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Research Article

Nonholonomic Geometry of Viscoanelastic Media and Experimental Confirmation

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A thermodynamical model for viscoanelastic media is analyzed using the nonholonomic geometry. A 27-dimensional manifold is introduced, and the differential equations for the geodetics are determined and analytically solved. It is shown that, in this manifold, the best specific entropy is a harmonic function. In the linear case the propagation of transverse acoustic waves is studied, and the theoretical results are compared with some experimental data from a polymeric material (polyisobutylene).

1. Introduction

From macroscopic point of view the most popular mathematical approaches to nonequilibrium thermodynamics are based both on the Caratheodory theory [1–3] which involves Pfaff equations and on the contact structure of thermodynamic state space [4–6]. If these equations are completely integrable, then the thermodynamics is called *holonomic* otherwise *nonholonomic*.

The theory of nonequilibrium holonomic thermodynamics for mechanical phenomena in continuous media was developed in the 90s (see e.g., [7] and references therein). The phenomenological equations were derived [8, 9] by introducing the tensorial internal variables $\varepsilon^{(1)}_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ which occur in the entropy production.

In particular, if one linearizes this theory by neglecting the cross effects among the irreversible phenomena (heat flow, mechanical viscosity, and anelastic deformations), the following rheological equations for distortional phenomena are obtained [10]:

$$\frac{d\widetilde{\tau}_{\alpha\beta}}{dt} + R^{(\tau)}_{(d)0}\widetilde{\tau}_{\alpha\beta} = R^{(\varepsilon)}_{(d)2}\frac{d^2\widetilde{\varepsilon}_{\alpha\beta}}{dt^2} + R^{(\varepsilon)}_{(d)1}\frac{d\widetilde{\varepsilon}_{\alpha\beta}}{dt} + R^{(\varepsilon)}_{(d)0}\widetilde{\varepsilon}_{\alpha\beta}, \tag{1}$$

where $\widetilde{\tau}_{\alpha\beta}$ and $\widetilde{\varepsilon}_{\alpha\beta}$ are the deviators of the stress $(\tau_{\alpha\beta})$ and the strain $(\varepsilon_{\alpha\beta})$ tensors, respectively.

The quantities R are given by

$$\begin{gathered} R^{(\tau)}_{(d)0} = a^{(1,1)}\eta^{(1,1)}_s = \frac{1}{\sigma} \geq 0, \\ R^{(\varepsilon)}_{(d)0} = a^{(0,0)}\left(a^{(1,1)} - a^{(0,0)}\right)\eta^{(1,1)}_s \geq 0, \\ R^{(\varepsilon)}_{(d)1} = a^{(0,0)} + a^{(1,1)}\eta^{(1,1)}_s\eta^{(0,0)}_s \geq 0, \\ R^{(\varepsilon)}_{(d)2} = \eta^{(0,0)}_s \geq 0, \end{gathered} \tag{2}$$

in which σ is the relaxation time and $a^{(1,1)} \geq a^{(0,0)} \geq 0$ are the state coefficients, while $\eta^{(0,0)}_s$ and $\eta^{(1,1)}_s$ are the phenomenological coefficients related to the following physical phenomena:

$$\begin{gathered} a^{(0,0)} \Longrightarrow \text{elasticity}, \\ a^{(1,1)} \Longrightarrow \text{anelasticity}, \\ \eta^{(0,0)}_s \Longrightarrow \text{viscosity}, \\ \eta^{(1,1)}_s \Longrightarrow \text{fluidity}. \end{gathered} \tag{3}$$

In this paper we reconsider the theory from the point of view of the nonholonomic geometry.

In Sections 2 and 3 the analytic properties of the entropy are discussed, and in Section 4 the differential equations of geodetics in the space of state are obtanied.

Finally, in Section 5 we study the transverse waves and we will show the connection between complex numbers and the shear complex modulus. By applying these results to a polymeric material, as polyisobutylene we will also show that the expected results of the theoretical model are in agreement with the experimental data.

2. Gibbs-Pfaff Equation of Viscoanelastic Media

Let us define the space of states

$$R^{27} = R^{2\cdot 13+1} = \left\{s, T, u, \tau_{\alpha\beta}^{(\mathrm{eq})}, \varepsilon_{\alpha\beta}, \tau_{\alpha\beta}^{(1)}, \varepsilon_{\alpha\beta}^{(1)}\right\}, \tag{4}$$

where s is the specific entropy, T is the absolute temperature, u is the specific internal energy, $\tau_{\alpha\beta}^{(\mathrm{eq})}$ is the symmetric equilibrium stress tensor, $\varepsilon_{\alpha\beta}$ is the total symmetric strain tensor, and $\tau_{\alpha\beta}^{(1)}$ is the symmetric affinity stress tensor conjugate to the anelastic tensor $\varepsilon_{\alpha\beta}^{(1)}$ (the symmetric tensorial thermodynamic internal variable).

Let ν be the specific volume related to the mass density ϱ = constant (homogeneous media) by $\nu\varrho = 1$. The Gibbs-Pfaff equation of viscoanelastic media is

$$T ds = du - \nu\tau_{\alpha\beta}^{(\mathrm{eq})} d\varepsilon_{\alpha\beta} + \nu\tau_{\alpha\beta}^{(1)} d\varepsilon_{\alpha\beta}^{(1)}. \tag{5}$$

This equation defines in R^{27} the nonholonomic contact distribution of dimension 26 so that the highest dimension of integral manifold is 13.

The C^{∞} representation of this integral manifold (of maximum dimension) is usually given as follows:

$$\begin{aligned} s &= \Phi\left(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)}\right), \\ T^{-1} &= \frac{\partial\Phi}{\partial u}\left(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)}\right), \\ \tau_{\alpha\beta}^{(\mathrm{eq})} &= -\varrho T \frac{\partial\Phi}{\partial\varepsilon_{\alpha\beta}}\left(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)}\right), \\ \tau_{\alpha\beta}^{(1)} &= \varrho T \frac{\partial\Phi}{\partial\varepsilon_{\alpha\beta}^{(1)}}\left(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)}\right). \end{aligned} \tag{6}$$

This parametrization is based on the arbitrary C^{∞}-function Φ. So that we have a family of integral manifolds of dimension 13 indexed on the arbitrary function Φ.

The state parameters

$$T^{-1}, T^{-1}\varrho^{-1}\tau_{\alpha\beta}^{(\mathrm{eq})}, T^{-1}\varrho^{-1}\tau_{\alpha\beta}^{(1)} \tag{7}$$

are related by the equations of motion and the equations of state [8, 9].

The C^{∞}-representation is a generic element in the 1-jet space converted into

$$\left(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)}, s\left(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)}\right), \frac{\partial s}{\partial u}, -\frac{\partial s}{\partial\varepsilon_{\alpha\beta}}, \frac{\partial s}{\partial\varepsilon_{\alpha\beta}^{(1)}}\right). \tag{8}$$

In order to fix a representative for the specific entropy $s = \Phi(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)})$, we need a supplementary condition as follows.

Theorem 1. *If $s = \Phi(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)})$ is an homogeneous function of order one, then*

$$\Phi = \frac{u}{T} - \frac{1}{\varrho T}\tau_{\alpha\beta}^{(\mathrm{eq})}\varepsilon_{\alpha\beta} + \frac{1}{\varrho T}\tau_{\alpha\beta}^{(1)}\varepsilon_{\alpha\beta}^{(1)}. \tag{9}$$

Proof. The condition of homogeneity of order one

$$\Phi\left(ku, k\varepsilon_{\alpha\beta}, k\varepsilon_{\alpha\beta}^{(1)}\right) = k\Phi\left(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)}\right) \tag{10}$$

gives the PDE

$$u\frac{\partial\Phi}{\partial u} + \varepsilon_{\alpha\beta}\frac{\partial\Phi}{\partial\varepsilon_{\alpha\beta}} + \varepsilon_{\alpha\beta}^{(1)}\frac{\partial\Phi}{\partial\varepsilon_{\alpha\beta}^{(1)}} = \Phi\left(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)}\right). \tag{11}$$

Then the entropy (solution of this PDE) appears as the potential (9). □

Corollary 2. *If the specific entropy $s = \Phi(u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)})$ is an homogeneous function of order one, then*

(1) *the variables $s, u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)}$ are conjugated to the intensive variables $T, \tau_{\alpha\beta}^{(eq)}, \tau_{\alpha\beta}^{(1)}$;*

(2) *the variables $s, u, \varepsilon_{\alpha\beta}, \varepsilon_{\alpha\beta}^{(1)}$ are not essential parameters (because they are not independent).*

Proof. From the expression (9) we find

$$\begin{aligned} T ds + s dT = du &- \frac{1}{\varrho}\left(\varepsilon_{\alpha\beta} d\tau_{\alpha\beta}^{(\mathrm{eq})} + \tau_{\alpha\beta}^{(\mathrm{eq})} d\varepsilon_{\alpha\beta}\right) \\ &+ \frac{1}{\varrho}\left(\varepsilon_{\alpha\beta}^{(1)} d\tau_{\alpha\beta}^{(1)} + \tau_{\alpha\beta}^{(1)} d\varepsilon_{\alpha\beta}^{(1)}\right). \end{aligned} \tag{12}$$

(1) Replacing the relation (5), we get

$$s dT = -\frac{1}{\varrho}\varepsilon_{\alpha\beta} d\tau_{\alpha\beta}^{(\mathrm{eq})} + \frac{1}{\varrho}\varepsilon_{\alpha\beta}^{(1)} d\tau_{\alpha\beta}^{(1)}, \tag{13}$$

and the first statement is true.

(2) The foregoing relation shows that, for example,

$$\frac{\varepsilon_{\alpha\beta}}{s}, \qquad \frac{\varepsilon_{\alpha\beta}^{(1)}}{s} \tag{14}$$

are essential parameters. □

3. Specific Entropy via Least Squares Lagrangian

The most convenient way to fix a representative Φ of the specific entropy s is to look at (5) as a partial derivative evolution equation and to build the least squares method Lagrangian L

$$2L = \left\| T^{-1} - \frac{\partial \Phi}{\partial u} \right\|^2 + \left\| \tau^{(\mathrm{eq})}_{\alpha\beta} + \rho T \frac{\partial \Phi}{\partial \varepsilon_{\alpha\beta}} \right\|^2 + \left\| \tau^{(1)}_{\alpha\beta} - \rho T \frac{\partial \Phi}{\partial \varepsilon^{(1)}_{\alpha\beta}} \right\|^2 \quad (15)$$

and the functional

$$\int_\Omega L\left(u, \varepsilon_{\alpha\beta}, \varepsilon^{(1)}_{\alpha\beta}, \Phi, \Phi_u, \Phi_{\varepsilon_{\alpha\beta}}, \Phi_{\varepsilon^{(1)}_{\alpha\beta}}\right) d\Omega, \quad (16)$$

where $\Phi_u = \partial\Phi/\partial u$, $\Phi_{\varepsilon_{\alpha\beta}} = \partial\Phi/\partial\varepsilon_{\alpha\beta}$, and $\Phi_{\varepsilon^{(1)}_{\alpha\beta}} = \partial\Phi/\partial\varepsilon^{(1)}_{\alpha\beta}$.

The extremals are solutions of the Euler-Lagrange PDE

$$\frac{\partial L}{\partial \Phi} - \left(D_u \frac{\partial L}{\partial \Phi_u} + D_{\varepsilon_{\alpha\beta}} \frac{\partial L}{\partial \Phi_{\varepsilon_{\alpha\beta}}} + D_{\varepsilon^{(1)}_{\alpha\beta}} \frac{\partial L}{\partial \Phi_{\varepsilon^{(1)}_{\alpha\beta}}} \right) = 0, \quad (17)$$

where D_* is the total derivative with respect to the variable $*$.

In our case, we get

$$\frac{\partial L}{\partial \Phi} = 0, \qquad \frac{\partial L}{\partial \Phi_u} = -\left(T^{-1} - \frac{\partial \Phi}{\partial u}\right),$$
$$\frac{\partial L}{\partial \Phi_{\varepsilon_{\alpha\beta}}} = \rho T \left(\tau^{(\mathrm{eq})}_{\alpha\beta} + \rho T \frac{\partial \Phi}{\partial \Phi_{\varepsilon_{\alpha\beta}}} \right), \quad (18)$$
$$\frac{\partial L}{\partial \Phi_{\varepsilon^{(1)}_{\alpha\beta}}} = -\rho T \left(\tau^{(1)}_{\alpha\beta} - \rho T \frac{\partial \Phi}{\partial \Phi_{\varepsilon^{(1)}_{\alpha\beta}}} \right).$$

So that, by replacing the partial derivatives of L in (17), there follows the Laplace equation for the entropy

$$\frac{\partial^2 \Phi}{\partial u^2} + \rho^2 T^2 \frac{\partial^2 \Phi}{\partial {\varepsilon_{\alpha\beta}}^2} + \rho^2 T^2 \frac{\partial^2 \Phi}{\partial {\varepsilon^{(1)}_{\alpha\beta}}^2} = 0. \quad (19)$$

Consequently, we have the following.

Theorem 3. *The best entropy for the nonholonomic nonequilibrium thermodynamics is an harmonic function.*

4. Geodesics

Any curve in the distribution (5) is described by

$$T(t)\, \dot{s}(t) = \dot{u}(t) - \nu \tau^{(\mathrm{eq})}_{\alpha\beta}(t)\, \dot{\varepsilon}_{\alpha\beta}(t) + \nu \tau^{(1)}_{\alpha\beta}(t)\, \dot{\varepsilon}^{(1)}_{\alpha\beta}(t). \quad (20)$$

In order to be a geodesic, this curve must minimize the energy functional

$$J = \frac{1}{2} \int_{t_0}^{t_1} \left(\dot{T}^2(t) + \dot{s}^2(t) + \dot{u}^2(t) + \delta_{\alpha\gamma}\delta_{\beta\delta} \dot{\tau}^{(\mathrm{eq})}_{\alpha\beta} \dot{\tau}^{(\mathrm{eq})}_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} \dot{\tau}^{(1)}_{\alpha\beta} \dot{\tau}^{(1)}_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} \dot{\varepsilon}_{\alpha\beta} \dot{\varepsilon}_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} \dot{\varepsilon}^{(1)}_{\alpha\beta} \dot{\varepsilon}^{(1)}_{\gamma\delta} \right) dt. \quad (21)$$

In short, we must solve the problem

$$\min J \text{ subject to } (20). \quad (22)$$

To solve this problem, we use the method of Lagrange multipliers. For this we defined the constrained Lagrangian

$$\begin{aligned} L_1 = \frac{1}{2} \Big(& \dot{T}^2(t) + \dot{s}^2(t) + \dot{u}^2(t) + \delta_{\alpha\gamma}\delta_{\beta\delta} \dot{\tau}^{(\mathrm{eq})}_{\alpha\beta}(t)\, \dot{\tau}^{(\mathrm{eq})}_{\gamma\delta}(t) \\ & + \delta_{\alpha\gamma}\delta_{\beta\delta} \dot{\tau}^{(1)}_{\alpha\beta}(t)\, \dot{\tau}^{(1)}_{\gamma\delta}(t) \\ & + \delta_{\alpha\gamma}\delta_{\beta\delta} \dot{\varepsilon}_{\alpha\beta}(t)\, \dot{\varepsilon}_{\gamma\delta}(t) \\ & + \delta_{\alpha\gamma}\delta_{\beta\delta} \dot{\varepsilon}^{(1)}_{\alpha\beta}(t)\, \dot{\varepsilon}^{(1)}_{\gamma\delta}(t) \Big) \\ & + p \Big(T(t)\, \dot{s}(t) - \dot{u}(t) + \nu \tau^{(\mathrm{eq})}_{\alpha\beta}(t)\, \dot{\varepsilon}_{\alpha\beta}(t) \\ & - \nu \tau^{(1)}_{\alpha\beta}(t)\, \dot{\varepsilon}^{(1)}_{\alpha\beta}(t) \Big), \end{aligned} \quad (23)$$

where p is the Lagrangian multiplier.

Theorem 4. *The geodesics of the nonholonomic viscoanelastic distribution are solutions of the Euler-Lagrange ODEs for the Lagrangian* (23)

$$\frac{\partial L_1}{\partial x^i} - D_t \frac{\partial L_1}{\partial \dot{x}^i} = 0, \quad (24)$$

where x^i are the generalized coordinates:

$$\begin{gathered} x^1 = T, \qquad x^2 = s, \qquad x^3 = u, \\ x^4_{\alpha\beta} = \tau^{(\mathrm{eq})}_{\alpha\beta}, \qquad x^5_{\alpha\beta} = \varepsilon_{\alpha\beta}, \qquad x^6_{\alpha\beta} = \tau^{(1)}_{\alpha\beta}, \\ x^7_{\alpha\beta} = \varepsilon^{(1)}_{\alpha\beta}. \end{gathered} \quad (25)$$

So that explicitly from (23) *and* (24) *we have*

$$\begin{gathered} p\dot{s} - \ddot{T} = 0, \qquad \frac{d}{dt}(\dot{s} + pT) = 0, \qquad \frac{d}{dt}(\dot{u} - p) = 0, \\ p\nu\dot{\varepsilon}_{\alpha\beta} - \frac{d}{dt}\dot{\tau}^{(\mathrm{eq})}_{\alpha\beta} = 0, \qquad p\nu\dot{\varepsilon}^{(1)}_{\alpha\beta} + \frac{d}{dt}\dot{\tau}^{(1)}_{\alpha\beta} = 0, \\ \frac{d}{dt}\left(\dot{\varepsilon}^{(1)}_{\alpha\beta} - p\nu\tau^{(1)}_{\alpha\beta}\right) = 0, \qquad \frac{d}{dt}\left(\dot{\varepsilon}_{\alpha\beta} + p\nu\tau^{(\mathrm{eq})}_{\alpha\beta}\right) = 0. \end{gathered} \quad (26)$$

The equations (26) with the condition (20) are the differential equations of geodesics.

Let

$$\begin{gathered} T_0 = T(0), \qquad \dot{T}_0 = \dot{T}(0), \qquad s_0 = s(0), \\ \dot{s}_0 = \dot{s}(0), \qquad u_0 = u(0), \qquad \dot{u}_0 = \dot{u}(0), \\ \underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta} = \tau^{(\mathrm{eq})}_{\alpha\beta}(0), \qquad \underset{0}{\dot{\tau}}{}^{(\mathrm{eq})}_{\alpha\beta} = \dot{\tau}^{(\mathrm{eq})}_{\alpha\beta}(0), \\ \underset{0}{\tau}{}^{(1)}_{\alpha\beta} = \tau^{(1)}_{\alpha\beta}(0), \qquad \underset{0}{\dot{\tau}}{}^{(1)}_{\alpha\beta} = \dot{\tau}^{(1)}_{\alpha\beta}(0), \\ \underset{0}{\varepsilon}{}_{\alpha\beta} = \varepsilon_{\alpha\beta}(0), \qquad \underset{0}{\dot{\varepsilon}}{}_{\alpha\beta} = \dot{\varepsilon}_{\alpha\beta}(0), \\ \underset{0}{\varepsilon}{}^{(1)}_{\alpha\beta} = \varepsilon^{(1)}_{\alpha\beta}(0), \qquad \underset{0}{\dot{\varepsilon}}{}^{(1)}_{\alpha\beta} = \dot{\varepsilon}^{(1)}_{\alpha\beta}(0) \end{gathered} \quad (27)$$

be the given (constant) initial values.

By some explicit computation we can easily show that the following.

Theorem 5. *The geodesics of the nonholonomic viscoanelastic distribution, as solution of the Cauchy problem* (26) *and* (27), *are the family of curves:*

$$
\begin{aligned}
u(t) &= \dot{u}_0 t + u_0 \\
s(t) &= \frac{\dot{T}_0}{p}\cos pt + \frac{\dot{s}_0}{p}\sin pt + \left(s_0 - \frac{\dot{T}_0}{p}\right) \\
T(t) &= \frac{\dot{T}_0}{p}\sin pt - \frac{\dot{s}_0}{p}\cos pt + \left(T_0 + \frac{\dot{s}_0}{p}\right) \\
\tau^{(\mathrm{eq})}_{\alpha\beta} &= \left[\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta} - \frac{1}{\nu p}\left(\underset{0}{\dot{\varepsilon}}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta}\right)\right]\cos\nu pt \\
&\quad + \frac{1}{\nu p}\underset{0}{\dot{\tau}}{}^{(\mathrm{eq})}_{\alpha\beta}\sin\nu pt + \frac{1}{\nu p}\left(\underset{0}{\dot{\varepsilon}}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta}\right) \\
\varepsilon_{\alpha\beta} &= -\left[\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta} - \frac{1}{\nu p}\left(\underset{0}{\dot{\varepsilon}}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta}\right)\right]\sin\nu pt \\
&\quad - \frac{1}{\nu p}\underset{0}{\dot{\tau}}{}^{(\mathrm{eq})}_{\alpha\beta}\cos\nu pt + \left(\underset{0}{\varepsilon}_{\alpha\beta} + \frac{1}{\nu p}\,\underset{0}{\dot{\tau}}{}^{(\mathrm{eq})}_{\alpha\beta}\right) \\
\tau^{(1)}_{\alpha\beta} &= \left[\underset{0}{\tau}{}^{(1)}_{\alpha\beta} + \frac{1}{\nu p}\left(\underset{0}{\dot{\varepsilon}}{}^{(1)}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(1)}_{\alpha\beta}\right)\right]\cos\nu pt \\
&\quad + \frac{1}{\nu p}\underset{0}{\dot{\tau}}{}^{(1)}_{\alpha\beta}\sin\nu pt - \frac{1}{\nu p}\left(\underset{0}{\dot{\varepsilon}}{}^{(1)}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(1)}_{\alpha\beta}\right) \\
\varepsilon^{(1)}_{\alpha\beta} &= -\left[\underset{0}{\tau}{}^{(1)}_{\alpha\beta} + \frac{1}{\nu p}\left(\underset{0}{\dot{\varepsilon}}{}^{(1)}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(1)}_{\alpha\beta}\right)\right]\sin\nu pt \\
&\quad - \frac{1}{\nu p}\underset{0}{\dot{\tau}}{}^{(1)}_{\alpha\beta}\cos\nu pt - \left(\underset{0}{\varepsilon}{}^{(1)}_{\alpha\beta} + \frac{1}{\nu p}\,\underset{0}{\dot{\tau}}{}^{(1)}_{\alpha\beta}\right).
\end{aligned}
\tag{28}
$$

Proof. Let us first solve the simplest equation: from $(26)_3$ we have

$$\dot{u} - p = \dot{u}_0 \Longrightarrow u = (\dot{u}_0 + p)\,t + u_0. \tag{29}$$

From $(26)_1$ it is

$$\frac{d}{dt}\left(ps - \dot{T}\right) = 0 \Longrightarrow \dot{T} = p\left(s - s_0\right) + \dot{T}_0 \tag{30}$$

and deriving $(26)_2$

$$\ddot{s} + p\dot{T} = 0. \tag{31}$$

By replacing $\dot{T}$ with the previous expression we get

$$\ddot{s} + p\left[p\left(s - s_0\right) + \dot{T}_0\right] = 0, \tag{32}$$

that is

$$\ddot{s} + p^2 s = \left(p^2 s_0 - p\dot{T}_0\right). \tag{33}$$

This is a linear nonhomogeneous second-order (harmonic) equation whose solution is

$$s(t) = \left(s_0 - \frac{\dot{T}_0}{p}\right) + \frac{\dot{T}_0}{p}\cos pt + \frac{\dot{s}_0}{p}\sin pt. \tag{34}$$

With this function, from the expression

$$\dot{T} = p\left(s - s_0\right) + \dot{T}_0, \tag{35}$$

we can compute also $T(t)$:

$$\dot{T} = p\left[\left(\left(s_0 - \frac{\dot{T}_0}{p}\right) + \frac{\dot{T}_0}{p}\cos pt + \frac{\dot{s}_0}{p}\sin pt\right) - s_0\right] + \dot{T}_0; \tag{36}$$

that is,

$$\dot{T} = \dot{T}_0\cos pt + \dot{s}_0\sin pt. \tag{37}$$

The solution is

$$T(t) = \frac{\dot{T}_0}{p}\sin pt - \frac{\dot{s}_0}{p}\cos pt + \left(T_0 + \frac{\dot{s}_0}{p}\right). \tag{38}$$

Concerning $(26)_{4,5,6,7}$ we can notice that it is enough to solve $(26)_{4,7}$ since their solutions are formally equal to the solutions of $(26)_{5,6}$ since these equations coincide with $(26)_{4,7}$ apart from the substitutions:

$$\varepsilon_{\alpha\beta} \Longrightarrow \varepsilon^{(1)}_{\alpha\beta}, \qquad \tau^{(\mathrm{eq})}_{\alpha\beta} \Longrightarrow -\tau^{(1)}_{\alpha\beta}. \tag{39}$$

Thus from $(26)_7$ it is

$$\dot{\varepsilon}_{\alpha\beta} + p\nu\tau^{(\mathrm{eq})}_{\alpha\beta} = \underset{0}{\dot{\varepsilon}}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta}; \tag{40}$$

that is,

$$\dot{\varepsilon}_{\alpha\beta} = -p\nu\tau^{(\mathrm{eq})}_{\alpha\beta} + \left(\underset{0}{\dot{\varepsilon}}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta}\right). \tag{41}$$

If we put this expression in $(26)_4$ we get

$$p\nu\left[-p\nu\tau^{(\mathrm{eq})}_{\alpha\beta} + \left(\underset{0}{\dot{\varepsilon}}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta}\right)\right] - \frac{d}{dt}\dot{\tau}^{(\mathrm{eq})}_{\alpha\beta} = 0, \tag{42}$$

and by some manipulation we get the (vectorial) linear second order harmonic equation for $\tau^{(\mathrm{eq})}_{\alpha\beta}$:

$$\frac{d}{dt}\dot{\tau}^{(\mathrm{eq})}_{\alpha\beta} + p^2\nu^2\tau^{(\mathrm{eq})}_{\alpha\beta} = p\nu\left(\underset{0}{\dot{\varepsilon}}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta}\right). \tag{43}$$

The solution is

$$
\begin{aligned}
\tau^{(\mathrm{eq})}_{\alpha\beta} &= \left[\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta} - \frac{1}{\nu p}\left(\underset{0}{\dot{\varepsilon}}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta}\right)\right]\cos\nu pt \\
&\quad + \frac{1}{\nu p}\underset{0}{\dot{\tau}}{}^{(\mathrm{eq})}_{\alpha\beta}\sin\nu pt + \frac{1}{\nu p}\left(\underset{0}{\dot{\varepsilon}}_{\alpha\beta} + p\nu\underset{0}{\tau}{}^{(\mathrm{eq})}_{\alpha\beta}\right),
\end{aligned}
\tag{44}
$$

so that by integrating (41) and using the previous equation, we can easily get the expression for $\varepsilon_{\alpha\beta}$

$$\varepsilon_{\alpha\beta} = -\left[\tau^{(\mathrm{eq})}_{0\alpha\beta} - \frac{1}{\nu p}\left(\dot{\varepsilon}_{0\alpha\beta} + p\nu\tau^{(\mathrm{eq})}_{0\alpha\beta}\right)\right]\sin\nu pt - \frac{1}{\nu p}\dot{\tau}^{(\mathrm{eq})}_{0\alpha\beta}\cos\nu pt + \left(\varepsilon_{0\alpha\beta} + \frac{1}{\nu p}\dot{\tau}^{(\mathrm{eq})}_{0\alpha\beta}\right). \tag{45}$$

With similar computations we get also the last two equations of (28). □

The Lagrangian multiplier p is obtained by inserting the functions (28) and derivatives into (20).

It should be noticed that the the projection of the geodesics (28) into different planes gives rise to well known curves. For instance, in the plane $\langle T, s\rangle$ (28) are the parametric equations of a cycloid. Moreover, by assuming that all the initial values are positive, the asymptotic limits give

$$\begin{aligned} &\lim_{t\to\infty} u(t) = +\infty, \\ &\lim_{t\to\infty} |T(t)| \le \left|T_0 - \frac{\dot{s}_0}{p}\right| + \left|\frac{\dot{T}_0}{p}\right| + \left|\frac{\dot{s}_0}{p}\right|, \\ &\lim_{t\to\infty} |s(t)| \le \left|s_0 - \frac{\dot{T}_0}{p}\right| + \left|\frac{\dot{T}_0}{p}\right| + \left|\frac{\dot{s}_0}{p}\right|, \end{aligned} \tag{46}$$

which are in agreement with physical consideration especially for the entropy s which is upper bounded. Analogously we have similar bounded asymptotic limits for the vectorial functions.

5. Experimental Approach to the Linear Response Theory

In a previous paper [11] by the application of the linear response theory [12–15] numerical values of (1) were considered, and the results were compared with experimental data.

In this section we study another aspect of the transversal waves propagation in viscoanelastic media, and we apply the theoretical results to a polymeric material as the polyisobutylene.

We consider, with respect to the Cartesian orthogonal axes (x_1, x_2, x_3), the following displacement $\mathbf{u}$

$$u_3 = Ae^{i(kx_1 - \omega t)}, \qquad u_1 = u_3 = 0 \tag{47}$$

being $i^2 = -1$ and $k = k_1 + ik_2$ the complex wave number, so that

$$v_s = \frac{\omega}{k_1} \tag{48}$$

is the phase velocity and k_2 is connected with the attenuation of the waves.

As $\varepsilon_{\alpha\beta} = 1/2(\partial u_\alpha/\partial x_\beta + \partial u_\beta/\partial x_\alpha)$, from (1) one obtains

$$\begin{aligned} k_1 &= \omega\sqrt{\varrho}\left\{B(\omega)\left(\sqrt{1 + D(\omega)} + 1\right)\right\}^{1/2}, \\ k_1 &= \omega\sqrt{\varrho}\left\{B(\omega)\left(\sqrt{1 + D(\omega)} - 1\right)\right\}^{1/2}, \end{aligned} \tag{49}$$

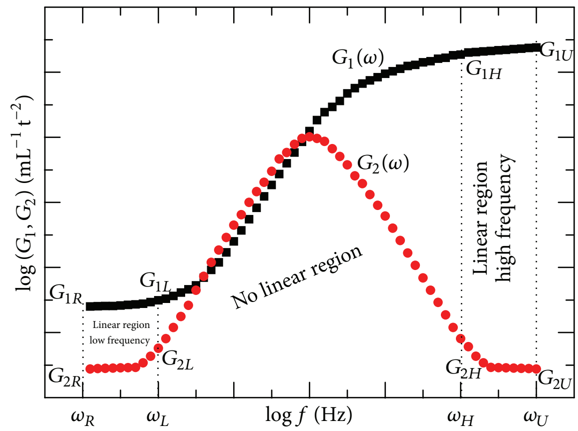

Figure 1: Generic storage and loss moduli.

where

$$\begin{aligned} B(\omega) &= \frac{R^{(\tau)}_{(d)0}R^{(\varepsilon)}_{(d)0} + \omega^2\left(R^{(\varepsilon)}_{(d)1} - R^{(\tau)}_{(d)0}R^{(\varepsilon)}_{(d)2}\right)}{\left(R^{(\varepsilon)}_{(d)0} - \omega^2 R^{(\varepsilon)}_{(d)2}\right)^2 + \left(\omega R^{(\varepsilon)}_{(d)1}\right)^2}, \\ B(\omega) &= \frac{\omega^2\left(R^{(\varepsilon)}_{(d)0} - R^{(\varepsilon)}_{(d)1}R^{(\tau)}_{(d)0} - \omega^2 R^{(\varepsilon)}_{(d)2}\right)^2}{\left\{R^{(\tau)}_{(d)0}R^{(\varepsilon)}_{(d)0} - \omega^2\left(R^{(\varepsilon)}_{(d)1} - R^{(\tau)}_{(d)0}R^{(\varepsilon)}_{(d)2}\right)\right\}^2}. \end{aligned} \tag{50}$$

The complex shear velocity is [14]

$$v_c = \frac{1}{v_s} + \frac{k_2}{i\omega} = \frac{G_1(\omega) + iG_2(\omega)}{\varrho}, \tag{51}$$

where $0 \le s \le 257$, $G_1(\omega)$ (storage modulus), and $G_2(\omega)$ (loss modulus) are, respectively, linked with the nondissipative and dissipative phenomena, and their experimental curves are plotted in Figure 1.

From (47) to (50), the following relations are obtained:

$$\begin{aligned} k_1 &= \frac{\omega\sqrt{(\varrho G_1/2)\left(\sqrt{1 + (G_2/G_1)^2} + 1\right)}}{G_1\sqrt{1 + (G_2/G_1)^2}}, \\ k_2 &= \frac{\omega\sqrt{(\varrho G_1/2)\left(\sqrt{1 + (G_2/G_1)^2} - 1\right)}}{G_1\sqrt{1 + (G_2/G_1)^2}}. \end{aligned} \tag{52}$$

Let us consider the range of high frequency $\omega_H \le \omega \le \omega_U$ and $\omega\tau \gg 1$ (of order 10^2) so that no relaxation phenomena occur (see Figure 1).

By putting

$$\begin{aligned} G_1 &= G_{1H}1,001^{s} \\ G_2 &= G_{2H}1,001^{-s}, \end{aligned} \tag{53}$$

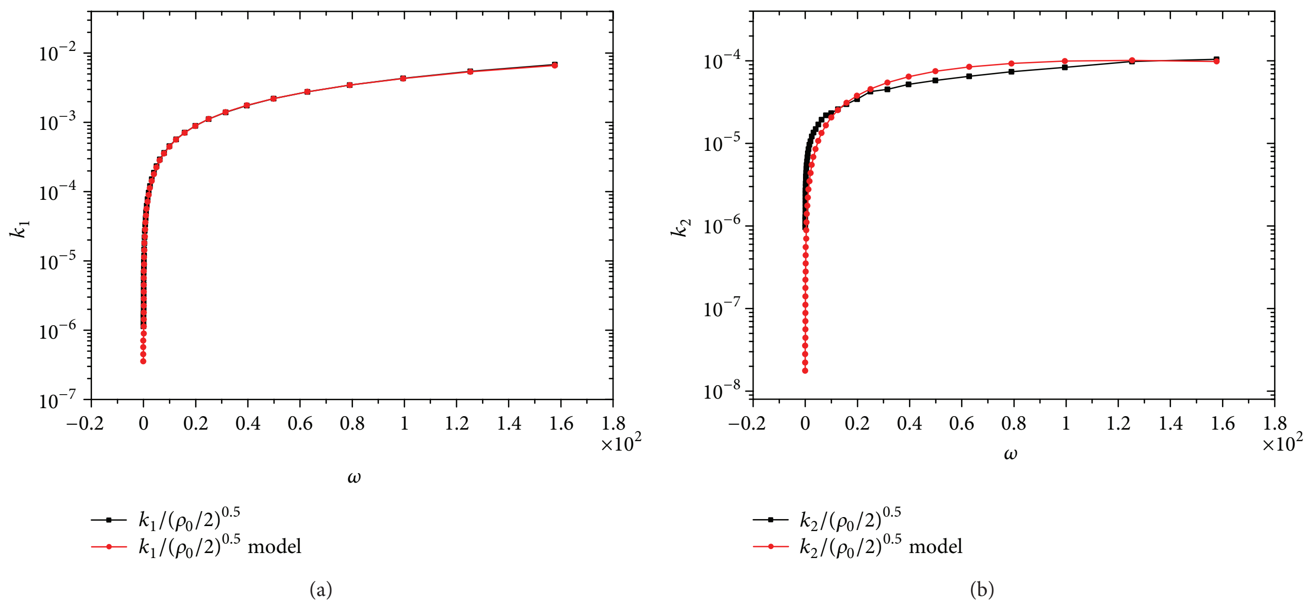

FIGURE 2: k_1 and k_2 for Polyisobutilene: (M.w. 10^6 g/mol; $T_0 = 273$ K) the experimental curves (in black) and the theoretical curves (in red) obtained by our model.

where $G_{1H} = G_1(\omega_H)$ and $G_{2H} = G_2(\omega_H)$, (52) becomes

$$k_1 = \omega\sqrt{\frac{\varrho}{2}}\sqrt{\frac{\sqrt{(G_{1H})^2 1.001^{2s} + (G_{2H})^2} + G_{1H}1.001^s}{(G_{1H})^2 1.001^{2s} + (G_{2H})^2}},$$

$$k_2 = \omega\sqrt{\frac{\varrho}{2}}\sqrt{\frac{\sqrt{(G_{1H})^2 1.001^{2s} + (G_{2H})^2} - G_{1H}1.001^s}{(G_{1H})^2 1.001^{2s} + (G_{2H})^2}}. \tag{54}$$

For the Polyisobutilene we have [15] the characteristic values which are

$$\sigma = 10^{-7}\text{ sec.},\qquad \omega_H = 3.2\cdot 10^{14}\text{ Hz},$$
$$\omega_U = 6\cdot 10^{14}\text{ Hz}, \tag{55}$$
$$G_{1U} = 2.4\cdot 10^{9}\text{ Pa},\qquad G_{2U} = 2.75\cdot 10^{4}\text{ Pa},$$

and the graphics confirm (see Figure 2) with experimental data the validity of the model proposed for viscoanelastic phenomena in continuous media.

6. Conclusions

From the viewpoint of nonholonomic irreversible thermodynamics, it is shown that the best specific entropy is an harmonic function in a 27-dimensional manifold. The differential equations of geodetics are obtained and the corresponding curves are explicitly computed. In the linearized theory it is shown that the theoretical results are in agreement with the experimental data in the case of polymeric material (Polyisobutilene) (Figure 2).

Conflict of Interests

The authors declare that there is no conflicts of interests regarding the publication of this paper.

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