# Tangent Orbital Rendezvous Using Linear Relative Motion with $J_{2}$ Perturbations 

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#### Abstract

The tangent-impulse coplanar orbit rendezvous problem is studied based on the linear relative motion for $J_{2}$-perturbed elliptic orbits. There are three cases: (1) only the first impulse is tangent; (2) only the second impulse is tangent; (3) both impulses are tangent. For a given initial impulse point, the first two problems can be transformed into finding all roots of a single variable function about the transfer time, which can be done by the secant method. The bitangent rendezvous problem requires the same solution for the first two problems. By considering the initial coasting time, the bitangent rendezvous solution is obtained with a difference function. A numerical example for two coplanar elliptic orbits with $J_{2}$ perturbations is given to verify the efficiency of these proposed techniques.


## 1. Introduction

The orbital rendezvous problem is a fundamental one in aerospace engineering for human space activities. If the impulse maneuver is adopted, a single impulse is enough to finish the orbital interception mission. But for the orbital rendezvous mission, it needs at least two impulses, in which the second impulse is to correct the terminal velocity. If the transfer time is assigned, the required initial velocity for the chaser can be obtained by solving Lambert's problem [14]. However, there is no constraint on the direction of the required initial velocity vector. If the required initial velocity vector is aligned with the current velocity of the chaser, the impulse is a tangent one, and then only the speed change will null the relative velocity.

Actually, the tangent orbit problem has existed for many years. The well-known Hohmann transfer is a classical twoimpulse bitangent transfer between two coplanar circular orbits [5]. Because of its simple tangent-impulse direction and the minimum-energy cost among all the two-impulse transfers, the Hohmann transfer has been widely used in engineering applications. However, the Hohmann transfer is
only available for coplanar circular orbits and coplanar coaxial elliptic orbits. Recently, for coplanar noncoaxial elliptic orbits, Adamyan et al. [6] used its geometric characteristics to obtain the analytical expression for the bitangent transfer orbital parameters. Based on the orbital hodograph theory, Thompson et al. [7] solved the coplanar bitangent orbit by a numerical iterative approach, that is, the regula falsi method, which is like the secant method. However, instead of retaining the last two points, the regula falsi method makes sure to keep one point on either side of the root. Furthermore, Zhang et al. [8] provided a simple and closed-form solution for the tangent transfer orbit problem associated with its solutionexistence conditions. In addition, Zhang and Zhou [9] studied a noncoplanar tangent orbit technique based on a new definition of orbit "tangency" condition. However, for the orbit rendezvous problem, the flight time of the chaser and that of the target are required to be equal. For this purpose, Zhang et al. [10, 11] solved the tangent orbit rendezvous problem with the same terminal velocity direction and the two-impulse bitangent rendezvous problem, respectively.

All the previous methods with tangent orbits are only for the two-body orbit problem, but not taking the $J_{2}$
perturbation into account. If the relative range between the chaser and the target is short, the linear relative motion equations can be used to solve the required velocity vector and the tangent orbit problem. The linear Hill-ClohessyWiltshire (HCW) equations [12, 13] and the TschaunerHempel (TH) equations [14] are used for the circular and elliptic target orbits, respectively. When the transfer time is fixed, the required initial relative velocity can be derived from the state transition matrix (STM). Because the coefficient matrices are constant for the HCW equations, the STM can be easily obtained. For the TH equations, by using the true anomaly of the target instead of the transfer time as the independent variable, simpler equations can be derived from the TH equations then the STM can also be obtained [15]. For the STM with the $J_{2}$ perturbation, Gim and Alfriend [16] used the geometric method to obtain an STM for the elliptic target orbit with the influence of the $J_{2}$ perturbation. In their method, the closed-form STMs for both mean elements and osculating elements were obtained. Moreover, Yamada et al. [17] derived an STM in eccentric orbits with $J_{2}$ by using the osculating orbital elements of the target in the initial state as the nominal orbital elements.

This paper studies the tangent-impulse orbital rendezvous problem between two coplanar elliptic orbits with the $J_{2}$ perturbation. Based on the STM in the elliptic target orbit with the influence of the $J_{2}$ perturbation [16], the required initial relative velocity vector can be obtained. Then two functions are defined for the tangent impulse velocity. By using the secant method, the tangent to initial/final orbit can be obtained. The bitangent orbit indicates that it is the same solution for the tangent to initial orbit problem and the tangent to final orbit problem.

## 2. Relative Motion with the $J_{2}$ Perturbation

2.1. Linear Equations of Relative Motion. Define an orthogonal coordinate system, where the origin of the coordinate system is the center of the target, the $X_{\mathscr{H}}$ axis is taken to be the radius vector from the center of the Earth to the center of the target, the $Z_{\mathscr{H}}$ axis is aligned with the angular momentum vector of the target orbit, and the $Y_{\mathscr{H}}$ axis completes the righthand system. This orthogonal coordinate system is called the local-vertical-local-horizontal (LVLH) frame of the target. Similarly, the LVLH frame of the chaser can also be defined. It should be notified that the LVLH coordinate system is a rotation frame. Since there is no maneuver on the target in orbit rendezvous, the linear relative motion equations with only the $J_{2}$ perturbation (other external accelerations are neglected) in the LVLH frame of target can be written as [18]
where $\mathbf{X}(t)=[x, y, z, \dot{x}, \dot{y}, \dot{z}]^{\mathrm{T}}$ denotes the state vector and the elements in the coefficient matrix are

$$
\begin{align*}
& a_{41}=\omega_{z}^{2}+2 \frac{\mu}{R_{t}^{3}}+G_{0}\left(1-3 \sin ^{2} \bar{i}_{t} \sin ^{2} \bar{u}_{t}\right),  \tag{2a}\\
& a_{42}=\dot{\omega}_{z}+G_{0} \sin ^{2} \bar{i}_{t} \sin \left(2 \bar{u}_{t}\right),  \tag{2b}\\
& a_{43}=-\omega_{x} \omega_{z}+G_{0} \sin \left(2 \bar{i}_{t}\right) \sin \bar{u}_{t},  \tag{2c}\\
& a_{51}=-\dot{\omega}_{z}+G_{0} \sin ^{2} \bar{i}_{t} \sin \left(2 \bar{u}_{t}\right),  \tag{2d}\\
& a_{52}=\omega_{x}^{2}+\omega_{z}^{2}-\frac{\mu}{R_{t}^{3}}+\frac{1}{4} G_{0}\left[-1+\sin ^{2} \bar{i}_{t}\left(7 \sin ^{2} \bar{u}_{t}-2\right)\right],  \tag{2e}\\
& a_{53}=\dot{\omega}_{x}-\frac{1}{4} G_{0} \sin \left(2 \bar{i}_{t}\right) \cos \bar{u}_{t},  \tag{2f}\\
& a_{61}=-\omega_{x} \omega_{z}+G_{0} \sin \left(2 \bar{i}_{t}\right) \sin \bar{u}_{t},  \tag{2~g}\\
& a_{62}=-\dot{\omega}_{x}-\frac{1}{4} G_{0} \sin \left(2 \bar{i}_{t}\right) \cos \bar{u}_{t},  \tag{2h}\\
& a_{63}=\omega_{x}^{2}-\frac{\mu}{R_{t}^{3}}+\frac{1}{4} G_{0}\left[-3+\sin 2 \bar{i}_{t}\left(5 \sin ^{2} \bar{u}_{t}+2\right)\right], \tag{2i}
\end{align*}
$$

where $G_{0}=6 n_{t}^{2} J_{2}\left(\bar{a}_{t} / R_{t}\right)^{3}, J_{2}=1.0826269 \times 10^{-3}, n_{t}$ is the mean motion, $R$ is the radius, $a$ is the semimajor axis, $i$ is the orbit inclination, and $u=\omega+\theta$ is the argument of latitude, where $\omega$ is the argument of perigee and $\theta$ is the true anomaly. The symbol $\overline{(\cdot)}$ denotes the mean orbit elements. The subscript $t$ denotes the target orbit.

The angular velocity vector of the LVLH frame of the target is

$$
\begin{equation*}
\boldsymbol{\omega}=\left[\omega_{x}, \omega_{y}, \omega_{z}\right]^{\mathrm{T}} \tag{3}
\end{equation*}
$$

whose components are

$$
\begin{align*}
& \omega_{x}=\dot{\Omega}_{t} \sin i_{t} \sin u_{t}+\dot{i}_{t} \cos u_{t}  \tag{4a}\\
& \omega_{y}=\dot{\Omega}_{t} \sin i_{t} \cos u_{t}-\dot{i}_{t} \sin u_{t}=0  \tag{4b}\\
& \omega_{z}=\dot{\Omega}_{t} \cos i_{t}+\dot{\omega}_{t}+\dot{\theta}_{t} \tag{4c}
\end{align*}
$$

where $\Omega$ denotes the right ascension of ascending node. Considering the $J_{2}$ perturbation, the derivatives of the orbital elements with respect to time are [16]

$$
\begin{gathered}
\dot{i}=-\frac{3 J_{2} R_{E}^{2}}{4} \frac{\sqrt{\mu}(1+e \cos \theta)^{3}}{\left[a\left(1-e^{2}\right)\right]^{7 / 2}} \sin (2 i) \sin (2 u), \\
\dot{\Omega}=-\frac{3 J_{2} R_{E}^{2} \sqrt{\mu}(1+e \cos \theta)^{3}}{\left[a\left(1-e^{2}\right)\right]^{7 / 2}} \cos i \sin ^{2} u,
\end{gathered}
$$

$$
\begin{align*}
\dot{\omega}= & -\frac{3 J_{2} R_{E}^{2}}{2} \frac{\sqrt{\mu}(1+e \cos \theta)^{4}}{e\left[a\left(1-e^{2}\right)\right]^{7 / 2}} \\
\times & {\left[\left(3 \sin ^{2} i \sin u-1\right) \cos \theta\right.} \\
& \left.+\left(\frac{2+e \cos \theta}{1+e \cos \theta}\right) \sin \theta \sin ^{2} i \sin (2 u)\right] \\
& -\dot{\Omega} \cos i, \\
\dot{\theta}= & \frac{3 \sqrt{\mu} J_{2} R_{E}^{2}(1+e \cos \theta)^{3}}{2 e\left[a\left(1-e^{2}\right)\right]^{7 / 2}} \\
\times & {\left[\left(3 \sin ^{2} i \sin ^{2} u-1\right)(1+e \cos \theta) \cos \theta\right.} \\
& \left.+(2+e \cos \theta) \sin \theta \sin ^{2} i \sin (2 u)\right] \\
& +\frac{\sqrt{\mu}(1+e \cos \theta)^{2}}{\left[a\left(1-e^{2}\right)\right]^{3 / 2}}, \tag{5}
\end{align*}
$$

where $R_{E}$ is the radius of the Earth, $e$ is the eccentricity, and $\mu$ is the gravity parameter. With (4a)-(5), the secondorder derivatives of $i_{t}, \Omega_{t}, \omega_{t}, \theta_{t}$ can be obtained; and then it is not difficult to obtain the derivatives of $\dot{\omega}_{x}$ and $\dot{\omega}_{z}$ in (2a)-(2i).
2.2. State Transition Matrix (STM). To obtain the STM for the linear relative motion equations with the $J_{2}$ perturbation, the nonsingular orbital elements are defined as $\boldsymbol{\kappa}=$ $\left[a, u, i, q_{1}, q_{2}, \Omega\right]^{\mathrm{T}}$, where $q_{1}=e \cos \omega$ and $q_{2}=e \sin \omega$. When the nonsingular orbital elements for the chaser and the target are both obtained, the differential orbital elements between the chaser and the target are denoted by $\delta \boldsymbol{\kappa}=\boldsymbol{\kappa}_{c}-\boldsymbol{\kappa}_{t}$. The subscripts $c$ and $t$ denote the chaser orbit and the target orbit, respectively.

The geometric transformation between the state vector $\mathbf{X}$ and the differential nonsingular osculating orbital elements $\delta \boldsymbol{\kappa}_{\text {osc }}$ is [16]

$$
\begin{equation*}
\mathbf{X}(t)=\left\{A(t)+3 J_{2} R_{E}^{2} B(t)\right\} \delta \boldsymbol{\kappa}_{\mathrm{osc}}(t) \tag{6}
\end{equation*}
$$

where $A(t)$ is the matrix for the two-body motion and $B(t)$ is the matrix perturbed by $J_{2}$.

Let the matrix $D(t)$ denote the transformation matrix from the relative mean elements to the relative osculating elements, and let the state transition $\bar{\phi}_{\kappa}$ be for the relative mean elements; then we have [16]

$$
\begin{gather*}
\delta \boldsymbol{\kappa}_{\text {osc }}(t)=D(t) \delta \boldsymbol{\kappa}_{\text {mean }}(t)=D(t) \bar{\phi}_{\kappa}\left(t, t_{0}\right) \delta \boldsymbol{\kappa}_{\text {mean }}\left(t_{0}\right) \\
=D(t) \bar{\phi}_{\kappa}\left(t, t_{0}\right) D^{-1}\left(t_{0}\right) \delta \boldsymbol{\kappa}_{\text {osc }}\left(t_{0}\right), \\
D(t)=\frac{\partial \boldsymbol{\kappa}_{\text {osc }}(t)}{\partial \boldsymbol{\kappa}_{\text {mean }}(t)} . \tag{7}
\end{gather*}
$$

All the elements of the matrices $A(t), B(t), \bar{\phi}_{\kappa}\left(t, t_{0}\right), D(t)$ can be obtained in [16]. Finally, the STM for the relative states with $J_{2}$ effects is

$$
\begin{aligned}
\Phi_{J_{2}}\left(t, t_{0}\right)= & \left\{A(t)+3 J_{2} R_{E}^{2} B(t)\right\} D(t) \bar{\phi}_{\kappa}\left(t, t_{0}\right) D^{-1}\left(t_{0}\right) \\
& \times\left\{A\left(t_{0}\right)+3 J_{2} R_{E}^{2} B\left(t_{0}\right)\right\}^{-1}
\end{aligned}
$$

and the state vector is

$$
\begin{equation*}
\mathbf{X}(t)=\Phi_{J_{2}}\left(t, t_{0}\right) \mathbf{X}\left(t_{0}\right) \tag{9}
\end{equation*}
$$

2.3. Relative State in the LVLH Frame of the Target. Note that the LVLH frame of the target is a rotation frame. The time derivative of an arbitrary vector $\rho$ in the earth-centeredinertial (ECI) frame can be written in terms of the derivative with respect to the LVLH frame of the target as

$$
\begin{equation*}
\frac{\mathrm{d} \boldsymbol{\rho}}{\mathrm{~d} t}=\left.\frac{\mathrm{d} \boldsymbol{\rho}}{\mathrm{~d} t}\right|_{\mathrm{rot}}+\boldsymbol{\omega} \times \boldsymbol{\rho} \tag{10}
\end{equation*}
$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the rotating frame which is obtained in (3). Assume that the position vector and the velocity vector of the chaser in the ECI frame are $\mathbf{R}_{c}$ and $\mathbf{V}_{c}$, respectively, and those of the target are $\mathbf{R}_{t}$ and $\mathbf{V}_{t}$, respectively. The transformation matrix $C^{\mathscr{H} \mathscr{I}}$ from the ECI to the LVLH frames is

$$
C^{\mathscr{H} \mathcal{F}}=\left[\begin{array}{ccc}
\cos u \cos \Omega-\sin u \cos i \sin \Omega & \cos u \sin \Omega+\sin u \cos i \cos \Omega & \sin u \sin i  \tag{11}\\
-\sin u \cos \Omega-\cos u \cos i \sin \Omega & -\sin u \sin \Omega+\cos u \cos i \cos \Omega & \cos u \sin i \\
\sin i \sin \Omega & -\sin i \cos \Omega & \cos i
\end{array}\right] .
$$

Using (11), the transformation matrices $C_{t}^{\mathscr{H} \mathscr{I}}$ for the LVLH frame of the target and $C_{c}^{\mathscr{C}}$ for the LVLH frame of the chaser are both obtained.

Then the relative position vector in the LVLH frame of the target is

$$
\begin{equation*}
\mathbf{r}=C_{t}^{\mathscr{H} \mathscr{I}} \cdot\left(\mathbf{R}_{c}-\mathbf{R}_{t}\right) . \tag{12}
\end{equation*}
$$

Leting $\boldsymbol{\rho}=\mathbf{r}$ in (10), the relative velocity vector in the LVLH frame of the target is

$$
\begin{equation*}
\mathbf{v}=\mathbf{C}_{t}^{\mathscr{H} \mathscr{I}} \cdot\left(\mathbf{R}_{c}-\mathbf{R}_{t}\right)-\boldsymbol{\omega} \times \mathbf{r} . \tag{13}
\end{equation*}
$$

## 3. Problem Analysis of Tangent Rendezvous

3.1. Tangent to the Initial Orbit. For the orbital interception and rendezvous problem, the required initial relative velocity can be obtained from the STM if the transfer time is fixed. The final state vector $\mathbf{X}\left(t_{f}\right)=\left[\mathbf{r}_{f}^{\mathrm{T}}, \mathbf{v}_{f}^{\mathrm{T}}\right]^{\mathrm{T}}$ can be obtained from the initial state vector as

$$
\left[\begin{array}{c}
\mathbf{r}_{f}  \tag{14}\\
\mathbf{v}_{f}
\end{array}\right]=\left[\begin{array}{cc}
\Phi_{11}\left(t_{f}, t_{0}\right) & \Phi_{12}\left(t_{f}, t_{0}\right) \\
\Phi_{21}\left(t_{f}, t_{0}\right) & \Phi_{22}\left(t_{f}, t_{0}\right)
\end{array}\right]\left[\begin{array}{c}
\mathbf{r}_{0} \\
\mathbf{v}_{0}
\end{array}\right]
$$

If the final time $t_{f}$ is fixed and the final relative position $\mathbf{r}_{f}$ is assigned, for example, $\mathbf{r}_{f}=[0,0,0]^{\mathrm{T}}$ for the interception and rendezvous problem, the required initial relative velocity in the LVLH frame of the target is

$$
\begin{equation*}
\mathbf{v}_{h 0}=-\Phi_{12}^{-1} \Phi_{11} \mathbf{r}_{0}+\Phi_{12}^{-1} \mathbf{r}_{f} . \tag{15}
\end{equation*}
$$

Then, the first velocity impulse in the LVLH frame of the target is

$$
\begin{equation*}
\Delta \mathbf{v}_{t 0}=\mathbf{v}_{h 0}-\mathbf{v}_{0} \tag{16}
\end{equation*}
$$

where $\mathbf{v}_{0}$ is the initial relative velocity. By transforming the first impulse velocity vector in the LVLH frame of the target to that in the LVLH frame of the chaser, we can obtain

$$
\begin{equation*}
\Delta \mathbf{v}_{c 0}=C_{c}^{\mathscr{H} \mathscr{I}}\left(C_{t}^{\mathscr{H} \mathscr{F}}\right)^{\mathrm{T}} \cdot \Delta \mathbf{v}_{t 0} . \tag{17}
\end{equation*}
$$

The normalized vector of $\Delta \mathbf{v}_{c 0}$ is

$$
\Delta \widehat{\mathbf{v}}_{c 0}=\frac{\Delta \mathbf{v}_{c 0}}{\left\|\Delta \mathbf{v}_{c 0}\right\|}=\left[\begin{array}{c}
\Delta \widehat{v}_{c 0 x}  \tag{18}\\
\Delta \widehat{v}_{c 0 y} \\
\Delta \widehat{v}_{c 0 z}
\end{array}\right]
$$

For the coplanar case, $\Delta \widehat{v}_{c 0 z}=0$ is satisfied.
The flight-direction angle $\gamma$ is defined as the angle from the position vector to the velocity vector in the ECI frame (see Figure 1). From the definition it is known that the range of the flight-direction angle is $\gamma \in(0, \pi)$. At the initial time, the flight-direction angle of the chaser $\gamma_{c 0}$ can be written as a function only of the initial true anomaly $\theta_{c 0}$

$$
\begin{equation*}
\gamma_{c 0}=\frac{\pi}{2}-\arctan \left(\frac{e_{c} \sin \theta_{c 0}}{1+e_{c} \cos \theta_{c 0}}\right) \tag{19}
\end{equation*}
$$

If the first velocity impulse is required to be tangent to the initial orbit of the chaser, the first impulse velocity vector in the ECI frame must be aligned with the initial velocity vector of the chaser in the ECI frame. Thus, the following expression should be satisfied:

$$
\begin{equation*}
F_{1} \triangleq-\Delta \widehat{v}_{c 0 x}+\Delta \widehat{v}_{c 0 y} \tan \gamma_{c 0}=0 \tag{20}
\end{equation*}
$$

Note that $\gamma_{c 0}$ is a function only of $t_{0}$, and $\Delta \widehat{\mathbf{v}}_{c 0}$ can be obtained by (18) if the final time $t_{f}$ is fixed. Therefore, $F_{1}$ is a function only of the final time $t_{f}$ or the transfer time $\Delta t_{\text {tran }} \triangleq t_{f}-t_{0}$ if the first impulse is at the initial time. Thus, the value of $\Delta t_{\text {tran }}$ for $F_{1}\left(\Delta t_{\text {tran }}\right)=0$ is required to be solved.


Figure 1: The flight-direction angle and the LVLH frame.
3.2. Tangent to the Final Orbit. After the first impulse, the relative velocity vector at the initial time reaches $\mathbf{v}_{h 0}$; then the relative velocity vector at the final time $t_{f}$ can be obtained from (14) as

$$
\begin{equation*}
\mathbf{v}_{f}=\Phi_{21} \mathbf{r}_{0}+\Phi_{22} \mathbf{v}_{h 0}=\left(\Phi_{21}-\Phi_{22} \Phi_{12}^{-1} \Phi_{11}\right) \mathbf{r}_{0}+\Phi_{22} \Phi_{12}^{-1} \mathbf{r}_{f} \tag{21}
\end{equation*}
$$

For the rendezvous problem, the required final relative velocity is zero. Thus, the second impulse in the LVLH frame of the target is

$$
\begin{equation*}
\Delta \mathbf{v}_{t f}=0-\mathbf{v}_{t f}=-\mathbf{v}_{t f} . \tag{22}
\end{equation*}
$$

The normalized vector of $\Delta \mathbf{v}_{t f}$ is

$$
\Delta \widehat{\mathbf{v}}_{t f}=\frac{\Delta \mathbf{v}_{t f}}{\left\|\Delta \mathbf{v}_{t f}\right\|} \equiv\left[\begin{array}{c}
\Delta \widehat{v}_{t f x}  \tag{23}\\
\Delta \widehat{v}_{t f y} \\
\Delta \widehat{v}_{t f z}
\end{array}\right]
$$

For the coplanar case, $\Delta \widehat{v}_{t f z}=0$ is also satisfied.
If the second impulse is required to be a tangent one, the second impulse vector in the ECI frame must be aligned with the final velocity vector of the chaser in the ECI frame; in other words, the direction of the final velocity vector of the chaser and that of the target are the same in the ECI frame. Thus, the second impulse in the ECI frame is aligned with the final velocity vector of the target in the ECI frame, which is equivalent to the condition that the second impulse in the LVLH frame of the target is aligned with the final relative velocity vector in the LVLH frame of the target, since $\mathbf{r}_{f}=$ $[0,0,0]^{\mathrm{T}}$ is satisfied for the rendezvous problem. At the final time, the flight-direction angle of the target is

$$
\begin{equation*}
\gamma_{t f}=\frac{\pi}{2}-\arctan \left(\frac{e_{t} \sin \theta_{t f}}{1+e_{t} \cos \theta_{t f}}\right) \tag{24}
\end{equation*}
$$

Therefore, the second impulse is tangent if and only if

$$
\begin{equation*}
F_{2} \triangleq-\Delta \widehat{v}_{t f x}+\Delta \widehat{v}_{t f y} \tan \gamma_{t f}=0 \tag{25}
\end{equation*}
$$

It is known that $F_{2}$ is also a function only of the transfer time $t_{f}$ or $\Delta t_{\text {tran }}$ if the first impulse is at the initial time. Thus, the value of $\Delta t_{\text {tran }}$ for $F_{2}\left(\Delta t_{\text {tran }}\right)=0$ will be solved for the tangent to final orbit problem.
3.3. Bitangent Orbit. The two-impulse method is the simplest and the most fundamental one to fulfill the rendezvous mission. For the tangent to initial orbit problem, only the first impulse is tangent, whereas for the tangent to final orbit problem, only the second impulse is tangent. Moreover, the bitangent rendezvous requires the transfer orbit to be tangent to the initial orbit and also tangent to the final orbit; that is, both impulses are tangent. Since both impulses are imposed on the chaser, then each impulse is required to be aligned with the velocity vector of chaser in the ECI frame. Thus, the same value of $\Delta t_{\text {tran }}$ is required to simultaneously satisfy two conditions, that is, $F_{1}=0$ and $F_{2}=0$. However, if the first impulse is at the initial time, there may not exist this value of $\Delta t_{\text {tran }}$.

When considering the initial coasting time $\Delta t_{1} \triangleq t_{1}-t_{0}$, the first impulse is not at the initial time $t_{0}$ but at the time $t_{1}$. Then $F_{1}$ and $F_{2}$ are functions of two variables, that is, $t_{1}$ and $t_{f}$ or $\Delta t_{1}$ and $\Delta t_{\text {tran }}$. Therefore, two variables $\Delta t_{1}$ and $\Delta t_{\text {tran }}$ are to be solved for the following equations:

$$
\begin{align*}
& F_{1}\left(\Delta t_{1}, \Delta t_{\text {tran }}\right)=0, \\
& F_{2}\left(\Delta t_{1}, \Delta t_{\text {tran }}\right)=0 . \tag{26}
\end{align*}
$$

The total energy cost for the two-impulse rendezvous is

$$
\begin{equation*}
\Delta v_{\text {total }}=\left\|\Delta \mathbf{v}_{c 0}\right\|+\left\|\Delta \mathbf{v}_{t f}\right\|=\left\|\Delta \mathbf{v}_{t 0}\right\|+\left\|\Delta \mathbf{v}_{t f}\right\| . \tag{27}
\end{equation*}
$$

## 4. Solution Procedure

4.1. Tangent to the Initial/Final Orbit Problem. Assume that the first impulse is at the initial time $t_{0}$. Then for the tangent to initial orbit problem, the value of $\Delta t_{\text {tran }}$ is to be solved for $F_{1}=0$. Since the linear relative motion equations cause large errors for the long-time case, the transfer time is assumed to be not greater than one mean orbit period of the target, that is, $\Delta t_{\operatorname{tran}} \in\left(0, \bar{T}_{t}\right]$, where $\bar{T}_{t}$ denotes the mean orbit period of the target. For different values of $\Delta t_{\text {tran }}$, the value of $F_{1}\left(\Delta t_{\text {tran }}\right)$ will change. Then, the extreme points of $F_{1}$ can be obtained by numerical methods. Assume that in the range $\left(0, \bar{T}_{t}\right]$ there are $m$ extreme points satisfying $\Delta t_{\text {tran,s1 }}<\Delta t_{\text {tran, } s 2}<\cdots<\Delta t_{\text {tran, } s m}$. Then there are altogether $m+1$ piecewise ranges in $\left(0, \bar{T}_{t}\right]$, that is, $\left(0, \Delta t_{\text {tran }, s 1}\right],\left(\Delta t_{\text {tran }, s 1}, \Delta t_{\text {tran,s2 }}\right], \ldots,\left(\Delta t_{\text {tran,sm }}, \bar{T}_{t}\right]$. If there is no extreme point in $\left(0, \bar{T}_{t}\right]$, then $F_{1}\left(\Delta t_{\text {tran }}\right)$ will decrease or increase monotonically. For each piecewise range ( $\Delta t_{\text {tran, } g_{1}}, \Delta t_{\text {tran, } g_{2}}$, there are two cases.
(1) If $F_{1}\left(\Delta t_{\text {tran }, g 1}+\delta\right) \cdot F_{1}\left(\Delta t_{\text {tran, } 22}\right)>0$, where $\delta$ is a positive constant small enough, there will be no solution in ( $\Delta t_{\text {tran }, g_{1}}, \Delta t_{\text {tran }, g_{2}}$ ].
(2) If $F_{1}\left(\Delta t_{\text {tran }, g 1}+\delta\right) \cdot F_{1}\left(\Delta t_{\operatorname{tran}, g 2}\right)<0$, there will be a single solution in ( $\Delta t_{\text {tran }, g 1}, \Delta t_{\text {tran }, g_{2}}$ ], which can be obtained by the secant method as

$$
\begin{align*}
\Delta t_{\text {tran }}^{(n+2)}= & \Delta t_{\text {tran }}^{(n+1)}-\left(\Delta t_{\text {tran }}^{(n+1)}-\Delta t_{\text {tran }}^{(n)}\right) \\
& \times \frac{F_{1}\left(\Delta t_{\text {tran }}^{(n+1)}\right)}{F_{1}\left(\Delta t_{\text {tran }}^{(n+1)}\right)-F_{1}\left(\Delta t_{\text {tran }}^{(n)}\right)}, \quad n=1,2,3, \ldots, \tag{28}
\end{align*}
$$

where $\Delta t_{\text {tran }}^{(1)}=\Delta t_{\text {tran, } g 1}+\delta, \Delta t_{\text {tran }}^{(2)}=\Delta t_{\text {tran }, g 2}$.
For the tangent to final orbit problem, $F_{2}\left(\Delta t_{\text {tran }}\right)=0$ is required to be solved. By using the same method for $F_{1}\left(\Delta t_{\text {tran }}\right)=0$, the tangent to final orbit problem can be solved with (28) by only replacing $F_{2}$ with $F_{1}$.
4.2. Bitangent Orbit Rendezvous Problem. For the bitangent orbit rendezvous problem, two variables $\Delta t_{1} \in\left(0, \bar{T}_{t}\right]$; and $\Delta t_{\text {tran }} \in\left(0, \bar{T}_{t}\right]$ in (26) are solved. Although the sequential two-point secant method and the sequential $N+1$ point secant method can be used to solve the nonlinear vector equation [19], the initial guesses are not easy to select. Herein, we will use an alternative method to solve the bitangent orbit rendezvous problem.

For a given initial coasting time $\Delta t_{1} \in\left(0, \bar{T}_{t}\right]$, the first impulse is at $t_{1}=t_{0}+\Delta t_{1} \in\left(t_{0}, t_{0}+\bar{T}_{t}\right]$; the transfer time $\Delta t_{\text {tran }}$ in one mean orbital period of the target for the tangent to initial/final orbit problem can be obtained by the proposed method as $\Delta t_{\text {tran, } F_{1}}$ and $\Delta t_{\operatorname{tran}, F_{2}}$. There may exist many solutions of $\Delta t_{\mathrm{tran}, F_{1}}$ and $\Delta t_{\mathrm{tran}, F_{2}}$ for a given initial coasting time $\Delta t_{1}$. Define a new function

$$
\begin{equation*}
\tau\left(\Delta t_{1}\right)=\Delta t_{\text {tran, } F_{1}}^{\left(j_{1}\right)}-\Delta t_{\text {tran, } F_{2},}^{\left(j_{2}\right)} \tag{29}
\end{equation*}
$$

where $j_{1}$ is the $j_{1}$ th solution of $\Delta t_{\text {tran }, F_{1}}$ and $j_{2}$ is the $j_{2}$ th solution of $\Delta t_{\text {tran, } F_{2}}$ such that the expression [ $\Delta t_{\text {tran, } F_{1}}-$ $\left.\Delta t_{\text {tran }, F_{2}}\right]^{2}$ is minimized. Then $\tau=0$ should be satisfied for the bitangent rendezvous problem. It should be notified that $\tau$ is a function only of $\Delta t_{1}$. Then, by using a numerical iterative algorithm, for example, the secant method, all solutions of $\Delta t_{1}$ can be obtained in the range $\left(0, \bar{T}_{t}\right]$.

## 5. Numerical Examples

At the initial time, the nonsingular mean orbital elements of the target are

$$
\begin{array}{ll}
\bar{a}_{t}=R_{E}+10000 \mathrm{~km}, \quad \bar{u}_{t}=20 \mathrm{deg}, & \bar{i}_{t}=90 \mathrm{deg}, \\
\bar{q}_{1 t}=0.295442, \quad \bar{q}_{2 t}=0.0520945, & \bar{\Omega}_{t}=0 \mathrm{deg}, \tag{30}
\end{array}
$$

where $R_{E}=6378.13 \mathrm{~km}$ is the radius of the Earth. Then, it is known that $\bar{e}_{t}=0.3, \bar{\omega}_{t}=10 \mathrm{deg}$ and $\bar{\theta}_{t 0}=10 \mathrm{deg}$. The


Figure 2: Functions $F_{1}$ and $F_{2}$ for different $\Delta t_{\text {tran }}$.
differential mean orbital elements between the chaser and the target are

$$
\begin{align*}
& \delta \bar{a}=-1 \mathrm{~km}, \quad \delta \bar{u}=-0.021 \mathrm{deg}, \quad \delta \bar{i}=0 \mathrm{deg}, \\
& \delta \bar{q}_{1}=-1.96053 \times 10^{-4}, \quad \delta \bar{q}_{2}=-3.98826 \times 10^{-5}  \tag{31}\\
& \delta \bar{\Omega}=0 \mathrm{deg}
\end{align*}
$$

The chaser orbit and the target orbit are coplanar. With the previous parameters, in the LVLH frame of the target, the initial relative position vector is $\mathbf{r}_{0}=[2.4020,-4.2185,0]^{\mathrm{T}} \mathrm{km}$, and the initial relative velocity vector is $\mathbf{v}_{0}=[-0.7223,-2.6583,0]^{\mathrm{T}} \mathrm{m} / \mathrm{s}$. The mean orbit period of the target is $\bar{T}_{t}=20859.684 \mathrm{~s}$. The range of the coasting time is $\Delta t_{1} \in\left(0, \bar{T}_{t}\right]$ and that of the transfer time is $\Delta t_{\text {tran }} \in\left(0, \bar{T}_{t}\right]$.
5.1. Tangent to Initial/Final Orbit. If the impulse is imposed at the initial time, the curves of functions $F_{1}$ and $F_{2}$ for different transfer time $\Delta t_{\text {tran }}$ can be plotted in Figure 2. By using (28), the solution for $F_{1}=0$ can be obtained as $\Delta t_{\text {tran, } F_{1}}=0.704717 \bar{T}_{t}$ and that for $F_{2}=0$ can be obtained as $\Delta t_{\text {tran }, F_{2}}=0.059408 \bar{T}_{t}$.

For the tangent to initial orbit problem, $\Delta t_{\text {tran, } F_{1}}=$ $0.704717 \bar{T}_{t}$, the first impulse is aligned with the initial velocity vector in the ECI frame, that is, $[-0.303824$, $0,0.952728]^{\mathrm{T}}$. Thus, with a tangent impulse whose magnitude is $0.148890 \mathrm{~m} / \mathrm{s}$, the "true" curve of relative position is plotted in Figure 3, which is obtained by the numerical integral for nonlinear equations with the $J_{2}$ perturbation. The final position error is $[11.9539,-31.7167,0]^{\mathrm{T}} \mathrm{m}$ in the LVLH frame of the target.

For the tangent to final orbit problem, $\Delta t_{\operatorname{tran}, F_{2}}=$ $0.059408 \bar{T}_{t}$, the first impulse is $[-5.09566,0,2.56639]^{\mathrm{T}} \mathrm{m} / \mathrm{s}$


Figure 3: Curves of relative position in the LVLH frame of the target for tangent to initial/final orbit.


Figure 4: $\tau$ versus $\Delta t_{1}$.
in the ECI frame and it is not a tangent impulse. The final velocity directions of the chaser and the target in the ECI frame are both $[-0.744802,0,0.667285]^{\mathrm{T}}$, which indicates that the second impulse is a tangent one. The magnitude of the second impulse is $4.13168 \mathrm{~m} / \mathrm{s}$. The curve of relative position is also plotted in Figure 3. The final position error is $[-0.1616,0.1757,0]^{\mathrm{T}} \mathrm{m}$ in the LVLH frame of the target.
5.2. Bitangent Rendezvous. If the first impulse is not at the initial time, the solutions of $\Delta t_{\operatorname{tran}, F_{1}}$ and $\Delta t_{\text {tran, } F_{2}}$ will change. For different initial coasting time $\Delta t_{1}$, all solutions of $\Delta t_{\text {tran, } F_{1}}$ and $\Delta t_{\text {tran }, F_{2}}$ in one (mean) orbit period of the target and the corresponding costs are listed in Table 1, which shows that there are many solutions of $\Delta t_{\mathrm{tran}, F_{1}}$ when $\Delta t_{1}=0.2 \bar{T}_{t}, \Delta t_{1}=0.8 \bar{T}_{t}$, and $\Delta t_{1}=0.9 \bar{T}_{t}$. The plot of


Figure 5: Curves of relative position in the LVLH frame of the target for bitangent rendezvous.

Table 1: Solutions of $\Delta t_{\text {tran, } F_{1}}$ and $\Delta t_{\text {tran, } F_{2}}$ and their costs for different $\Delta t_{1}$.

| $\Delta t_{1}$ (Orbit period) | Tangent to initial orbit |  | Tangent to final orbit |  |
| :---: | :---: | :---: | :---: | :---: |
|  | $\Delta t_{\text {tran }, F_{1}}$ <br> (Orbit period) | $\begin{aligned} & \Delta v_{\text {total }} \\ & (\mathrm{m} / \mathrm{s}) \end{aligned}$ | $\Delta t_{\text {tran, } F_{2}}$ <br> (Orbit period) | $\begin{gathered} \Delta v_{\text {total }} \\ (\mathrm{m} / \mathrm{s}) \end{gathered}$ |
| 0 | 0.704717 | 0.825201 | 0.059408 | 9.837243 |
| 0.1 | 0.502039 | 0.982136 | 0.033730 | 22.20027 |
| 0.2 | 0.004492 | 158.3517 | 0.918635 | 1.034115 |
|  | 0.260872 | 1.949606 |  |  |
| 0.3 | 0.897311 | 1.111503 | 0.561613 | 1.039989 |
| 0.4 | 0.801650 | 1.142299 | 0.342601 | 1.704567 |
| 0.5 | 0.707957 | 1.160767 | 0.193872 | 3.458322 |
| 0.6 | 0.615953 | 1.163889 | 0.097424 | 8.342175 |
| 0.7 | 0.526142 | 1.147815 | 0.034026 | 28.92826 |
| 0.8 | 0.006292 | 184.3432 | 0.875984 | 1.035060 |
|  | 0.440795 | 1.101123 |  |  |
|  | 0.983633 | 1.313181 |  |  |
| 0.9 | 0.022954 | 51.75954 | 0.644049 | 0.655018 |
|  | 0.370941 | 0.977307 |  |  |
|  | 0.837305 | 0.828427 |  |  |

$\tau$ versus $\Delta t_{1}$ can be plotted in Figure 4. The final solution for $\tau=0$ can be obtained as $\Delta t_{1}=0.974968 \overline{T_{t}}$, which is the coasting time for the bitangent rendezvous and the corresponding rendezvous time is $\Delta t_{\text {tran }}=0.540606 \bar{T}_{t}$, and the first impulse is a tangent impulse whose magnitude is $0.258068 \mathrm{~m} / \mathrm{s}$. The final velocity direction of the chaser in the ECI frame is $[0.317860,0,-0.948138]^{\mathrm{T}}$ and that of the target is $[0.317854,0,-0.948140]^{\mathrm{T}}$; thus they are almost the same. The magnitude of the second impulse is $0.258676 \mathrm{~m} / \mathrm{s}$. The curve of relative position is plotted in Figure 5, which includes the initial coasting arc and the rendezvous arc. The
final position error is $[61.0958,-105.1616,0]^{\mathrm{T}} \mathrm{m}$ in the LVLH frame of the target. The total cost of two impulses for the bitangent rendezvous is $0.516744 \mathrm{~m} / \mathrm{s}$.

## 6. Conclusions

In this paper the short-range coplanar orbital rendezvous problem with tangent impulses on the chaser is studied using the linear relative motion with the $J_{2}$ perturbation. Three cases are analyzed with either the first impulse or the second impulse or both impulses being tangent. For the first and the second tangent impulse problems, the numerical solutions are obtained by the secant method for a single variable function. The first tangent impulse can be used in the single-impulse interception problem. The bitangent rendezvous indicates that there exists the same solution for the first and the second tangent impulse problems, which can be obtained by considering the initial coasting time. This method is available for two coplanar elliptic orbits with the $J_{2}$ perturbation. However, since this method is based on the linear relative motion equations, it is only suitable for the short-range case between two spacecraft.

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