta_{p}-(1-T \alpha)^{\bar{d}+1} x^{T}(k-\bar{d}) \vartheta_{p} \\
\leq
\end{array} \quad x^{T}(k)\left[A_{p}^{T} v_{p}+\alpha v_{p}+(1-T \alpha) \quad \times(\bar{d}-\underline{d}+1) v_{p}+(1-T \alpha) \vartheta_{p}\right]\right)
$$

From (13), we obtain

$$
\begin{align*}
\delta V_{p}(k)+\alpha V_{p}(k) & \leq 0 \\
& \Longrightarrow \delta V_{p}(k)=\frac{V_{p}(k+1)-V_{p}(k)}{T} \\
& \leq-\alpha V_{p}(k) \\
& \Longrightarrow V_{p}(k+1) \leq V_{p}(k)-T \alpha V_{p}(k) \\
& \Longrightarrow V_{p}(k+1) \leq(1-T \alpha) V_{p}(k) . \tag{22}
\end{align*}
$$

Let $k_{1}<\cdots<k_{g}$ denote the switching instants of $\sigma(k)$ over the interval $\left[k_{0}, k\right)$. Consider the following piecewise Lyapunov functional candidate for system (12):

$$
\begin{equation*}
V(k)=V_{\sigma(k)}(k) \tag{23}
\end{equation*}
$$

From (15) and (18), we obtain

$$
\begin{equation*}
V_{\sigma\left(k_{i}\right)}(k) \leq \mu V_{\sigma\left(k_{i}^{-}\right)}(k), \quad i=1,2, \ldots, g \tag{24}
\end{equation*}
$$

Then, it follows from (22), (24), and the relation $N_{\sigma}\left(k_{0}, k\right) \leq$ $\left(k-k_{0}\right) / \tau_{a}$ that, for $\left[k_{i}, k_{i+1}\right)$,

$$
\begin{align*}
V_{\sigma(k)}(k) & =V_{\sigma\left(k_{i}\right)}(k) \leq(1-T \alpha)^{\left(k-k_{i}\right)} V_{\sigma\left(k_{i}\right)}\left(k_{i}\right) \\
& \leq \mu(1-T \alpha)^{\left(k-k_{i}\right)} V_{\sigma\left(k_{i}^{-}\right)}\left(k_{i}^{-}\right) \\
& \leq \mu(1-T \alpha)^{\left(k-k_{i}\right)+\left(k_{i}-k_{i-1}\right)} V_{\sigma\left(k_{i-1}\right)}\left(k_{i-1}\right) \\
& =\mu(1-T \alpha)^{\left(k-k_{i-1}\right)} V_{\sigma\left(k_{i-1}\right)}\left(k_{i-1}\right) \\
& =\mu^{N_{\sigma}\left(k_{i-1}, k\right)}(1-T \alpha)^{\left(k-k_{i-1}\right)} V_{\sigma\left(k_{i-1}\right)}\left(k_{i-1}\right)  \tag{25}\\
& \leq \cdots \\
& \leq \mu^{N_{\sigma}\left(k_{0}, k\right)}(1-T \alpha)^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \\
& \leq \mu^{\left(k-k_{0}\right) / \tau_{a}}(1-T \alpha)^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \\
& \leq\left(\mu^{1 / \tau_{a}}(1-T \alpha)\right)^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) .
\end{align*}
$$

Considering the definition of $V_{\sigma(k)}(k), \varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}$, and $\varepsilon_{4}$ in Theorem 12, it yields that

$$
\begin{gather*}
V_{\sigma(k)}(k) \geq \varepsilon_{1}\|x(k)\|, \\
V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \leq\left(\varepsilon_{2}+T \varepsilon_{3} \bar{d}+0.5 T \varepsilon_{3}(\bar{d}+\underline{d}-1)\right. \\
\left.\times(\bar{d}-\underline{d})+T \varepsilon_{4} \bar{d}\right) \sup _{-\bar{d} \leq \theta \leq 0}\left\|x\left(k_{0}+\theta\right)\right\| . \tag{26}
\end{gather*}
$$

Combining (25)-(26), we obtain

$$
\begin{align*}
\|x(k)\| \leq & \left(\frac{\varepsilon_{2}}{\varepsilon_{1}}+\frac{T \varepsilon_{3} \bar{d}}{\varepsilon_{1}}+\frac{0.5 T \varepsilon_{3}(\bar{d}+\underline{d}-1)(\bar{d}-\underline{d})}{\varepsilon_{1}}+\frac{T \varepsilon_{4} \bar{d}}{\varepsilon_{1}}\right) \\
& \times\left(\mu^{1 / \tau_{a}}(1-T \alpha)\right)^{\left(k-k_{0}\right)}\left\|x\left(k_{0}\right)\right\|_{c}, \tag{27}
\end{align*}
$$

where $\left\|x\left(k_{0}\right)\right\|_{c}=$ sup $_{-\bar{d} \leq \theta \leq 0}\left\|x\left(k_{0}+\theta\right)\right\|$.
Therefore, according to Definition 8, system (12) is exponentially stable for any switching signals $\sigma(k)$ with average dwell time (14).

This completes the proof.
Remark 13. When $\mu=1$ in (15), which leads to $v_{p}=v_{q}, v_{p}=$ $v_{q}, \vartheta_{p}=\vartheta_{q}$, for all $p, q \in \underline{m}$, and $\tau_{a}^{*}=0$ by (14), system (12) possesses a common copositive type Lyapunov-Krasovskii functional, and the switching signal can be arbitrary.

When $d_{k}=0$, system (12) can be represented by

$$
\begin{equation*}
\delta x(k)=\widetilde{A}_{\sigma(k)} x(k) \tag{28}
\end{equation*}
$$

where $\widetilde{A}_{p}=A_{p}+A_{d p}$ satisfies $I+T \widetilde{A}_{p} \geq 0$, for all $p \in \underline{m}$. Then we have the following corollary.

Corollary 14. Given a positive constant $0<\alpha<1 / T$, if there exist $\nu_{p} \in R_{+}^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{equation*}
\widetilde{A}_{p}^{T} v_{p}+\alpha v_{p} \leq 0, \tag{29}
\end{equation*}
$$

then system (28) is exponentially stable for any switching signals $\sigma(k)$ with average dwell time (14), where $\mu \geq 1$ satisfies

$$
\begin{equation*}
v_{p} \preceq \mu v_{q}, \quad \forall p, q \in \underline{m} . \tag{30}
\end{equation*}
$$

When the sampling period $T \rightarrow 0$, system (12) becomes a continuous-time system as follows:

$$
\begin{gather*}
\dot{x}(t)=A_{\sigma(t)} x(t)+A_{d \sigma(t)} x(t-d(t)), \\
x\left(t_{0}+\theta\right)=\varphi(\theta), \quad \theta \in\left[-d_{2}, 0\right] \tag{31}
\end{gather*}
$$

where $A_{p}$ are Metzler matrices and $A_{d p} \succeq 0$, for all $p \in \underline{m}$. $d(t)$ denotes the time-varying delay which satisfies $0 \leq d_{1} \leq$ $d(t) \leq d_{2}$ and $\dot{d}(t) \leq d_{d}<1$ for known constants $d_{1}, d_{2}$, and $d_{d}$.

We can obtain sufficient conditions of exponential stability of system (31) by Theorem 12.

Corollary 15. Given a positive constant $\alpha$, if there exist $v_{p}, v_{p}, \vartheta_{p} \in R_{+}^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
A_{p}^{T} v_{p}+\alpha v_{p}+\left(d_{2}-d_{1}+1\right) v_{p}+\vartheta_{p} \preceq 0 \\
A_{d p}^{T} v_{p}-\left(1-d_{d}\right) e^{-\alpha d_{2}} v_{p} \preceq 0 \tag{32}
\end{gather*}
$$

then system (31) is exponentially stable for any switching signals $\sigma(t)$ with average dwell time

$$
\begin{equation*}
\tau_{a}>\tau_{a}^{*}=\frac{\ln \mu}{\alpha} \tag{33}
\end{equation*}
$$

where $\mu \geq 1$ satisfies ( 15 ).
Let $\bar{A}_{\sigma(k)}=A_{\sigma(k)}+I$. When the sampling period $T=1$, system (12) becomes a discrete-time system as follows:

$$
\begin{align*}
& x(k+1)=\bar{A}_{\sigma(k)} x(k)+A_{d \sigma(k)} x\left(k-d_{k}\right), \\
& x\left(k_{0}+\theta\right)=\varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0 \tag{34}
\end{align*}
$$

where $\bar{A}_{p} \geq 0$ and $A_{d p} \geq 0$, for all $p \in \underline{m}$. One can obtain sufficient conditions of exponential stability of system (34) by Theorem 12.

Corollary 16. Given a positive constant $0<\alpha<1$, if there exist $v_{p}, v_{p}, \vartheta_{p} \in R_{+}^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
\bar{A}_{p}^{T} v_{p}+\alpha v_{p}+(1-\alpha)(\bar{d}-\underline{d}+1) v_{p}+(1-\alpha) \vartheta_{p} \leq 0,  \tag{35}\\
A_{d p}^{T} v_{p}-(1-\alpha)^{\bar{d}+1} v_{p} \preceq 0,
\end{gather*}
$$

then system (34) is exponentially stable for any switching signals $\sigma(k)$ with average dwell time

$$
\begin{equation*}
\tau_{a}>\tau_{a}^{*}=-\frac{\ln \mu}{\ln (1-\alpha)} \tag{36}
\end{equation*}
$$

where $\mu \geq 1$ satisfies (15).
3.2. $l_{1}$-Gain Analysis. The following theorem establishes sufficient conditions of exponential stability with $l_{1}$-gain property for system (2).

Theorem 17. For given positive constants $0<\alpha<1 / T$ and $\gamma$, if there exist $v_{p}, v_{p}, \vartheta_{p} \in R_{+}^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
A_{p}^{T} v_{p}+\alpha v_{p}+(1-T \alpha)(\bar{d}-\underline{d}+1) v_{p} \\
+(1-T \alpha) \vartheta_{p}+\widetilde{c}_{p} \preceq 0,  \tag{37}\\
A_{d p}^{T} v_{p}-(1-T \alpha)^{\bar{d}+1} v_{p} \preceq 0, \\
D_{p}^{T} v_{p}+\widetilde{e}_{p}-\gamma 1_{w} \leq 0, \tag{38}
\end{gather*}
$$

where $1_{w}=[\underbrace{[1,1, \ldots, 1}_{w}]^{T}, \widetilde{c}_{p}=\left[\left\|c_{p 1}\right\|,\left\|c_{p 2}\right\|, \ldots,\left\|c_{p n}\right\|\right]^{T}, c_{p j}$ represents the $j$ th column of matrix $C_{p}, j \in \underline{n}=\{1,2, \ldots, n\}$, $\widetilde{e}_{p}=\left[\left\|e_{p 1}\right\|,\left\|e_{p 2}\right\|, \ldots,\left\|e_{p w}\right\|\right]^{T}$, and $e_{p j}$ represents the $j$ th column of matrix $E_{p}, j \in \underline{w}=\{1,2, \ldots, w\}$, then system (2) is exponentially stable with an $l_{1}$-gain performance for any switching signals $\sigma(k)$ with average dwell time (14), where $\mu \geq 1$ satisfies (15).

Proof. By Theorem 12, the exponential stability of system (2) with $w(k)=0$ is ensured if (37) holds. To show the weighted $l_{1}$-gain performance, we choose the Lyapunov functional (18). From (15), we have

$$
\begin{equation*}
V_{\sigma\left(k_{i}\right)}\left(k_{i}\right) \leq \mu V_{\sigma\left(k_{i}^{-}\right)}\left(k_{i}^{-}\right), \quad \forall i=1,2, \ldots \tag{39}
\end{equation*}
$$

For any $k \in\left[k_{i}, k_{i+1}\right)$, noticing (37)-(38), we have

$$
\begin{equation*}
V(k) \leq(1-T \alpha)^{\left(k-k_{i}\right)} V_{\sigma\left(k_{i}\right)}\left(k_{i}\right)-\sum_{s=k_{i}}^{k}(1-T \alpha)^{(k-s)} \Lambda(s), \tag{40}
\end{equation*}
$$

where $\Lambda(s)=\|z(s)\|-\gamma\|w(s)\|$.
Combining (39) and (40) leads to

$$
\begin{aligned}
V(k) \leq & \mu(1-T \alpha)^{\left(k-k_{i}\right)} V_{\sigma\left(k_{i}^{-}\right)}\left(k_{i}^{-}\right)-\sum_{s=k_{i}}^{k}(1-T \alpha)^{(k-s)} \Lambda(s) \\
\leq & \mu(1-T \alpha)^{\left(k-k_{i-1}\right)} V_{\sigma\left(k_{i-1}\right)}\left(k_{i-1}\right) \\
& -\mu \sum_{s=k_{i-1}}^{k_{i}}(1-T \alpha)^{(k-s)} \Lambda(s) \\
& -\sum_{s=k_{i}}^{k}(1-T \alpha)^{(k-s)} \Lambda(s) \\
= & \mu^{N_{\sigma}\left(k_{i-1}, k\right)}(1-T \alpha)^{\left(k-k_{i-1}\right)} V_{\sigma\left(k_{i-1}\right)}\left(k_{i-1}\right) \\
& -\mu^{N_{\sigma}\left(k_{i-1}, k\right)} \sum_{s=k_{i-1}}^{k_{i}}(1-T \alpha)^{(k-s)} \Lambda(s) \\
& \quad-\sum_{s=k_{i}}^{k}(1-T \alpha)^{(k-s)} \Lambda(s) \\
\leq & \cdots
\end{aligned}
$$

$$
\begin{align*}
\leq & \mu^{N_{\sigma}\left(k_{0}, k\right)}(1-T \alpha)^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \\
& -\mu^{N_{\sigma}\left(k_{0}, k\right)} \sum_{s=k_{0}}^{k_{1}}(1-T \alpha)^{(k-s)} \Lambda(s) \\
& -\mu^{N_{\sigma}\left(k_{1}, k\right)} \sum_{s=k_{1}}^{k_{2}}(1-T \alpha)^{(k-s)} \Lambda(s)-\cdots \\
& -\sum_{s=k_{i}}^{k}(1-T \alpha)^{(k-s)} \Lambda(s) \\
= & \mu^{N_{\sigma}\left(k_{0}, k\right)}(1-T \alpha)^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \\
& -\sum_{s=k_{0}}^{k} \mu^{N_{\sigma}(s, k)}(1-T \alpha)^{(k-s)} \Lambda(s) . \tag{41}
\end{align*}
$$

Under the zero initial condition, we obtain from (41) that

$$
\begin{equation*}
0 \leq-\sum_{s=k_{0}}^{k} \mu^{N_{\sigma}(s, k)}(1-T \alpha)^{(k-s)} \Lambda(s) \tag{42}
\end{equation*}
$$

namely,

$$
\begin{align*}
& \sum_{s=k_{0}}^{k} \mu^{N_{\sigma}(s, k)}(1-T \alpha)^{(k-s)}\|z(s)\| \\
& \quad \leq \gamma \sum_{s=k_{0}}^{k} \mu^{N_{\sigma}(s, k)}(1-T \alpha)^{(k-s)}\|w(s)\| . \tag{43}
\end{align*}
$$

Multiplying both sides of (43) by $\mu^{-N_{\sigma}\left(k_{0}, k\right)}$ yields

$$
\begin{align*}
& \sum_{s=k_{0}}^{k} \mu^{-N_{\sigma}\left(k_{0}, s\right)}(1-T \alpha)^{(k-s)}\|z(s)\| \\
& \quad \leq \gamma \sum_{s=k_{0}}^{k} \mu^{-N_{\sigma}\left(k_{0}, s\right)}(1-T \alpha)^{(k-s)}\|w(s)\| . \tag{44}
\end{align*}
$$

Noticing that $N_{\sigma(k)}\left(k_{0}, s\right) \leq\left(s-k_{0}\right) / \tau_{a}$, we have

$$
\begin{equation*}
\mu^{-N_{\sigma}\left(k_{0}, s\right)} \geq(1-T \alpha)^{\left(s-k_{0}\right)} . \tag{45}
\end{equation*}
$$

Combining (44) and (45) leads to

$$
\begin{align*}
& \sum_{s=k_{0}}^{k}(1-T \alpha)^{\left(s-k_{0}\right)}(1-T \alpha)^{(k-s)}\|z(s)\|  \tag{46}\\
& \quad \leq \gamma \sum_{s=k_{0}}^{k}(1-T \alpha)^{(k-s)}\|w(s)\| .
\end{align*}
$$

Summing both sides of (46) from $k=k_{0}$ to $\infty$ leads to

$$
\begin{equation*}
\sum_{k=k_{0}}^{\infty}(1-T \alpha)^{\left(k-k_{0}\right)}\|z(k)\| \leq \gamma \sum_{k=k_{0}}^{\infty}\|w(k)\| . \tag{47}
\end{equation*}
$$

From Definition 10, it can be concluded that system (2) is exponentially stable with a prescribed $l_{1}$-gain performance level $\gamma$.

This completes the proof.
Remark 18. When $\mu=1$ in Theorem 17 , summing both sides of (44) from $k=k_{0}$ to $\infty$ leads to

$$
\begin{equation*}
\sum_{k=k_{0}}^{\infty}\|z(k)\| \leq \gamma \sum_{k=k_{0}}^{\infty}\|w(k)\| \tag{48}
\end{equation*}
$$

which gives the standard $l_{1}$-gain performance.
3.3. Controller Design. In this section, we are interested in designing a state feedback controller $u(k)=K_{\sigma(k)} x(k)$ for positive switched system (11) such that the corresponding closed-loop system

$$
\begin{align*}
& \delta x(k)=\left(A_{\sigma(k)}+B_{\sigma(k)} K_{\sigma(k)}\right) x(k) \\
&+A_{d \sigma(k)} x\left(k-d_{k}\right)+D_{\sigma(k)} w(k), \\
& x\left(k_{0}+\theta\right)= \varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0  \tag{49}\\
& z(k)=C_{\sigma(k)} x(k)+E_{\sigma(k)} w(k)
\end{align*}
$$

is exponentially stable with an $l_{1}$-gain performance.
Theorem 19. Considering system (11), for given positive scalars $0<\alpha<1 / T$ and $\gamma$, if there exist $v_{p}, v_{p}, \vartheta_{p} \in R_{+}^{n}$ and $g_{p} \in R^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
A_{p}^{T} v_{p}+g_{p}+\alpha v_{p}+(1-T \alpha)(\bar{d}-\underline{d}+1) v_{p}  \tag{50}\\
+(1-T \alpha) \vartheta_{p}+\widetilde{c}_{p} \preceq 0 \\
A_{d p}^{T} v_{p}-(1-T \alpha)^{\bar{d}+1} v_{p} \preceq 0  \tag{51}\\
D_{p}^{T} v_{p}+\widetilde{e}_{p}-\gamma 1_{w} \preceq 0  \tag{52}\\
I+T\left(A_{p}+B_{p} K_{p}\right) \succeq 0 \tag{53}
\end{gather*}
$$

where $\tilde{c}_{p}$ and $\tilde{e}_{p}$ have been defined in Theorem 17 and $g_{p}=$ $K_{p}^{T} B_{p}^{T} v_{p}$, then the corresponding closed-loop system (49) is positive and exponentially stable with a prescribed $l_{1}$-gain performance level $\gamma$ for any switching signals $\sigma(k)$ with average dwell time (14), where $\mu \geq 1$ satisfies (15).

Proof. Denote $g_{p}=K_{p}^{T} B_{p}^{T} v_{p}$. Following the proof line of Theorem 17, one can exactly obtain Theorem 19. It is omitted here.

This completes the proof.
Consider the controller design of the following positive switched delta operator system without time delay:

$$
\begin{gather*}
\delta x(k)=\widetilde{A}_{\sigma(k)} x(k)+B_{\sigma(k)} u(k)+D_{\sigma(k)} w(k),  \tag{54}\\
z(k)=C_{\sigma(k)} x(k)+E_{\sigma(k)} w(k),
\end{gather*}
$$

where $\widetilde{A}_{\sigma(k)}=A_{\sigma(k)}+A_{d \sigma(k)}$. Then we directly have the following corollary.

Corollary 20. Considering system (54), for given positive scalars $0<\alpha<1 / T$ and $\gamma$, if there exist $\nu_{p} \in R_{+}^{n}$ and $g_{p} \in R^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
\widetilde{A}_{p}^{T} v_{p}+\alpha v_{p}+g_{p}+\widetilde{c}_{p} \preceq 0, \\
D_{p}^{T} v_{p}+\widetilde{e}_{p}-\gamma 1_{w} \preceq 0,  \tag{55}\\
I+T\left(\widetilde{A}_{p}+B_{p} K_{p}\right) \succeq 0,
\end{gather*}
$$

where $\widetilde{c}_{p}$ and $\widetilde{e}_{p}$ have been defined in Theorem 17 and $g_{p}=$ $K_{p}^{T} B_{p}^{T} v_{p}$, then the corresponding closed-loop system is positive and exponentially stable with a prescribed $l_{1}$-gain performance level $\gamma$ for any switching signals $\sigma(k)$ with average dwell time (14), where $\mu \geq 1$ satisfies (30).

Based on Theorem 19, one is now in a position to present an effective algorithm for constructing the desired controller.

Algorithm 21. Consider the following.
Step 1. Input the matrices $A_{p}, A_{d p}, B_{p}, C_{p}, D_{p}$, and $E_{p}$.
Step 2. Choose the parameters $0<\alpha<1 / T$ and $\gamma>0$. By solving (50)-(52), one can obtain the solutions of $v_{p}, v_{p}, \vartheta_{p}$, and $g_{p}$.

Step 3. By the equation $g_{p}=K_{p}^{T} B_{p}^{T} v_{p}$ with the obtained $g_{p}$ and $\nu_{p}$, one can get the gain matrices $K_{p}$.

Step 4. Check condition (53) in Theorem 19. If it holds, go to Step 5; otherwise, adjust the parameter $\alpha$ and return to Step 2.

Step 5. Construct the feedback controller $u(k)=K_{\sigma(k)} x(k)$, where $K_{p}, p \in \underline{m}$, are the gain matrices.

## 4. Numerical Example

Consider positive switched delta operator system (11) consisting of two subsystems described by the following.

Subsystem 1:

$$
\begin{array}{cc}
A_{1}=\left[\begin{array}{cc}
1.8 & 4.5 \\
1.5 & -2.8
\end{array}\right], & A_{d 1}=\left[\begin{array}{ll}
0.5 & 0.0 \\
0.1 & 0.0
\end{array}\right], \quad B_{1}=\left[\begin{array}{l}
0.4 \\
0.1
\end{array}\right], \\
C_{1}=\left[\begin{array}{ll}
0.1 & 0.2
\end{array}\right], \quad D_{1}=\left[\begin{array}{l}
0.1 \\
0.2
\end{array}\right], \quad E_{1}=[0.1], \tag{56}
\end{array}
$$

Subsystem 2:

$$
\begin{gather*}
A_{2}=\left[\begin{array}{cc}
5.2 & 4.5 \\
3.8 & -1.8
\end{array}\right], \quad A_{d 2}=\left[\begin{array}{ll}
0.5 & 0.0 \\
0.1 & 0.0
\end{array}\right], \quad B_{2}=\left[\begin{array}{l}
0.5 \\
0.2
\end{array}\right], \\
C_{2}=\left[\begin{array}{ll}
0.2 & 0.3
\end{array}\right], \quad D_{2}=\left[\begin{array}{l}
0.2 \\
0.1
\end{array}\right], \quad E_{2}=[0.2], \tag{57}
\end{gather*}
$$



Figure 1: Switching signal.
and $\bar{d}=2, \underline{d}=0, \alpha=1.3, \gamma=2$, and $T=0.25$. Then, by solving (50)-(52) in Theorem 19, we can obtain the following solutions:

$$
\begin{array}{ll}
v_{1}=\left[\begin{array}{l}
9.6472 \\
2.5911
\end{array}\right], & v_{1}=\left[\begin{array}{c}
17.4916 \\
1.9008
\end{array}\right],
\end{array} \vartheta_{1}=\left[\begin{array}{l}
0.7312 \\
1.4427 \tag{58}
\end{array}\right],
$$

and the state feedback gain matrices can be obtained as follows:

$$
\begin{align*}
K_{1} & =\left[\begin{array}{ll}
-14.2263 & -10.7904
\end{array}\right]  \tag{59}\\
K_{2} & =\left[\begin{array}{ll}
-18.2172 & -8.8405
\end{array}\right]
\end{align*}
$$

Obviously, condition (53) is satisfied.
According to (15), we have $\mu=2.0193$. Then from (14), we get $\tau_{a}>\tau_{a}^{*}=1.7880$. Choosing $\tau_{a}=2$, the simulation results are shown in Figures 1 and 2, where the initial conditions are $x(0)=\left[\begin{array}{cc}0.2 & 0.3\end{array}\right]^{T}$ and $x(k)=\left[\begin{array}{ll}0 & 0\end{array}\right]^{T}, k=-2,-1$, and the exogenous disturbance input is $w(k)=0.05 e^{-0.5 k}$ which belongs to $l_{1}[0, \infty)$. The switching signal with average dwell time $\tau_{a}=2$ is shown in Figure 1 and the state responses of the corresponding closed-loop system are given in Figure 2. From the simulation results, it can been seen that the closedloop system is exponentially stable with a prescribed $l_{1}$-gain performance level $\gamma=2$.

## 5. Conclusions

In this paper, the stability and $l_{1}$-gain controller design problems for positive switched systems with time-varying delays via delta operator approach have been investigated. By constructing a copositive type Lyapunov-Krasovskii functional and using the average dwell time approach, we proposed sufficient conditions of exponential stability and $l_{1}-$ gain performance for the considered system. The desired


Figure 2: State responses of the closed-loop system.
state feedback $l_{1}$-gain controller was designed such that the corresponding closed-loop system is exponentially stable and satisfies an $l_{1}$-gain performance. Finally, a numerical example was presented to demonstrate the feasibility of the obtained results. In our future work, we will study the robust stabilization problem of positive switched systems with uncertainties and time-varying delays via delta operator approach.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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